Int. J. Adv. Appl. Math. and Mech. 3(4) (2016) 65 – 75 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Liftable vector fields over Mond's map-germs

Research Article

M. S. J. Al-Bahadeli*

Department of Mathematics, College of Science, University of Baghdad, Baghdad, Iraq

Received 31 May 2016; accepted (in revised version) 17 June 2016

Abstract: In this paper we describe generators for the module of holomorphic vector fields tangent to the image of corank 1 holomorphic map-germs from an 2-manifold to an 3-manifold.

MSC: 58K20 • 58K40

Keywords: Liftable and Tangent vector fields • Mond's singularities

1. Introduction

A holomorphic vector field ξ on $(\mathbb{C}^p,0)$ is liftable over a holomorphic map-germ $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ if there is a holomorphic vector field η on $(\mathbb{C}^n,0)$ such that $df\circ\eta=\xi\circ f$. That is, the following diagram commutes

$$\begin{array}{ccc} T(\mathbb{C}^n,0) & \xrightarrow{df} & T(\mathbb{C}^p,0) \\ \eta\uparrow & & \uparrow\xi \\ (\mathbb{C}^n,0) & \xrightarrow{f} & (\mathbb{C}^p,0) \end{array}$$

where $T(\mathbb{C}^q,0)$ denotes the tangent bundle germ for $(\mathbb{C}^q,0)$.

The notion of liftable vector fields over map-germs introduced by Arnol'd [1] as a technical tool to aid with the classification problems of singularities of map-germs since as usual in singularity theory, one can integrate these vector fields to produce a one-parameter family of diffeomorphisms preserving a sub-variety.

For holomorphic map-germs $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ with $n\geq p$, many authors have studied the modules of liftable vector fields and show that the importance of these vector fields in the study of classification problems for more details see ([2], [3], [4], [5], [6], [7] and [8]). However, a little is known about the modules of liftable vector fields with $n<p$.

Houston and Littlestone in [9] study vector fields liftable over corank 1 stable map-germs from an n-manifold to $n+1$-manifold and they gave three families of vector fields with the Euler vector field that are liftable over the minimal cross cap mapping $\varphi_d:(\mathbb{C}^{2d},0)\to(\mathbb{C}^{2d-1},0)$ with $d\geq 2$. Also, in [10], Nishimura and others obtained generators for the module of liftable vector fields over map-germs $f:(\mathbb{C}^n,S)\to(\mathbb{C}^p,0)$ of corank at most one admitting a one-parameter stable unfolding by using a systematic method. However, in this method we need more calculations for concrete liftable vector fields.

Liftable and tangent vector fields on the discriminant are equivalent for holomorphic map-germs $f:(\mathbb{C}^n,0)\to(\mathbb{C}^p,0)$ (for more details see [11], [5] and [12]).In this paper, we well give generators for the module of holomorphic vector fields tangent to the image of corank 1 holomorphic map-germs from an 2-manifold to an 3-manifold. These tangent vector fields agree with the calculation in [10]. Throughout the paper all map-germs and vector fields which we consider will be holomorphic.

* *E-mail address:* m.s.jbara@gmail.com

2. Mond's singularities

In this section we introduce some basic notation and the main result of Mond's classification. For more details see [14].

Let $\mathscr{O}_n$ be the ring of all function-germs $(\mathbb{C}^n, 0) \to \mathbb{C}$. This ring has a maximal ideal $\mathfrak{m}_n$ consisting of germs of functions $f \in \mathscr{O}_n$ with $f(0) = 0$. The set of all map-germs $f : (\mathbb{C}^n, 0) \to \mathbb{C}^p$ is an $\mathscr{O}_n$-module and will be denoted $\mathscr{O}_n^p$. The set of all tangent vector fields in $(\mathbb{C}^p, 0)$ is a free $\mathscr{O}_p$-module of rank p and will be denoted θ_p. The group of all diffeomorphisms $(\mathbb{C}^p, 0) \to (\mathbb{C}^p, 0)$ is denoted $\mathrm{Diff}(\mathbb{C}^n, 0)$.

Definition 2.1.
Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be two map-germs. We say that f and g are $\mathscr{A}$**-equivalent** if there exist diffeomorphism germs $\varphi \in \mathrm{Diff}(\mathbb{C}^n, 0)$ and $\psi \in \mathrm{Diff}(\mathbb{C}^p, 0)$ for which the following diagram commutes

$$\begin{array}{ccc} (\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}^p, 0) \\ \varphi \downarrow & & \psi \downarrow \\ (\mathbb{C}^n, 0) & \xrightarrow{g} & (\mathbb{C}^p, 0) \end{array}$$

i.e. $\psi \circ f = g \circ \varphi$.

Mond classified $\mathscr{A}$-simple map-germs $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$. The models for these germs are the following:

Theorem 2.1.
Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a simple map-germ. Then f is $\mathscr{A}$-equivalent to one of the map-germs in the following:

Label	Normal form (Singularity)
Immersion	$(x, y, 0)$
Cross-cap	(x, y^2, xy)
$\mathrm{S_k}$	$(x, y^2, y^3 + x^{k+1} y), \quad k \geq 1$
$\mathrm{B_k}$	$(x, y^2, x^2 y + y^{2k+1}), \quad k \geq 2$
$\mathrm{C_k}$	$(x, y^2, xy^3 + x^k y), \quad k \geq 3$
$\mathrm{F_4}$	$(x, y^2, x^3 + y^5)$
$\mathrm{H_k}$	$(x, y^3, xy + y^{3k-1}), \quad k \geq 2$

The map-germs $\mathrm{S_k}$, $\mathrm{B_k}$, $\mathrm{C_k}$, $\mathrm{F_4}$ and $\mathrm{H_k}$ are called **Mond's map-germs** or **Mond's singularities**.

