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Fractional order generalized thermoelasticity theories: A review

Review Article

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Abstract: In the present article, a comprehensive review of relevant literature is presented to highlight the role of fractional calculus in the field of thermoelasticity. This review is devoted to the generalizations of the classical heat conduction equation and formulation of associated theories of fractional thermoelasticity. The recently developed fractional order thermoelastic models are described with their basic mathematical formulation and characteristic features of these models are illustrated. Finally, the paper concludes with a discussion on the future potential of the use of fractional order theory of thermoelasticity for the analysis of thermodynamical interactions in solid continuum.

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1. Introduction

The subject fractional calculus is the generalization of the ordinary differential and integration to non-integer order. The elegance and the conciseness of fractional operators in describing the transition between distinct differential equations and hence between different models, are attracting an increasing numbers of scientists and researchers in many fields like economics, mathematical physics or engineering.

The fractional calculus became a very attractive subject to mathematicians. Before a systematic study of fractional calculus was undertaken, Euler [1], Lagrange [2] and Fourier [3] mentioned the concept of derivatives of arbitrary order in their studies. Consequently, many different forms of fractional (i.e. non-integer) differential operators were introduced, mainly: Riemann-Liouville derivative, Caputo derivative and Riesz derivative ([4–8]).

Notable contributions have been made to both the theory and application of fractional calculus during the 20th century, when some rather special but natural properties of fractional order derivatives were examined with respect to arbitrary functions. Firstly, Abel [9] has applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. The generalization of the concept of fractional calculus has been subjected to several approaches and some various alternative definitions of fractional derivatives have been explained in Oldham and Spanier [4], Miller and Ross [5], Podlubny [6], Hilfer [7] and Harmann [8]. There are remarkably comprehensive encyclopedic-type monographs written by Samko et al. [10], in which one can find a modern encyclopedic, detailed and rigorous theory of fractional differential equations.

In recent years, fractional calculus has been applied in an increasing number of fields, such as electromagnetism, control engineering and signal processing, chemistry, astrophysics, quantum mechanics, nuclear physics and quantum field theory etc. Models for anomalous transport process in the form of time and/or space fractional convection diffusion wave equation attained much attention and have been considered by many researchers. Moreover, the global dependency and non-local property of fractional derivative is one of the main reasons for its growing popularity. The concept of non-locality utilizing fractional differential operator in thermoelastic models opens new perspective

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Nomenclature

$\Gamma(.)$	Gamma function
α	Fractional order parameter
D_t^α	Fractional derivative of order α with respect to time t
q_i	Components of heat flux vector
v	Thermal displacement
τ_0	Thermal relaxation time
k_{ij}	Thermal conductivity
$T = \theta - \theta_0$,	θ is absolute temperature, T_0 is reference temperature assumed to obey the inequality $\left\lvert \frac{T}{\theta_0} \right\rvert \ll 1$
I^α	Riemann-Liouville fractional integral of order
ϕ	Conductive temperature
η_j	Components of diffusive mass vector
$P = p - P_0$	is the chemical potential per unit mass at non-equilibrium, is the chemical potential per unit mass at natural state
D	Diffusion constant
τ_q	Phase-lag of the heat flux
τ_T	Phase-lag of the temperature gradient
τ_v	Phase-lag for thermal displacement gradient
k^*_{ij}	Material constant
σ_{ij}	Components of stress tensor
e_{ij}	Components of strain tensor
τ^*	Material constant

in the study of thermoelastic deformations in solid mechanics. Moreover, non-standard formulations are essential for a constant progress in the development of new materials.

Caputo [11] and Caputo and Mainardi [12, 13] have employed fractional order derivatives for the description of viscoelastic materials and they have successfully established the relation between the fractional order derivatives and the linear theory of viscoelasticity. They also obtained a very good agreement with the experimental results successfully. Some applications of fractional calculus to various problems of mechanics of solids are reviewed in the book of Mainardi [14].

