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Generalized Non-Newtonian fluid flow analysis of heat transfer in natural convection: A deductive group symmetry approach

Research Article

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Abstract: A rigorous analysis of heat transfer in natural convection boundary layer flow of the generalized Non-Newtonian fluid over a vertical flat plate is made using deductive group symmetry method. Similarity transformations are derived to reduce the governing equations to a set of nonlinear ordinary differential equations. The system of equations for special Non-Newtonian fluids namely, Powell-Eyring and Williamson are solved numerically employing `bvp4c` algorithmic in MATLAB. The graphical profiles for the interested physical quantities like velocity, temperature, wall skin friction coefficient, for various values of flow parameters are analyzed and discussed. The mutual comparison of numerical solutions is made and discussed for both fluid models. The numerical comparison of local skin friction and local Nusselt number in terms of similarity functions is made to study the effect of Prandtl number.

MSC: 76A05 • 76M55 • 54H15

Keywords: Deductive group symmetry • Non-Newtonian fluid • Similarity solution • Powell-Eyring fluid • Williamson fluid • Skin friction • Flow parameter

1. Introduction

There has been remarkable interest in the boundary layer theory for the flow of Non-Newtonian fluids. Flow of Non-Newtonian fluids has attained a great success in the theory of fluid mechanics due to its applications in biological sciences and industry. Problems of natural convection involving Non-Newtonian fluid models are studied by the many researchers since last many decades. Several techniques are found to analyze and derive the solutions of governing equations. Some of these are cited in Refs. [1–11]. To investigate the Non-Newtonian effects the class of solutions known as similarity solutions plays an important role. This is because; it is only a class of exact solution for the differential equations governing motion of non-Newtonian fluids, which are usually non-linear partial differential equations (PDEs) of the boundary layer type. Further this class of solutions also serves as a reference to check approximate solutions.

The classical theory of fluid mechanics is based upon the hypothesis of a linear relationship between two tensor components, shearing stress and rate of strain,

$$\tau = -\mu \frac{\partial u}{\partial y} \tag{1}$$

The fluids with properties different from that described by Eq. (1), called Non-Newtonian fluids. Non-Newtonian fluids are generally subdivided into following two categories; Viscoelastic fluids and Viscoinelastic fluids. The viscoelastic fluids are those in which stress tensor is related to both the instantaneous strain rate component and the past strain history, whereas in viscoinelastic fluids the stress tensor depends upon the current state of strain component only. Under the impact of both academic curiosity and practical necessity, considerable efforts have been expanded towards investigation of the Non-Newtonian effect for different types of Non-Newtonian fluids for various problems of real world. (See [12–18])

In the literature several information are available on the natural convective heat transfer from a vertical isothermal surface to a Non-Newtonian fluids. Lee and Ames [13], Acrivos [14] and Na and Hansen [15] have obtained similarity solutions of power law fluid in case of flow over a non-isothermal vertical flat plate. It was also found by Acrivos

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Nomenclature

A, B, C	Flow consistency indices
F, G, H	Transformed dependent variables
g	Gravitational force
L	Reference length
Gr	Grashof number
Nu	Nusselt number
Pr	Prandtl number
Re	Reynolds number
u, v	Component of velocity in $x-$ and $y-$ directions respectively
U	Main stream velocity
x, y	Rectangular coordinates
$\mathscr{F}$	Functional relation of Shearing stress and velocity gradient

Greek symbols

α	Thermal diffusivity
β	Volumetric thermal expansion coefficient
ρ	Density
ν	Kinematics viscosity
μ	Viscosity
ψ	Stream function
τ	Shearing stress
Ω	Functional relation of τ and velocity gradient
η	Transformed independent variables
θ	Temperature distribution
θ_w	Temperature distribution near plate wall

Subscripts

∞	Conditions at infinity in the y-direction
x	Local
0	Reference condition

Table 1. Some strain-stress functional relationship of Non-Newtonian fluids.