3. A defining equation for the image of map-germ

In this section we shall compute the defining equation for the image of the minimal cross cap of multiplicity $d \geq 2$. We define the local algebra of f to be

$$Q(f) := \frac{\mathscr{O}_n}{f^*(\mathfrak{m}_p)} = \frac{\mathscr{O}_n}{\langle f_1, \ldots, f_p \rangle}.$$

Definition 3.1.
A map-germ $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is **finite** if it is continuous, closed and the fiber $F^{-1}(y)$ is finite for all $y \in (\mathbb{C}^p, 0)$.

Let X be a Cohen-Macaulay space of dimension n and $F : (X, x) \to (\mathbb{C}^{n+1}, 0)$ be a finite map-germ. We can use the algorithm of Mond and Pellikaan to determine the corresponding defining equation for the image (see [15], section 2). An algorithm consists basically of the following steps:

1. Choose a projection $\pi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ such that $\widetilde{F} = \pi \circ F$ is finite.

2. After a coordinate change we may suppose that $F(x) = (\widetilde{F}(x), F_{n+1}(x))$. Let X_{n+1} denote the last component of the coordinate system on $\mathbb{C}^{n+1}$ so that $F_{n+1} = X_{n+1} \circ F$.

3. Let $1, g_1, g_2, \ldots, g_k$ be generators of $Q(\widetilde{F})$, where $Q(\widetilde{F})$ is the local algebra of $\widetilde{F}$. Put $g_0 = 1$ and find elements $\alpha_{i,j} \in \mathscr{O}_n$, $0 \le i, j \le k$, such that

$$g_j F_{n+1} = \sum_{i=0}^{k} \left(\alpha_{i,j} \circ \widetilde{F}\right) g_i.$$

4. Define a matrix $\lambda = \left(\lambda_{i,j}\right)$ by letting
   - $\lambda_{i,j} = \alpha_{i,j} \circ \pi$ for $i \ne j$,
   - $\lambda_{i,i} = \alpha_{i,i} \circ \pi - X_{n+1}$.
5. A defining equation for the image of F is given by the determinant of the matrix λ.

Example 3.1.
Consider the mapping $f(x,y) = (x, y^2, yp(x,y^2))$. We choose a projection $\pi : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ such that $\pi(X,Y,Z) = (X,Y)$. Then we have

$$\begin{aligned}\widetilde{f}(x,y) =& \pi \circ f(x,y) \\ =& (x, y^2).\end{aligned}$$

We find that $Q(\widetilde{f})$ is generated by 1 and y. By solving the following equations

$$\begin{aligned} yp(x,y^2) =& \alpha_{0,0}(x,y^2) + \alpha_{1,0}(x,y^2)y \quad \text{and} \\ y^2 p(x,y^2) =& \alpha_{0,1}(x,y^2) + \alpha_{1,1}(x,y^2)y. \end{aligned}$$

We find $\alpha_{0,0}(x,y^2) = 0$, $\alpha_{1,0}(x,y^2) = p(x,y^2)$, $\alpha_{0,1}(x,y^2) = y^2 p(x,y^2)$ and $\alpha_{1,1}(x,y^2) = 0$.
Now,

$$\begin{aligned}\lambda_{0,0} =& \alpha_{0,0} \circ \pi(X,Y) - Z \\ =& 0 - Z \\ =& -Z,\end{aligned}$$

$$\begin{aligned}\lambda_{1,1} =& \alpha_{1,1} \circ \pi(X,Y) - Z \\ =& 0 - Z \\ =& -Z,\end{aligned}$$

$$\begin{aligned}\lambda_{1,0} =& \alpha_{1,0} \circ \pi(X,Y) \\ =& p(X,Y),\end{aligned}$$

$$\begin{aligned}\lambda_{0,1} =& \alpha_{0,1} \circ \pi(X,Y) \\ =& Y p(X,Y).\end{aligned}$$

We obtain the matrix

$$\lambda = \begin{pmatrix} -Z & p(X,Y) \\ Y p(X,Y) & -Z \end{pmatrix}.$$

A defining equation for the image of f is given by the determinant of the matrix λ, i.e.,

$$\begin{aligned}\varphi(X,Y,Z) =& \det(\lambda) \\ =& Z^2 - Y p(X,Y)^2.\end{aligned}$$

Example 3.2.
Consider $\mathrm{H_k}$ singularities, i.e., $f(x,y) = (x, y^3, xy - y^{3k-1})$. It can be show in a similar way in Example 3.1 that

$$\lambda = \begin{pmatrix} -Z & X & Y^{k-1} \\ Y^k & -Z & X \\ XY & Y^k & -Z \end{pmatrix}.$$

A defining equation for the image of f is given by

$$\begin{aligned}\varphi(X,Y,Z) =& \det(\lambda) \\ =& Z^3 - 3XY^k Z - X^3 Y - Y^{3k-1}.\end{aligned}$$

4. Tangent vector fields

Definition 4.1.
Let $\varphi : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be a map-germ. We say that h is a **quasihomogeneous** or **weighted homogeneous** of type $(a_1, \ldots, a_n; d_1, \ldots, d_p)$, with $a_i, d_j \in \mathbb{N}$ if the relation

$$\varphi_j(t^{a_1} x_1, \ldots, t^{a_n} x_n) = t^{d_j} \varphi_j(x_1, \ldots, x_n)$$

holds for each coordinate function φ_j of f for all $t \in (\mathbb{C}, 0)$. The number a_i is called the weight of the variable x_i and the number d_j is the degree of the function φ_j.

Example 4.1.
Consider the defining equation of $\mathrm{S_k}$ map-germ, i.e., $\varphi(X, Y, Z) = Z^2 - Y(Y + X^{k+1})^2$. Then we check that

$$\begin{aligned}\varphi\left(t^2 X, t^{2(k+1)} Y, t^{3(k+1)} Z\right) &= (t^{3(k+1)} Z)^2 - t^{2(k+1)} Y (t^{2(k+1)} Y + (t^2 X)^{k+1})^2 \\ &= t^{6(k+1)} Z^2 - t^{2(k+1)} . t^{4(k+1)} \left(Y(Y + X^{k+1})^2\right) \\ &= t^{6(k+1)} Z^2 - t^{6(k+1)} \left(Y(Y + X^{k+1})^2\right) \\ &= t^{6(k+1)} \left(Z^2 - Y(Y + X^{k+1})^2\right) \\ &= t^{6(k+1)} \varphi(X, Y, Z).\end{aligned}$$

Hence, the defining equation of $\mathrm{S_k}$ map-germ is a quasihomogeneous of type $(2, 2(k+1), 3(k+1); 6(k+1))$.