In some of the materials like: porous materials, biological materials/polymers and colloids, glassy etc. and in some physical situations such as: low-temperature, amorphous media and transient loading etc., the classical conventional coupled thermoelasticity theory based on the classical Fourier's law (Biot's theory [15]) is unsuitable (Dreyer and Struchtrup [16]). In such cases, a generalized thermoelastic model based on an anomalous heat conduction theory involving fractional time-derivatives is more useful (see [Ignaczak and Ostoja-Starzewski [17]). Povstenko [18] made a survey of non-local generalization of the Fourier's law and presented a problem of thermoelasticity that uses fractional heat conduction equation. Povstenko [19] investigated thermal stresses in the framework of a quasi-static uncoupled theory of thermoelasticity based on heat conduction equation with a time fractional derivative in an infinite medium. The review article by Rossikhin and Shitikova [20] is devoted to the analysis of new trends and results in the field of fractional calculus and its application to dynamic problems of solid mechanics. Povstenko [21, 22] has studied the problems in a composite medium based on fractional heat conduction equations with different orders of fractional derivatives. Salehbhai and Timol [23] solved some fractional differential equation using the method of Laplace transform and discussed some special cases. Aslefallah et al. [24] proposed a numerical method to solve the time-fractional diffusion equation and discussed the consistency, stability and convergence analysis. Aslefallah and Rostamy [25] presented a numerical scheme in the general framework of $\theta-$ and spatial extrapolation method for solving space-fractional differential equation. Sumelka [26] presented the application of fractional continuum mechanics to the problem of the theory of thermoelasticity and obtained a solution of the boundary value problem. The book of Povestenko [27] is devoted to the time- and space-nonlocal generalization of Fourier's law, corresponding generalization of the heat conduction equation and formulation of the associated theories of fractional thermoelasticity. This book also presents a picture of state-of-art of fractional thermoelasticity and some problems for the time-fractional heat conduction equation.

In the last few years, fractional calculus theory has been employed successfully in theories of thermoelasticity and several models of fractional order generalized thermoelasticity are established by many authors. Sherief et al. [28] introduced the fractional order theory of thermoelasticity by using the methodology of fractional calculus, proved uniqueness theorem and derived variational principle and reciprocity theorem. Youssef [29] formulated the theory of fractional order generalized thermoelasticity by introducing the Riemann-Liouville fractional integral operator into the generalized heat conduction equation, proved uniqueness theorem and solved one dimensional problem. Ezzat [30] constructed a new mathematical model of fractional heat conduction law in which the generalized Fourier's law of heat conduction is modified by using the new Taylor's series expansion of time fractional order [31]. El-Karamany and Ezzat [32] introduced the two-temperature fractional thermoelasticity theory for non-homogeneous anisotropic elastic solid, proved uniqueness and reciprocal theorems and established the convolutional variational principle. Ezzat and Fayik [33] derived the fractional order theory of thermoelastic diffusion in elastic solids and established variational principle, uniqueness theorem and reciprocity theorem.

A new theory of generalized micropolar thermoelasticity with two temperatures using fractional calculus has been derived by Shaw and Mukhopadhyay [34]. Ezzat et al. [35] formulated the field equations of three-phase lag heat conduction model of linear theory of thermoelasticity with time-fractional order derivatives and proved uniqueness and reciprocity theorems. Hamza et al. [36] established the theory of thermoelasticity associated with two relaxation times using the methodology of fractional calculus and derived uniqueness and reciprocity theorems. Wang et al. [37] constructed fractional order theory of generalized thermoelasticity for elastic media with variable properties. The authors used the Clausius inequality and the higher expansions of the free energy to derive the formulations of anisotropic heterogeneous material with temperature dependent material properties. Recently, Youssef [38] derived a new theory of thermoelasticity with fractional order strain which is considered as a new modification to Duhamel-Neumann's stress-strain relation. In this paper, the author postulated a new unified system of equations that govern seven different models of thermoelasticity in the context of one temperature and two temperature type and solved a one-dimensional problem for an isotropic and homogeneous elastic half-space.

Several authors studied a number of problems by employing the above mentioned theories of fractional order generalized thermoelasticity in solid medium. Some of them are found in the references [39–54]. This paper presents a review of fractional order thermoelastic models that look very promising for future development of fractional order theories. We have presented a short introduction to fractional calculus as a theory of integration and differentiation of non-integer order.

2. Fractional order derivatives: definition and formulation

In literature, there exists a number of definitions of fractional derivative but two of them are popularly used, namely: Riemann-Liouville fractional order derivative and Caputo fractional order derivative, which are given as:

Definition 2.1 (Riemann-Liouville fractional order derivative [5–9]:).
Riemann-Liouville fractional derivative $D_t^\alpha f(x,t)$ of order α with respect to time t is defined as:

$$D_t^\alpha f(x,t) = \frac{\partial^\alpha f(x,t)}{\partial t^\alpha} = \frac{\partial^n}{\partial t^n}\left(\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} f(x,\tau)\,d\tau\right), \quad \text{for} \quad (\mathrm{n}-1<\alpha\le \mathrm{n}). \tag{1}$$