	Fluid Model	Strain-Stress relation
1	Prandtl	$\tau = A\sin^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)$
2	Powell Eyring	$\tau = \mu\frac{\partial u}{\partial y} + \frac{1}{B}\sinh^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right)$
3	Williamson	$\tau = \left(\frac{A}{B+\frac{\partial u}{\partial y}} + \mu_\infty\right)\frac{\partial u}{\partial y}$
4	Power-law	$\tau = A\left\lvert\frac{\partial u}{\partial y}\right\rvert^{n-1}\frac{\partial u}{\partial y}$
5	Eyring	$\tau = \frac{1}{B}\frac{\partial u}{\partial y} + C\sin\left(\frac{\tau}{A}\right)$
6	Ellis	$-\frac{\partial u}{\partial y} = \left(A + B\lvert\tau\rvert^{\alpha-1}\right)\tau$
7	Prandtl-Eyring	$\tau = A\sinh^{-1}\left(\frac{1}{B}\frac{\partial u}{\partial y}\right)$
8	Sisko	$\tau = A\frac{\partial u}{\partial y} + B\left\lvert\frac{\partial u}{\partial y}\right\rvert^{n}$
9	Reiner-Philippoff	$\tau = \left[\mu_\infty + \frac{\mu_0 - \mu_\infty}{1+(\tau/\tau_s)^2}\right]\frac{\partial u}{\partial y}$
10	Sutterby	$\tau = \mu_0\left[\left(B\frac{\partial u}{\partial y}\right)^{-1}\sinh^{-1}\left(B\frac{\partial u}{\partial y}\right)\right]^{A}\frac{\partial u}{\partial y}.$

[14] that similarity solution does not exist in case of isothermal plate. Na and Hansen [15] have also confirmed the Acrivos' results. But all those previous cases were limited to Power-law fluid only. Here we will concentrate our discussion upon the similarity solution of laminar natural convection flows of all time independent Non-Newtonian viscoinelastic fluids, such a class of fluids are severely omitted in literature due to mathematical complicity of its nonlinear stress-strain relationship.

The similarity technique plays an important role in problem analysis, especially in the boundary layer flows. The similarity method involves the determination of similarity variables which reduce the system of governing partial differential equations in to ordinary differential equations. Indeed similarity solution is the only class of more accurate solution for the governing differential equations. In present work we have analyzed the particular boundary layer problem for the generalized Non-Newtonian fluid flow and numerical solutions are produced for particular Non-Newtonian fluids. Several information are available on similarity solutions for the natural convective heat transfer of a Non-Newtonian fluids [11–15]. Researchers have presented works on natural convection flow [Some of these are [19–22]]. At this point it is worth to note that most of the work has been done for Non-Newtonian power-law fluids, this is because of its mathematical simplicity. However there are empirical Non-Newtonian fluid models based on functional relationship between shear stress and rate of the strain are available [20]. In present work we concentrate our discussion on the similarity solution of steady laminar natural convection flows of generalized Non-Newtonian fluid. Such a class of fluids are severely omitted in analysis due to mathematical complicity of its non-linear stress-strain relationship. Further, from these chart, we noticed that all the similarity solutions presented there in are derived either by adopting or by ad-hoc assumption on similarity variables. In context of these work it is necessary to develop the systematically group transformation for similarity solution. Hence, present work focused on deductive group symmetry analysis based on general group of transformations. The analysis is applied to the particular problem of boundary layer theory. In the present paper similarity analysis is made using deductive group symmetry approach based on general group transformation is, to analyze the steady of the laminar natural convection heat transfer of all time independent non-Newtonian fluids over a non-isothermal vertical flat plate, characterized by the property that its stress tensor component τ_{ij} can be related to the strain rate component e_{ij} by the arbitrary continuous function of the type

$$\mathscr{F}\left(\tau_{ij}, e_{ij}\right) = 0 \tag{2}$$

The analysis carried out in present paper is more general and systematic along with auxiliary conditions. This method has been successfully applied to various non-linear problems by Abd-el-Malek et al.[16], Darji and Timol [17, 23], Adnan et al.[24].