Let $\mathscr{G} = \mathrm{B_k}, \mathrm{C_k}, \mathrm{F_4}$ or $\mathrm{H_k}$. Then, it can be show in a similar way that the defining equation of $\mathscr{G}$ map-germ is a quasihomogeneous

Definition 4.2.
Suppose that V is a $\mathbb{C}$-analytic variety of $(\mathbb{C}^p, 0)$. We denote by $I(V)$ the ideal of germs vanishing on V. A vector field $\xi \in \theta_p$ is said to be **tangent** to V if

$$\xi(I(V)) \subseteq I(V).$$

The module of such vector fields is denoted $\mathrm{Der}(-\log \mathrm{V})$.

Remark 4.1.

1. When $I(V) = \langle \varphi_1, \ldots, \varphi_q \rangle$, we write

   $$\mathrm{Der}(-\log \mathrm{V}) = \left\{\xi \in \theta_p : \exists g_{ij} \in \mathscr{O}_p \text{ such that } \xi\left(\varphi_j\right) = \sum_{i=1}^{q} g_{ij} \varphi_i, \quad j = 1, \ldots, q\right\}.$$

   Let $\varphi : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ be any defining equation for V, i.e, $V = \varphi^{-1}(0)$. Then we define a submodule of $\mathrm{Der}(-\log \mathrm{V})$ by

   $$\mathrm{Der}_0(-\log \mathrm{V}) = \left\{\xi \in \theta_p : \xi\left(\varphi\right) = 0\right\}.$$

2. The module $\mathrm{Der}(-\log \mathrm{V})$ depends on the choice of equation for V, and not only on V itself. In [11], Damon shows that this module is a finitely generated $\mathscr{O}_p$-module.

3. Let $X_1, \ldots, X_p$ denote the standard coordinates on $\mathbb{C}^p$. Then the **Euler vector field** denoted by ξ_e is given by

   $$\xi_e = \sum_{i=1}^{p} a_i X_i \frac{\partial}{\partial X}.$$

Example 4.2.
From Example 4.1 we have $a_1 = 2$, $a_2 = 2(k+1)$ and $a_3 = 3(k+1)$. Then we can see that the Euler vector field $\xi_e = 2X\frac{\partial}{\partial X} + 2(k+1)Y\frac{\partial}{\partial Y} + 3(k+1)Z\frac{\partial}{\partial Z}$ is tangent to the image of $\mathrm{S_k}$ as follows:

$$\begin{aligned}\xi_e(\varphi) &= 2X\frac{\partial \varphi}{\partial X} + 2(k+1)Y\frac{\partial \varphi}{\partial Y} + 3(k+1)Z\frac{\partial \varphi}{\partial Z} \\ &= 2X\left(-2(k+1)X^k Y^2 - (2k+2)X^{2k+1} Y\right) + 2(k+1)Y\left(-3Y^2 - 4X^{k+1} Y - X^{2k+2}\right) + 3(k+1)Z(2Z) \\ &= 6(k+1)\varphi.\end{aligned}$$

I.e., $\xi_e \in \mathrm{Der}(-\log \mathrm{V})$ where V is the image of $\mathrm{S_k}$.

Theorem 4.1 ([13]).
Let $\varphi : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ be a quasihomogeneous map-germ and $V = \varphi^{-1}(0)$. Then

$$\mathrm{Der}(-\log \mathrm{V}) = \langle \xi_e \rangle \oplus \mathrm{Der}_0(-\log \mathrm{V}).$$

That is, we can conclude that one vector field is Euler and the other annihilate the defining equation.

Remark 4.2.
Let $f, g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be two smooth map-germs with their discriminants V and W, respectively. If f and g are $\mathscr{A}$-equivalent with $f = \psi \circ g \circ \varphi$, then $\psi_* \left(\mathrm{Der}(-\log \mathrm{V})\right) = \mathrm{Der}(-\log \mathrm{W})$ where $\psi_*(\xi) = d\psi.\xi \circ \psi^{-1}$ (see [5] and [10]).

From Theorem 2.1 and Remark 4.2 if we need to find the module tangent vector fields on the discriminant of a map-germ $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, then we need to find the module tangent vector fields on Mond singularities only.

Proposition 4.1.
Let $\varphi : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ be a map-germ. Consider the mapping

$$\Phi : \mathscr{O}_p^p \to \mathscr{O}_p$$

defined by

$$\Phi(\lambda) = \Phi(\lambda_1, \ldots, \lambda_p) = \sum_{i=1}^{p} \lambda_i \frac{\partial \varphi}{\partial x_i},$$

where $\lambda_1, \ldots, \lambda_p$ are the components of λ. Then $\ker(\Phi)$ is spanned as an $\mathscr{O}_p$-module by the set of mappings

$$\left\{ \gamma_{ij} \in \mathscr{O}_p^p | \gamma_{ij} = \frac{\partial \varphi}{\partial x_i} \mathrm{e}_j - \frac{\partial \varphi}{\partial x_j} \mathrm{e}_i, \quad 1 \le i < j \le p \right\},$$

where $x = (x_1, \ldots, x_p)$ and e_k is the vector in $\mathbb{C}^n$ with a 1 in the kth position and zeros elsewhere.

Proof. We the result by induction on p. For $p = 1$, $\ker(\Phi) = \left\langle \frac{\partial \varphi}{\partial x_1} \mathrm{e}_1 \right\rangle$, and the hypothesis is satisfied. For $p = 2$, $\ker(\Phi) = \left\langle (\lambda_1, \lambda_2) | \lambda_1 \frac{\partial \varphi}{\partial x_1} + \lambda_2 \frac{\partial \varphi}{\partial x_2} = 0 \right\rangle$ if and only if $\lambda_1 = -\frac{\partial \varphi}{\partial x_2}$ and $\lambda_2 = \frac{\partial \varphi}{\partial x_1}$ and the hypothesis is satisfied.