Definition 2.2 (Caputo fractional order derivative [5–9]:).
Caputo fractional derivative $D_t^\alpha f(x,t)$ of order α with respect to time t is defined as:

$$D_t^\alpha f(x,t) = \frac{\partial^\alpha f(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-\tau)^{n-\alpha-1} \frac{\partial^n f(x,\tau)}{\partial \tau^n}\,d\tau, \quad \text{for} \quad (\mathrm{n}-1<\alpha\le \mathrm{n}). \tag{2}$$

3. Fractional order generalized thermoelastic models

- **Sherief's Model of Fractional Order Generalized Thermoelastic Heat Conduction [28]:**

  The heat conduction equation in this model takes the form:

$$q_i + \tau_0 \frac{\partial^\alpha}{\partial t^\alpha} q_i = -k_{ij} T,_j \qquad 0<\alpha\le 1. \tag{3}$$

- **Youssef's Model of Fractional Order Generalized Thermoelastic Heat Conduction [29]:**

  The heat conduction equation in this model takes the form:

$$q_i + \tau_0 \frac{\partial}{\partial t} q_i = -k_{ij} I^{\alpha-1} T,_j \qquad 0<\alpha\le 2, \tag{4}$$

where $\quad I^\alpha f(x,t) = \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_0^t (t-s)^{\alpha-1} f(x,s)\,ds.$

- **Ezzat's Model of Fractional Order Generalized Thermoelastic Heat Conduction [30]:**

  The heat conduction equation in this model takes the form:

$$q_i + \frac{\tau_0}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} q_i = -k_{ij} T,_j \qquad 0 < \alpha \leq 1. \tag{5}$$

- **Fractional Order Two-Temperature Generalized Thermoelastic Heat Conduction Model (El-Karamany and Ezzat [32]):**

  The heat conduction equation in this model takes the form:

$$q_i + \frac{\tau_0}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} q_i = -k_{ij} \phi,_j \qquad 0 < \alpha \leq 1. \tag{6}$$

- **Fractional Order Generalized Thermoelastic Diffusion Model (Ezzat and Fayik [33]):**

  The equation of mass flux vector in this model takes the form:

$$\eta_j + \frac{\tau_0}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} \eta_j = -DP,_i \qquad 0 < \alpha \leq 1. \tag{7}$$

- **Fractional Order Generalized Thermoelastic Model with Three-Phase-Lag (Ezzat et al. [35]):**

  The equation of heat conduction in this model takes the form:

$$\left(1 + \frac{\tau_q^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} + \frac{\tau_q^{2\alpha}}{\Gamma(2\alpha+1)} \frac{\partial^{2\alpha}}{\partial t^{2\alpha}}\right) q_j = \left(\tau_v^* T,_i + k_{ij} \frac{\tau_T^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha} + k_{ij}^* v,_i\right) \qquad 0 < \alpha \leq 1, \tag{8}$$

  where $\tau_v^* = k_{ij} + k_{ij}^* \frac{\tau_v^\alpha}{\Gamma(\alpha+1)} \frac{\partial^{\alpha-1}}{\partial t^{\alpha-1}}.$

- **Generalized Thermoelastic Model with Fractional Order Strain (Youssef [38]):**

  The stress-strain relation in this model is given as:

$$\sigma_{ij} = E^* \tau^{*^\alpha} \frac{\partial^\alpha e_{ij}}{\partial t^\alpha}, \qquad 0 < \alpha \leq 1. \tag{9}$$

4. Summary

The present article is devoted to the development of theories of generalized thermoelasicity with fractional order of derivative and its application to dynamic problems of solid mechanics and structural mechanics. The detailed description of the generalized thermoelastic models involving fractional derivatives is presented along with the definitions of fractional derivative and fractional integral. Thus, this state-of-the-art article has shown that during the last decade, fractional calculus entered the mainstream of engineering analysis and has been widely applied to generalized thermoelastic problems in solid mechanics.

Theory of thermoelasticity with fractional order time derivatives is a new branch of research. In literature, there are a number of investigations with fractional order thermoelasticity. The review presented in this paper should prove useful for researchers in material science, designers of new materials, low temperature physicists as well as for those working on the development of a theory of hyperbolic thermoelasticity with fractional order. The fractional order strain problem may be applicable in the fields of biomechanics, biomedical problems and skin tissues where knowledge of such changes would enable early diagnostic monitoring for the onset of disease and better assessment of the effectiveness of new drugs or therapies.

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