2. Governing equations

Consider the steady laminar natural convection flow of a non-Newtonian fluid over a vertical permeable surface. Consider the vertical upward along the surface as positive $x-$direction, and the origin is fixed (Fig. 1). Also for a class

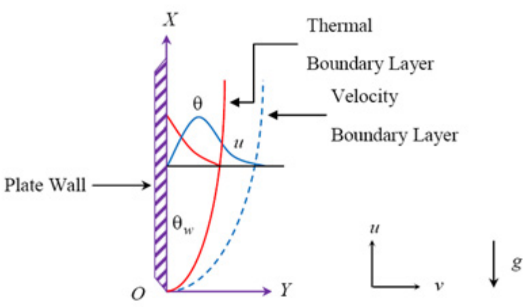


Fig. 1. Schematic diagram of the natural convection.

of Non-Newtonian fluids, the stress-strain relation, under the boundary layer assumption can be found in the form of arbitrary function with only non-vanishing component τ_{yx}. Then the relation (2) can be given by (see [25])

$$\mathscr{F}\left(\tau_{yx}, \frac{\partial u}{\partial y}\right) = 0 \tag{3}$$

Examples of some such models are tabulated in Table 1.

The basic governing equation of such a system can be written (See [26, 27]) after employing Boussinesq approximation along with an order analysis, as

Continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{4}$$

Momentum Equation:

$$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\tau_{yx}\right) + g\beta\theta \tag{5}$$

Energy Equation:

$$u\frac{\partial \theta}{\partial x} + v\frac{\partial \theta}{\partial y} = \alpha\frac{\partial^2 \theta}{\partial y^2} \tag{6}$$

Together with boundary conditions

$$\left.\begin{array}{ll} \forall x, & u = v = 0, \quad \theta = \theta_w(x) \text{ at } y = 0 \\ \forall x, & u = \theta = 0 \text{ as } y \to \infty \end{array}\right\} \tag{7}$$

where α is thermal diffusivity, β is the volumetric thermal expansion coefficient.

Introducing following dimensionless quantities,

$$\begin{aligned} & x^* = \frac{Gr}{L}x, \quad y^* = \left(\frac{Re}{3}\cdot Gr\right)^{1/2}\frac{y}{L}, \\ & u^* = \frac{u}{U_\infty}, \quad v^* = \left(\frac{Re}{3Gr}\right)^{1/2}\frac{v}{U_\infty}, \\ & \tau^*_{yx} = \left(\frac{Re}{3Gr}\right)^{1/2}\frac{\tau_{yx}}{\rho U_\infty^2}, \quad \theta^* = \frac{\theta}{(\theta_w - \theta_\infty)}, \quad \theta^*_w = \frac{\theta_w}{(\theta_w - \theta_\infty)}, \\ & Re = \frac{U_\infty L}{v}, \quad P_r = \frac{U_0 L}{\alpha Re}, \quad Gr = \frac{L}{U_\infty^2} g\beta(\theta_w - \theta_\infty) \end{aligned}$$

where L is the reference length of plate, U_0 is the reference velocity, U_∞ is the far velocity (near boundary layer), θ_w and $\theta_\infty$are the absolute temperatures of fluid near plate wall and near boundary layer respectively. Substitute the values in (3) to (7) along with non-dimensional stream function $\psi^*\left(x^*, y^*\right)$ such that $u^* = \frac{\partial \psi^*}{\partial y^*}$ and $v^* = -\frac{\partial \psi^*}{\partial x^*}$, and dropping the asterisks (for simplicity), we get

$$\mathscr{F}\left(\tau_{yx}, \frac{\partial u}{\partial y}\right) = 0 \tag{8}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial y \partial x} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \frac{\partial}{\partial y}\left(\tau_{yx}\right) + \theta \tag{9}$$

$$\frac{\partial \psi}{\partial y}\frac{\partial \theta}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \theta}{\partial y} = \frac{1}{3Pr}\frac{\partial^2 \theta}{\partial y^2} \tag{10}$$