Now suppose the result holds for $p = k$, i.e.,

$$\ker(\Phi) = \left\langle \gamma_{ij} \in \mathscr{O}_k^k | \gamma_{ij} = \frac{\partial \varphi}{\partial x_i} \mathrm{e}_j - \frac{\partial \varphi}{\partial x_j} \mathrm{e}_i, \quad 1 \le i < j \le k \right\rangle.$$

Consider $\lambda = (\lambda_1, \ldots, \lambda_{k+1})$ with $\lambda \in \ker(\Phi)$. If $\lambda_{k+1} = 0$, then λ can be viewed as a linear combination of the set $\left\{ \gamma_{ij} \in \mathscr{O}_k^k | \gamma_{ij} = \frac{\partial \varphi}{\partial x_i} \mathrm{e}_j - \frac{\partial \varphi}{\partial x_j} \mathrm{e}_i, \quad 1 \le i < j \le k \right\}$ by hypothesis. Otherwise, let $\widetilde{\lambda} = (\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_{k+1})$. We can find an appropriate $\widetilde{\lambda} \in \ker(\Phi)$ in the form of $\frac{\partial \varphi}{\partial x_i} \mathrm{e}_{k+1} - \frac{\partial \varphi}{\partial x_{k+1}} \mathrm{e}_i$ and then $\lambda = a\widetilde{\lambda} + (\lambda_1, \ldots, \lambda_k, 0)$. Therefore,

$$\lambda \in \left\langle \gamma_{ij} \in \mathscr{O}_{k+1}^{k+1} | \gamma_{ij} = \frac{\partial \varphi}{\partial x_i} \mathrm{e}_j - \frac{\partial \varphi}{\partial x_j} \mathrm{e}_i, \quad 1 \le i < j \le k+1 \right\rangle.$$

The result follows. □

5. Vector fields on $\mathrm{S_k}$ singularities

Theorem 5.1.
Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a S_k-map germ, i.e., a map-germ of the form $f(x, y) = (x, y^2, y^3 + x^{k+1} y)$. Then

$$\mathrm{Der}(-\log \mathrm{V}) = \langle \xi_e, \xi_1, \xi_2, \xi_3 \rangle,$$

where

$$\begin{aligned}
\xi_e &= 2X \frac{\partial}{\partial X} + 2(k+1) Y \frac{\partial}{\partial Y} + 3(k+1) Z \frac{\partial}{\partial Z} \\
\xi_1 &= 3Z \frac{\partial}{\partial X} - 2(k+1) X^k Y \frac{\partial}{\partial Y} - (k+1) \left(X^{k+1} Y + X^{3k+2} \right) \frac{\partial}{\partial Z}. \\
\xi_2 &= 2Z \frac{\partial}{\partial Y} + \left(3Y^2 + 4X^{k+1} Y + X^{2k+2} \right) \frac{\partial}{\partial Z} \\
\xi_3 &= \left(3Y + X^{k+1} \right) \frac{\partial}{\partial X} - 2(k+1) X^k Y \frac{\partial}{\partial Y}
\end{aligned}$$

Proof. Let V be the image of S_k. Then, we have $p(x,y^2)=y^2+x^{k+1}$ and the defining equation of V is given by

$$\varphi(X,Y,Z)=Z^2-Y\left(Y+X^{k+1}\right)^2.$$

From Example 4.1 we have $\xi_e(\varphi)=6(k+1)h$ and

$$\begin{aligned}\xi_1(\varphi)=&3Z\frac{\partial\varphi}{\partial X}-2(k+1)X^kY\frac{\partial\varphi}{\partial Y}-(k+1)\left(X^{k+1}Y+X^{3k+2}\right)\frac{\partial\varphi}{\partial Z}\\=&3Z\left(-2(k+1)X^kY^2-(2k+2)X^{2k+1}Y\right)-2(k+1)X^kY\left(-3Y^2-4X^{k+1}Y-X^{2k+2}\right)\\&\quad-(k+1)\left(X^{k+1}Y+X^{3k+2}\right)(2Z)\\=&0.\end{aligned}$$

In the same way, we can see that $\xi_2(\varphi)=0$ and $\xi_3(\varphi)=0$. Therefore, ξ_e,ξ_1,ξ_2 and ξ_3 are certainly tangent to V. If η is a vector field tangent to the image of S_k, then $\eta(\varphi)=gh$ for some polynomial g. Therefore,

$$\left(\eta-\frac{1}{6(k+1)}g\xi_e\right)(\varphi)=\eta(\varphi)-\left(\frac{1}{6(k+1)}g\xi_e\right)(\varphi)=g\varphi-g\varphi=0.$$

We need to check that ξ_2,ξ_3 and ξ_4 generate all vector fields $\zeta=\zeta_1\frac{\partial}{\partial X}+\zeta_2\frac{\partial}{\partial Y}+\zeta_3\frac{\partial}{\partial Z}$ such that $\zeta\in\mathrm{Der}_0(-\log V)$, i.e., we solve

$$\zeta_1\frac{\partial\varphi}{\partial X}+\zeta_2\frac{\partial\varphi}{\partial Y}+\zeta_3\frac{\partial\varphi}{\partial Z}=0.$$

Put

$$\Phi(\zeta_1,\zeta_2,\zeta_3)=\zeta_1\frac{\partial\varphi}{\partial X}+\zeta_2\frac{\partial\varphi}{\partial Y}+\zeta_3\frac{\partial\varphi}{\partial Z}.$$