Together with boundary conditions,

$$\left.\begin{array}{l} \frac{\partial \psi}{\partial y}(x,0) = \frac{\partial \psi}{\partial x}(x,0) = 0, \quad \theta(x,0) = \theta_w(x) \\ \frac{\partial \psi}{\partial y}(x,\infty) = \theta(x,\infty) = 0 \end{array}\right\} \tag{11}$$

3. Deductive group symmetry analysis

We now seek some sort of transformation namely, similarity transformation which transforms the partial differential Eqs. (8)-(10) into the ordinary differential equations along with appropriate auxiliary conditions (11). To search this transformation we introduce one parameter general deductive group of transformations as,

$$G: \quad \overline{Q}(a) = \aleph^Q(a)\, s + \Re^Q \tag{12}$$

where Q stands for $x, y, \psi, \theta, \tau_{yx}$ $\aleph's$ and $\Re's$ are real-valued and at least differential in the real argument a. To transform the differential equation, transformations of the derivatives of ψ can be obtained obtained from G via chain-rule operations.

3.1. Invariance analysis

Equations (8) to (10) are said to be invariantly transformed, for some functions $\xi_i(a)\,(i = 1, 2, 3)$, whenever

$$\mathscr{F}\left(\overline{\tau}_{\overline{y}\overline{x}}, \frac{\partial^2 \overline{\psi}}{\partial \overline{y}^2}\right) = \xi_1(a)\,\mathscr{F}\left(\tau_{yx}, \frac{\partial^2 \psi}{\partial y^2}\right) \tag{13}$$

$$\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial^2\overline{\psi}}{\partial\overline{y}\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial^2\overline{\psi}}{\partial\overline{y}^2}-\frac{\partial}{\partial\overline{y}}\left(\overline{\tau}_{\overline{yx}}\right)-\overline{\theta} = \xi_2(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial}{\partial y}\left(\tau_{yx}\right)-\theta\right] \quad (14)$$

$$\frac{\partial\overline{\psi}}{\partial\overline{y}}\frac{\partial\overline{\theta}}{\partial\overline{x}}-\frac{\partial\overline{\psi}}{\partial\overline{x}}\frac{\partial\overline{\theta}}{\partial\overline{y}}-\frac{1}{3P_r}\frac{\partial^2\overline{\theta}}{\partial\overline{y}^2} = \xi_3(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{1}{3P_r}\frac{\partial^2\theta}{\partial y^2}\right] \quad (15)$$

Substituting the values from (12), using chain rule operation in above Eqs. (13)-(15), yields

$$\mathscr{F}\left(\aleph^{\tau_{yx}}\tau_{yx}+\Re^{\tau_{yx}},\frac{\aleph^{\psi}}{(\aleph^{y})^2}\frac{\partial^2\psi}{\partial y^2}\right)=\xi_1(a)\mathscr{F}\left(\tau_{yx},\frac{\partial^2\psi}{\partial y^2}\right) \quad (16)$$

$$\frac{(\aleph^{\psi})^2}{\aleph^{x}(\aleph^{y})^2}\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}\right]-\frac{\aleph^{\tau_{yx}}}{\aleph^{y}}\frac{\partial}{\partial y}\left(\tau_{yx}\right)-\left(\aleph^{\theta}\theta+\Re^{\theta}\right)+b\left(\aleph^{B}B+\Re^{B}\right)^2\frac{\aleph^{\psi}}{\aleph^{y}}\frac{\partial\psi}{\partial y}=\xi_2(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial^2\psi}{\partial y\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial^2\psi}{\partial y^2}-\frac{\partial}{\partial y}\left(\tau_{yx}\right)-\theta\right] \quad (17)$$

$$\frac{\aleph^{\psi}\aleph^{\theta}}{\aleph^{x}\aleph^{y}}\left[\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}\right]-\frac{\aleph^{\theta}}{(\aleph^{y})^2}\left[\frac{1}{3P_r}\frac{\partial^2\theta}{\partial y^2}\right] = \xi_3(a)\left[\frac{\partial\psi}{\partial y}\frac{\partial\theta}{\partial x}-\frac{\partial\psi}{\partial x}\frac{\partial\theta}{\partial y}-\frac{1}{3P_r}\frac{\partial^2\theta}{\partial y^2}\right] \quad (18)$$

The invariance of Eqs. (16)-(18) together with boundary conditions (11), implies that

$$\left.\begin{aligned}&\Re^{\theta}=\Re^{\tau_{yx}}=\Re^{x}=\Re^{y}=\Re^{\psi}=0,\\&\aleph^{\tau_{yx}}=\frac{\aleph^{\psi}}{(\aleph^{y})^2}=1=\xi_1(a),\\&\frac{(\aleph^{\psi})^2}{\aleph^{x}(\aleph^{y})^2}=\frac{\aleph^{\tau_{yx}}}{\aleph^{y}}=\aleph^{\theta}=\xi_2(a),\\&\frac{\aleph^{\psi}\aleph^{\theta}}{\aleph^{x}\aleph^{y}}=\frac{\aleph^{\theta}}{(\aleph^{y})^2}=\xi_3(a),\end{aligned}\right\} \quad (19)$$

These yields,

$$\aleph^{x}=\left(\aleph^{y}\right)^3,\ \aleph^{\psi}=\left(\aleph^{y}\right)^2,\ \aleph^{\theta}=\frac{1}{\aleph^{y}},\ \aleph^{\tau_{yx}}=1. \quad (20)$$

Finally, we get the one-parameter group G, which transforms invariantly the system of Eqs. (8)-(10) along with the auxiliary conditions (11), as

$$G\ :\left\{\begin{aligned}&\overline{x}=\left(\aleph^{y}\right)^3x,\quad \overline{y}=\aleph^{y}y\\&\overline{\psi}=\left(\aleph^{y}\right)^2\psi,\quad \overline{\theta}=\frac{1}{\aleph^{y}}\theta,\quad \overline{\tau}_{\overline{yx}}=\tau_{yx}\end{aligned}\right. \quad (21)$$

3.2. The complete set of absolute invariants

We develop a complete set of absolute invariants so that the original problem (8)-(10) will transformed into similarity equations under derived deductive group (21).

If $\eta=\eta(x,y)$ is the absolute invariant of the independent variables then variables then, the three absolute invariants of for the dependent variables ψ,θ,τ_{yx} are given by

$$g_j\left(x,y,\psi,\theta,\tau_{yx}\right)=F_j(\eta),\quad j=1,2,3. \quad (22)$$

and can be obtained by following first-order linear partial differential equation: (see [28, 29])

$$\sum_{i=1}^{5}\left(\alpha_iQ_i+\beta_i\right)\frac{\partial g}{\partial Q_i}=0,\quad Q_i=x,y,\psi,\theta,\tau_{yx} \quad (23)$$

where

$$\alpha_i=\left.\frac{\partial\aleph^{i}}{\partial a}\right|_{a=a^0}\ \text{and}\ \beta_i=\left.\frac{\partial\Re^{i}}{\partial a}\right|_{a=a^0}\quad (i=1,2,\ldots5) \quad (24)$$

and a^0 denotes the value of a which yields the identity element of the group G.