Now Proposition 4.1 implies that $\ker(\Phi)=\langle\eta_1,\eta_2,\eta_3\rangle$ where

$$\begin{aligned}\eta_1=&\left(3Y^2+4X^{k+1}Y+X^{2k+2}\right)\frac{\partial}{\partial X}-\left(2(k+1)X^kY^2+(2k+2)X^{2k+1}Y\right)\frac{\partial}{\partial Y}\\=&\left(Y+X^{k+1}\right)\xi_3.\\\eta_2=&-2Z\frac{\partial}{\partial X}-\left(2(k+1)X^kY^2+(2k+2)X^{2k+1}Y\right)\frac{\partial}{\partial Z}\\=&\frac{1}{3}\left(\xi_1+(k+1)X^k\xi_2\right).\\\eta_3=&-2Z\frac{\partial}{\partial Y}-\left(3Y^2+4X^{k+1}Y+X^{2k+2}\right)\frac{\partial}{\partial Z}\\=&-\xi_2.\end{aligned}$$

For all $\zeta\in\mathrm{Der}_0(-\log V)$, we have $\zeta\in\ker(\Phi)$. We can see that ζ is a linear combination of the form $g_1\xi_1+g_2\xi_2+g_3\xi_3$ with $g_i\in\mathscr{O}_3$ for $i=1,2,3$. □

We can see that the tangent vector fields on S_k singularities agree with the calculation in [10], Section 6.8.

6. Vector fields on B_k singularities

Theorem 6.1.
Let $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ be a B_k-map germ with $k\geq 2$, i.e., a map-germ of the form $f(x,y)=(x,y^2,x^2y+y^{2k+1})$. Then

$$\mathrm{Der}(-\log V)=\langle\xi_e,\xi_1,\xi_2,\xi_3\rangle,$$

where

$$\begin{aligned}\xi_e=&kX\frac{\partial}{\partial X}+2Y\frac{\partial}{\partial Y}+(2k+1)Z\frac{\partial}{\partial Z}\\\xi_1=&Z\frac{\partial}{\partial X}+\left(2X^3Y+2XY^{k+1}\right)\frac{\partial}{\partial Z}\\\xi_2=&2Z\frac{\partial}{\partial Y}+\left(X^4+(2k+2)X^2Y^k+(2k+1)Y^{2k}\right)\frac{\partial}{\partial Z}\\\xi_3=&\left(X^2+(2k+1)Y^k\right)\frac{\partial}{\partial X}-4XY\frac{\partial}{\partial Y}.\end{aligned}$$

Proof. Let V be the image of $\mathrm{B_k}$. Then, we have $p(x,y^2)=x^2+y^{2k}$ and the defining equation of V is given by

$$\varphi(X,Y,Z)=Z^2-Y\left(X^2+Y^k\right)^2.$$

We see that

$$\begin{aligned}\xi_e(\varphi)=&kX\frac{\partial\varphi}{\partial X}+2Y\frac{\partial\varphi}{\partial Y}+(2k+1)Z\frac{\partial\varphi}{\partial Z}\\ =&kX\left(-4X^3Y-4XY^{k+1}\right)+2Y\left(-X^4-2(k+1)X^2Y^k-(2k+1)Y^{2k}\right)+(2k+1)Z(2Z)\\ =&2(2k+1)h.\\ \xi_1(\varphi)=&2Z\frac{\partial\varphi}{\partial Y}-2Y\frac{\partial\varphi}{\partial Z}\\ =&2Z\left(-X^4-2(k+1)X^2Y^k-(2k+1)Y^{2k}\right)-\left(-X^4-2(k+1)X^2Y^k-(2k+1)Y^{2k}\right)(2Z)=\quad 0.\end{aligned}$$

In the same way, we can see that $\xi_2(\varphi)=0$ and $\xi_3(\varphi)=0$. Therefore, ξ_e,ξ_1,ξ_2 and ξ_3 are certainly tangent to V. If η is a vector field tangent to the image of $\mathrm{B_k}$, then $\eta(h)=gh$ for some polynomial g. Therefore,

$$\left(\eta-\frac{1}{2(2k+1)}g\xi_1\right)(\varphi)=\eta(\varphi)-\left(\frac{1}{2(2k+1)}g\xi_1\right)(\varphi)=g\varphi-g\varphi=0.$$

We need to check that ξ_2,ξ_3 and ξ_4 generate all vector fields $\zeta=(\zeta_1,\zeta_3,\zeta_3)$ such that $\zeta\in\mathrm{Der}_0(-\log \mathrm{V})$, i.e., we solve

$$\zeta_1\frac{\partial\varphi}{\partial X}+\zeta_2\frac{\partial\varphi}{\partial Y}+\zeta_3\frac{\partial\varphi}{\partial Z}=0.$$

Put

$$\Phi(\zeta_1,\zeta_2,\zeta_3)=\zeta_1\frac{\partial\varphi}{\partial X}+\zeta_2\frac{\partial\varphi}{\partial Y}+\zeta_3\frac{\partial\varphi}{\partial Z}.$$

Now Proposition 4.1 implies that $\ker(\Phi)$ is spanned by

$$\begin{aligned}\eta_1=&\left(X^4+2(k+1)X^2Y^k+(2k+1)Y^{2k}\right)\frac{\partial}{\partial X}-\left(4X^3Y+4XY^{k+1}\right)\frac{\partial}{\partial Y}\\ =&\left(X^2+Y^k\right)\xi_3.\\ \eta_2=&-2Z\frac{\partial}{\partial X}-\left(4X^3Y+4XY^{k+1}\right)\frac{\partial}{\partial Z}\\ =&-\frac{1}{2}\xi_1.\\ \eta_3=&-2Z\frac{\partial}{\partial Y}-\left(X^4+2(k+1)X^2Y^k+(2k+1)Y^{2k}\right)\frac{\partial}{\partial Z}\\ =&-\xi_2.\end{aligned}$$