Since $\Re^{\theta}=\Re^{\tau_{yx}}=\Re^{\psi}=\Re^{x}=\Re^{y}=0$ implies that $\beta_1=\beta_2=\beta_3=\beta_4=\beta_5=0$ and from (24), we get $\alpha_1=3\alpha_2=\frac{3}{2}\alpha_3=-\alpha_4$, $\alpha_5=0$, Eq. (23) yields

$$(\alpha_1x)\frac{\partial g}{\partial x}+\left(\frac{\alpha_1y}{3}\right)\frac{\partial g}{\partial y}+\left(\frac{2\alpha_1\psi}{3}\right)\frac{\partial g}{\partial\psi}-\left(\frac{\alpha_1\theta}{3}\right)\frac{\partial g}{\partial\theta}+(0)\frac{\partial g}{\partial\tau_{yx}}=0. \quad (25)$$

Corresponding characteristic equation of (25) is

$$\frac{dx}{x}=\frac{3dy}{y}=\frac{3d\psi}{2\psi}=-\frac{3d\theta}{\theta}=\frac{d\tau_{yx}}{0} \quad (26)$$

Applying variable separable method, the absolute invariants of independent and dependent variables owing the Eq. (26) are given by

$$\left.\begin{aligned}&\eta(x,y)=yx^{-1/3}\\&F_1(\eta)=\psi x^{-2/3}=F(\eta)\\&F_2(\eta)=\theta x^{1/3}=G(\eta)\\&F_3(\eta)=\tau_{yx}=H(\eta)\end{aligned}\right\} \quad (27)$$

The deductive group of transformations of absolute invariants is

$$G:\left\{\begin{aligned}&\eta(x,y)=yx^{-1/3}\\&\psi=x^{2/3}F(\eta)\\&\theta=x^{-1/3}G(\eta)\\&\tau_{yx}=H(\eta)\end{aligned}\right. \quad (28)$$

3.3. Reduction to ordinary differential equations

Substituting the values of derivatives from (28) in Eqs. (8)-(10), yields to following system of non-linear ordinary differential equations.

$$\left(F'\right)^2-2FF'-3H'-G=0. \quad (29)$$

$$2FG'+F'G+\frac{1}{P_r}G'=0. \quad (30)$$

Together with boundary conditions (11), subject to the wall temperature $\theta_w(x) = x^{-\frac{1}{3}}$ as

$$\left.\begin{array}{l} F(0) = F'(0) = 0, \quad G(0) = 1, \\ F'(\infty) = G(\infty) = 0 \end{array}\right\} \tag{31}$$

where $H(\eta)$ is similarity variable related to non-dimensional strain-stress relation $\mathscr{F}(H, F'') = 0$.

The reduced similarity system (29)-(31) corresponding to governing system for considered particular boundary layer flow is validate for all types of available Non-Newtonian fluids. Hence the analysis is more general for a class of Non-Newtonian fluid flow and any problem of particular Non-Newtonian fluid can be directly derived from similarity system (29)-(31).

4. Important physical quantities

4.1. Velocity components

The dimensionless components of velocity along x- and along y- direction in terms of similarity variable are given by,

$$u(x,y) = \frac{\partial\psi}{\partial y} = x^{1/3} F'(\eta). \tag{32}$$

$$v(x,y) = -\frac{\partial\psi}{\partial x} = -\frac{1}{3} x^{-1/3} \left[2F(\eta) - \eta F'(\eta)\right]. \tag{33}$$

4.2. Skin friction coefficient

The local skin friction coefficient C_{f_x}, is defined by,

$$C_{f_x} \equiv \frac{\tau_w}{\frac{1}{2}\rho U_w^2} \tag{34}$$

where τ_w is the local wall (i.e. at $\tau_w = \tau_{yx}$ at $y = 0$) shear stress, ρ is the fluid density and U_w is the free-stream velocity near plate wall.

Using dimensionless quantity from section 2 and dropping the asterisk, we get

$$\frac{1}{2} C_{f_x} \sqrt{\mathrm{Re}_x/\mathrm{Gr}_x} \equiv \tau_w = \tau_{yx}\big|_{y=0} \tag{35}$$

where Re_x is the local Reynolds number and Gr_x is local Grashof number.

4.3. Nusselt number

The local Nusselt Nu_x number, is defined by, (The rate of heat transfer in terms of the Nusselt number at the wall)

$$Nu_x \equiv \frac{x \cdot h(x)}{k}; \text{ where } h(x) = \frac{q_w(x)}{T_w - T_\infty} \tag{36}$$

where $q_w(x)$ is the local surface heat flux and is given by $q_w(x) = -k\left(\frac{\partial\theta}{\partial y}\right)_{y=0}$ and is the thermal conductivity of the fluid, hence

$$Nu_x \equiv \frac{x}{(T_w - T_\infty)}\left(-\frac{\partial\theta}{\partial y}\right)_{y=0}$$

Using dimensionless quantity and dropping the asterisk, we get

$$Nu_x \left(\frac{\mathrm{Re}_x}{\mathrm{Gr}_x}\right)^{-1/2} \equiv x^{1/3}\left(-\frac{\partial\theta}{\partial y}\right)_{y=0} \tag{37}$$

In terms of similarity transformation, Eq. (37) reduces to

$$Nu_x \left(\frac{\mathrm{Re}_x}{\mathrm{Gr}_x}\right)^{-1/2} \equiv -G'(0) \tag{38}$$

Eq. (38) gives local Nusselt number expression for all types of Non-Newtonian fluid models.