□

7. Vector fields on $\mathrm{C_k}$ singularities

Theorem 7.1.
Let $f:(\mathbb{C}^2,0)\to(\mathbb{C}^3,0)$ be a C_k-map germ with $k\geq 3$, i.e., a map-germ of the form $f(x,y)=(x,y^2,xy^3+x^ky)$. Then

$$\mathrm{Der}(-\log \mathrm{V})=\langle\xi_e,\xi_1,\xi_2,\xi_3\rangle,$$

where

$$\begin{aligned}\xi_e=&2X\frac{\partial}{\partial X}+2(k-1)Y\frac{\partial}{\partial Y}+(3k-1)Z\frac{\partial}{\partial Z}\\ \xi_1=&Z\frac{\partial}{\partial X}+(XY^3+(k+1)X^kY^2+kX^{2k-1}Y)\frac{\partial}{\partial Z}\\ \xi_2=&2Z\frac{\partial}{\partial Y}+(3X^2Y^2+4X^{k+1}y+X^{2k})\frac{\partial}{\partial Z}\\ \xi_3=&(3XY+X^k)\frac{\partial}{\partial X}-(2Y^2+2kX^{k-1}Y)\frac{\partial}{\partial Y}.\end{aligned}$$

Proof. Let V be the image of C_k. Then, we have $p(x,y^2) = xy^2 + x^k$ and the defining equation of V is given by

$$\varphi(X,Y,Z) = Z^2 - Y\left(XY + X^k\right)^2.$$

We see that

$$\begin{aligned}\xi_1(\varphi) =& 2X\frac{\partial\varphi}{\partial X} + 2(k-1)Y\frac{\partial\varphi}{\partial Y} + (3k-1)Z\frac{\partial\varphi}{\partial Z}\\ =& 2X\left(-2XY^3 - 2(k+1)X^kY^2 - 2kX^{2k-1}Y\right) + 2(k-1)Y\left(-3X^2Y^2 - 4X^{k+1}Y - X^{2k}\right) + (3k-1)Z(2Z)\\ =& 2(3k-1)\varphi.\end{aligned}$$

$$\begin{aligned}\xi_1(\varphi) =& 2Z\frac{\partial\varphi}{\partial Y} - 2Y\frac{\partial\varphi}{\partial Z}\\ =& 2Z\left(-3X^2Y^2 - 4X^{k+1}Y - X^{2k}\right) + \left(3X^2Y^2 + 4X^{k+1}Y + X^{2k}\right)(2Z)\\ =& 0.\end{aligned}$$

In the same way, we can see that $\xi_2(\varphi) = 0$ and $\xi_3(\varphi) = 0$. Therefore, ξ_e, ξ_1, ξ_2 and ξ_3 are certainly tangent to V. If η is a vector field tangent to the image of C_k, then $\eta(\varphi) = g\varphi$ for some polynomial g. Therefore,

$$\left(\eta - \frac{1}{2(3k-1)}g\xi_1\right)(\varphi) = \eta(\varphi) - \left(\frac{1}{2(3k-1)}g\xi_1\right)(\varphi) = g\varphi - g\varphi = 0.$$

We need to check that ξ_2, ξ_3 and ξ_4 generate all vector fields $\zeta = (\zeta_1, \zeta_3, \zeta_3)$ such that $\zeta \in \mathrm{Der}_0(-\log \mathrm{V})$, i.e., we solve

$$\zeta_1\frac{\partial\varphi}{\partial X} + \zeta_2\frac{\partial\varphi}{\partial Y} + \zeta_3\frac{\partial\varphi}{\partial Z} = 0.$$

Put

$$\Phi(\zeta_1,\zeta_2,\zeta_3) = \zeta_1\frac{\partial\varphi}{\partial X} + \zeta_2\frac{\partial\varphi}{\partial Y} + \zeta_3\frac{\partial\varphi}{\partial Z}.$$

Now Proposition 4.1 implies that $\ker(\Phi) = \langle\eta_1,\eta_2,\eta_3\rangle$ where

$$\begin{aligned}\eta_1 =& \left(3X^2Y^2 + 4X^{k+1}Y + X^{2k}\right)\frac{\partial}{\partial X} - 2\left(XY^3 + (k+1)X^kY + kX^{2k-1}Y\right)\frac{\partial}{\partial Y}\\ =& \left(XY + X^k\right)\xi_3.\end{aligned}$$

$$\begin{aligned}\eta_2 =& -2Z\frac{\partial}{\partial X} - 2\left(XY^3 + (k+1)X^kY + kX^{2k-1}Y\right)\frac{\partial}{\partial Z}\\ =& -\frac{1}{2}\xi_1.\\ \eta_3 =& -2Z\frac{\partial}{\partial Y} - \left(3X^2Y^2 + 4X^{k+1}Y + X^{2k}\right)\frac{\partial}{\partial Z}\\ =& -\xi_2.\end{aligned}$$

□

8. Vector fields on F_4 singularities

Theorem 8.1.
Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a F_4-map germ, i.e., a map-germ of the form $f(x,y) = (x, y^2, x^3y + y^5)$. Then

$$\mathrm{Der}(-\log \mathrm{V}) = \langle\xi_e, \xi_1, \xi_2, \xi_3\rangle,$$

where

$$\begin{aligned}\xi_e =& 4X\frac{\partial}{\partial X} + 6(k-1)Y\frac{\partial}{\partial Y} + 15Z\frac{\partial}{\partial Z}\\ \xi_1 =& Z\frac{\partial}{\partial X} + 3(X^2Y^3 + X^5Y)\frac{\partial}{\partial Z}\\ \xi_2 =& 2Z\frac{\partial}{\partial Y} + (5Y^4 + 6X^3Y^2 + X^6)\frac{\partial}{\partial Z}\\ \xi_3 =& (5Y^2 + X^3)\frac{\partial}{\partial X} - 6X^2Y\frac{\partial}{\partial Y}.\end{aligned}$$

Proof. Let V be the image of F_4. Then, we have $p(x,y^2) = x^3 + y^4$ and the defining equation of V is given by

$$\varphi(X,Y,Z) = Z^2 - Y\left(X^3+Y^2\right)^2.$$

We see that

$$\begin{aligned}\xi_e(\varphi) =& 4X\frac{\partial\varphi}{\partial X} + 6Y\frac{\partial\varphi}{\partial Y} + 15Z\frac{\partial\varphi}{\partial Z}\\ =& 4X\left(-6X^5Y - 6X^2Y^3\right) + 6Y\left(-X^6 - 6X^3Y^2 - 5Y^4\right) + 15Z(2Z)\\ =& 30\varphi.\end{aligned}$$

$$\begin{aligned}\xi_1(\varphi) =& 2Z\frac{\partial\varphi}{\partial Y} + \left(5Y^4 + 4X^3Y^2 + X^6\right)\frac{\partial\varphi}{\partial Z}\\ =& 2Z\left(-X^6 - 6X^3Y^2 - 5Y^4\right) + \left(5Y^4 + 4X^3Y^2 + X^6\right)(2Z)\\ =& 0.\end{aligned}$$

In the same way, we can see that $\xi_2(\varphi) = 0$ and $\xi_3(\varphi) = 0$. Therefore, ξ_e, ξ_1, ξ_2 and ξ_3 are certainly tangent to V. If η is a vector field tangent to the image of F_4, then $\eta(\varphi) = g\varphi$ for some polynomial g. Therefore,

$$\left(\eta - \frac{1}{2(3k-1)}g\xi_1\right)(\varphi) = \eta(\varphi) - \left(\frac{1}{2(3k-1)}g\xi_1\right)(\varphi) = g\varphi - g\varphi = 0.$$

We need to check that ξ_2, ξ_3 and ξ_4 generate all vector fields $\zeta = (\zeta_1, \zeta_3, \zeta_3)$ such that $\zeta \in \mathrm{Der}_0(-\log \mathrm{V})$, i.e., we solve

$$\zeta_1\frac{\partial\varphi}{\partial X} + \zeta_2\frac{\partial\varphi}{\partial Y} + \zeta_3\frac{\partial\varphi}{\partial Z} = 0.$$

Put

$$\Phi(\zeta_1,\zeta_2,\zeta_3) = \zeta_1\frac{\partial\varphi}{\partial X} + \zeta_2\frac{\partial\varphi}{\partial Y} + \zeta_3\frac{\partial\varphi}{\partial Z}.$$

Now Proposition 4.1 implies that $\ker(\Phi) = \langle\eta_1,\eta_2,\eta_3\rangle$ where

$$\begin{aligned}\eta_1 =& \left(X^6 + 6X^3Y^2 + 5Y^4\right)\frac{\partial}{\partial X} - 6\left(X^5Y + X^2Y^3\right)\frac{\partial}{\partial Y}\\ =& \left(X^3 + Y^2\right)\xi_3.\end{aligned}$$

$$\begin{aligned}\eta_2 =& -2Z\frac{\partial}{\partial X} - 6\left(X^5Y + X^2Y^3\right)\frac{\partial}{\partial Z}\\ =& -\frac{1}{2}\xi_1.\\ \eta_3 =& -2Z\frac{\partial}{\partial Y} - \left(X^6 + 6X^3Y^2 + 5Y^4\right)\frac{\partial}{\partial Z}\\ =& -\xi_2.\end{aligned}$$

□

9. Vector fields on H_k singularities

Theorem 9.1.
Let $f : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ be a H_k-map germ with $k \geq 3$, i.e., a map-germ of the form $f(x,y) = (x, y^3, xy + y^{3k-1})$. Then

$$\mathrm{Der}(-\log \mathrm{V}) = \langle\xi_e, \xi_1, \xi_2, \xi_3, \xi_4\rangle,$$

where

$$\begin{aligned}\xi_e =& (3k-2)X\frac{\partial}{\partial X} + 3Y\frac{\partial}{\partial Y} + (3k-1)Z\frac{\partial}{\partial Z}\\ \xi_1 =& \left(X^2 + (3k-1)Y^{k-1}Z\right)\frac{\partial}{\partial X} - 3XY\frac{\partial}{\partial Y} - (3k-1)Y^{2k-1}\frac{\partial}{\partial Z}\\ \xi_2 =& \left(XZ + (3k-1)Y^{2k-1}\right)\frac{\partial}{\partial X} - 3YZ\frac{\partial}{\partial Y} - (3k-1)XY^k\frac{\partial}{\partial Z}\\ \xi_3 =& -\left(X^2Y^{k-1} + (3k-1)yY2k - 2Z\right)\frac{\partial}{\partial X} + 3Z^2\frac{\partial}{\partial Y} + \left(X^3 + 3kXY^{k-1}Z\right)\frac{\partial}{\partial Z}\\ \xi_4 =& \left(Z^2 - XY^k\right)\frac{\partial}{\partial X} + \left(X^2Y + Y^kZ\right)\frac{\partial}{\partial Z}\end{aligned}$$

Proof. Let V be the image of $\mathrm{H_k}$. Then from Example 3.2 the defining equation of V is given by

$$\varphi(X,Y,Z) = Z^3 - 3XY^kZ - X^3Y - Y^{3k-1}.$$

We can see that

$$\begin{aligned}
\xi_e(\varphi) =&(3k-2)X\frac{\partial\varphi}{\partial X} + 3Y\frac{\partial\varphi}{\partial Y} + (3k-1)Z\frac{\partial\varphi}{\partial Z}\\
=&(3k-2)X\left(13Y^kZ + 3X^2Y\right) + 3Y\left(-3kY^{k-1}Z - X^3 - (3k-1)Y^{3k-2}\right) + (3k-1)Z\left(3Z^2 - 3XY^k\right)\\
=&3(3k-1)\varphi.\\
\xi_1(\varphi) =&\left(X^2 + (3k-1)Y^{k-1}Z\right)\frac{\partial\varphi}{\partial X} - 3Y\frac{\partial\varphi}{\partial Y} - (3k-1)Z\frac{\partial\varphi}{\partial Z}\\
=&\left(X^2 + (3k-1)Y^{k-1}Z\right)\left(13Y^kZ - 3X^2Y\right) - 3XY\left(-3kY^{k-1}Z - X^3 - (3k-1)Y^{3k-2}\right)\\
&\quad - \left((3k-1)Y^{2k-1}\right)\left(3Z^2 - 3XY^k\right)\\
=&0.
\end{aligned}$$

In the same way, we can see that $\xi_2(\varphi) = 0$, $\xi_3(\varphi) = 0$ and $\xi_4(\varphi) = 0$. Therefore, $\xi_e, \xi_1, \xi_2, \xi_3$ and ξ_4 are certainly tangent to V. If η is a vector field tangent to the image of $\mathrm{H_k}$, then $\eta(\varphi) = g\varphi$ for some polynomial g. Therefore,

$$\left(\eta + \frac{1}{3(3k-1)}g\xi_1\right)(\varphi) = \eta(\varphi) + \left(\frac{1}{3(3k-1)}g\xi_1\right)(\varphi) = g\varphi - g\varphi = 0.$$

We need to check that ξ_2, ξ_3, ξ_4 and ξ_5 generate all vector fields $\zeta = (\zeta_1, \zeta_3, \zeta_3)$ such that $\zeta \in \mathrm{Der}_0(-\log \mathrm{V})$, i.e., we solve

$$\zeta_1\frac{\partial\varphi}{\partial X} + \zeta_2\frac{\partial\varphi}{\partial Y} + \zeta_3\frac{\partial\varphi}{\partial Z} = 0.$$

Put

$$\Phi(\zeta_1,\zeta_2,\zeta_3) = \zeta_1\frac{\partial\varphi}{\partial X} + \zeta_2\frac{\partial\varphi}{\partial Y} + \zeta_3\frac{\partial\varphi}{\partial Z}.$$

Now Proposition 4.1 implies that $\ker(\Phi) = \langle \eta_1, \eta_2, \eta_3 \rangle$ where

$$\begin{aligned}
\eta_1 =&\left(3kY^{k-1}Z + X^3 + (3k-1)Y^{3k-2}\right)\frac{\partial}{\partial X} - 3\left(Y^kz + X^2Y\right)\frac{\partial}{\partial Y}\\
=&X\xi_1 + Y^{k-1}\xi_2.
\end{aligned}$$

$$\begin{aligned}
\eta_2 =&\left(-3Z^2 + 3XY^k\right)\frac{\partial}{\partial X} - 3\left(Y^kz + X^2Y\right)\frac{\partial}{\partial Z}\\
=&-\frac{1}{3}\xi_4.\\
\eta_3 =&\left(-3Z^2 + 3XY^k\right)\frac{\partial}{\partial Y} - \left(3kY^{k-1}Z + X^3 + (3k-1)Y^{3k-2}\right)\frac{\partial}{\partial Z}\\
=&-\left(\xi_3 + Y^{k-1}\xi_1\right).
\end{aligned}$$

□

Remark 9.1.

1. In [10], they show that the minimal number of generators for $\mathrm{Der}(-\log \mathrm{V})$ where V be the image of $\mathrm{B_k}, \mathrm{C_k}, \mathrm{F_4}$ is less than or equal to 5 and of $\mathrm{S_k}$ map-germ is always 4. We can see that from our theorem above the minimal number of generators for $\mathrm{Der}(-\log \mathrm{V})$ is exactly equal to 4. In this case all map-germs admit one-parameter stable unfolding.
2. We have primary results for map-germs admit two-parameter stable unfolding and we hope to complete these results in a subsequent paper.

References

[1] V.I. Arnol'd, Wave front evolution and equivariant Morse lemma, Comm. Pure Appl. Math. 29 (1976) 557-582.
[2] K. Saito, Theory of logarithmic differential forms and logarithmic vector fields, J. Fac. Sci. Univ. Tokyo. 27 (1980) 265-291.
[3] J.W. Bruce, Functions on Discriminants, J. London Math. Soc. 3 (1984) 551-567.
[4] J.W. Bruce, R.M. Roberts, Critical points of functions on analytic varieties, Topology 27 (1988) 57-90.
[5] J.W. Bruce, A.A. du Plessis, L.C. Wilson, Discriminants and liftable vector fields, J. Algebraic Geometry 3 (1994) 725-753.
[6] J.W. Bruce, N.P. Kirk and A.A. du Plessis, Complete transversals and the classifcation of singularities , Nonlinearity 10 (1997) 253-275.
[7] J.W. Bruce, J.M. West, Functions on cross-caps, Math. Proc. Cambridge Philos. Soc. 123 (1998) 19-39.
[8] M.S.J. Al-Bahadeli, Classification of Real and Complex Analytic Map-Germs on the Generalized Cross Cap, PhD thesis, University of Leeds, 2012.
[9] K. Houston, D. Littlestone, Vector fields liftable over corank 1 stable maps, preprint 2009. arXiv:0905.0556v1 [math.DG]
[10] T. Nishimura, R. Oset Sinha, M.A.S. Ruas, R. Wik Atique, Liftable vector fields over corank one multigerms, Math. Ann. Octoper (2015) 1-39.
[11] J. Damon, Deformations of sections of singularities and Gorenstein surface singularities, Amer. J. Math. 109 (1987) 695-722.
[12] J. Damon, $\mathscr{A}$-equivalence and the equivalence of sections of images and discriminants, in Singularity Theory and its Applications, SLNM 1462, D. Mond, J. Montaldi (Eds.), Springer Verlag Berlin, (1991) 93-121.
[13] J. Damon, D. Mond, $\mathscr{A}$-codimension and vanishing topology of discriminants, Invent. Math. 106 (1991), 217-242.
[14] D. Mond, On the classification of germs of maps from $\mathbb{R}^2$ to $\mathbb{R}^3$, Proc. London Math. Soc. 50 (1985) 333-369.
[15] D. Mond and R. Pellikaan, Fitting ideals and Multiple points of analytic mappings, Algebraic Geometry and complex analysis (Pátzcuaro, 1987), Lecture Notes in Mathematics, Springer-Berlin vol. 1414 (1990) 107-161.