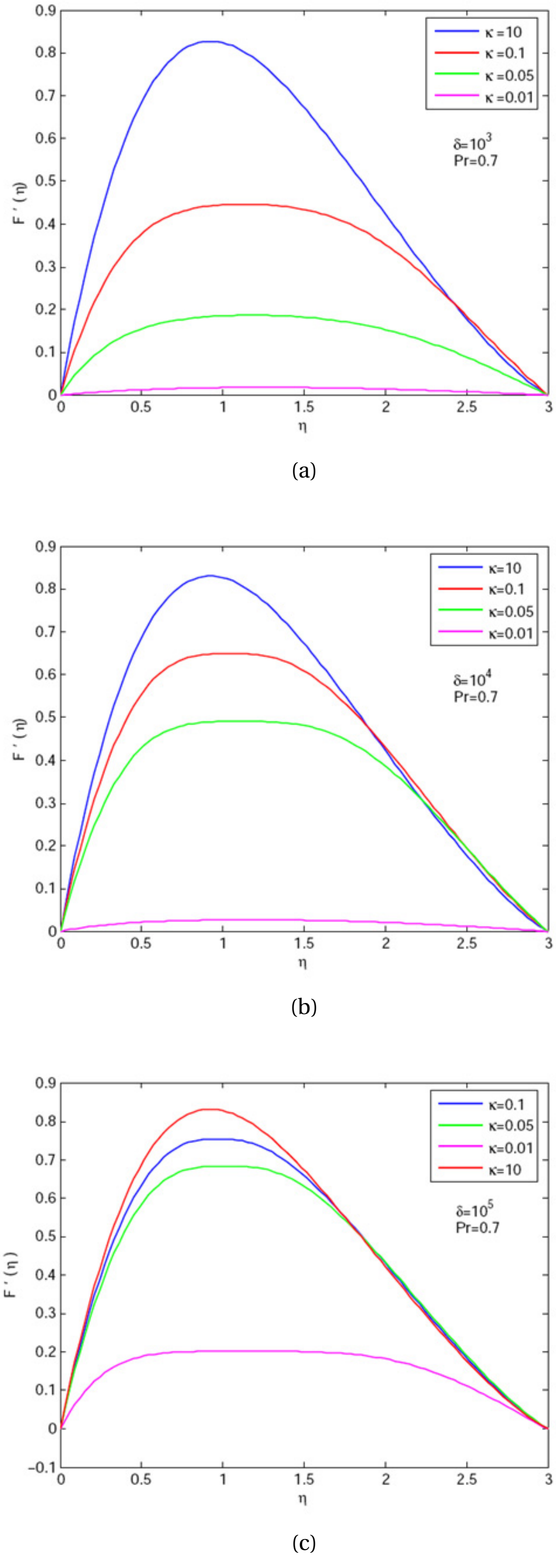


Fig. 2. Influence of δ on similarity velovity function of Powell-Eyring fluid.

5. Particular Non-Newtonian fluids

Non-Newtonian fluid models based on functional relationship between shear-stress and rate of the strain, shown by equation (3) are listed in Table 1. Among these models most research work is so far carried out on power-law fluid model (Model-4 in Table 1), this is because of its math-

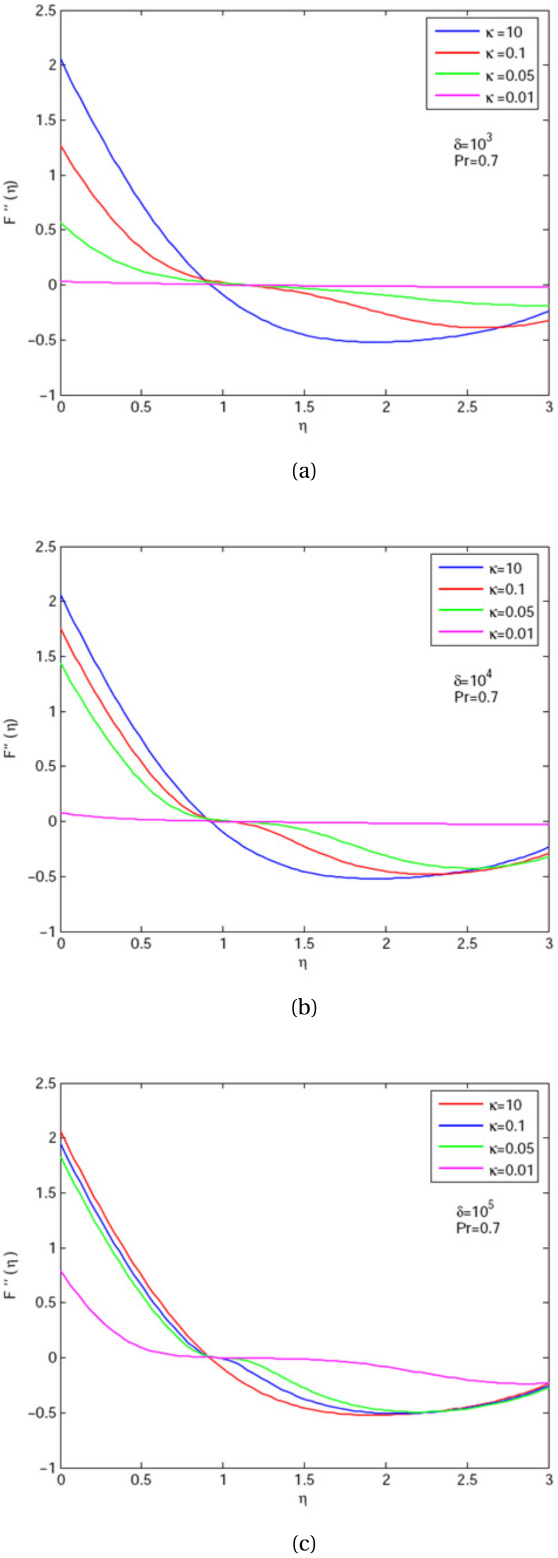


Fig. 3. Influence of κ and δ on similaratity skin coefficient function of Powell-Eyring fluid.

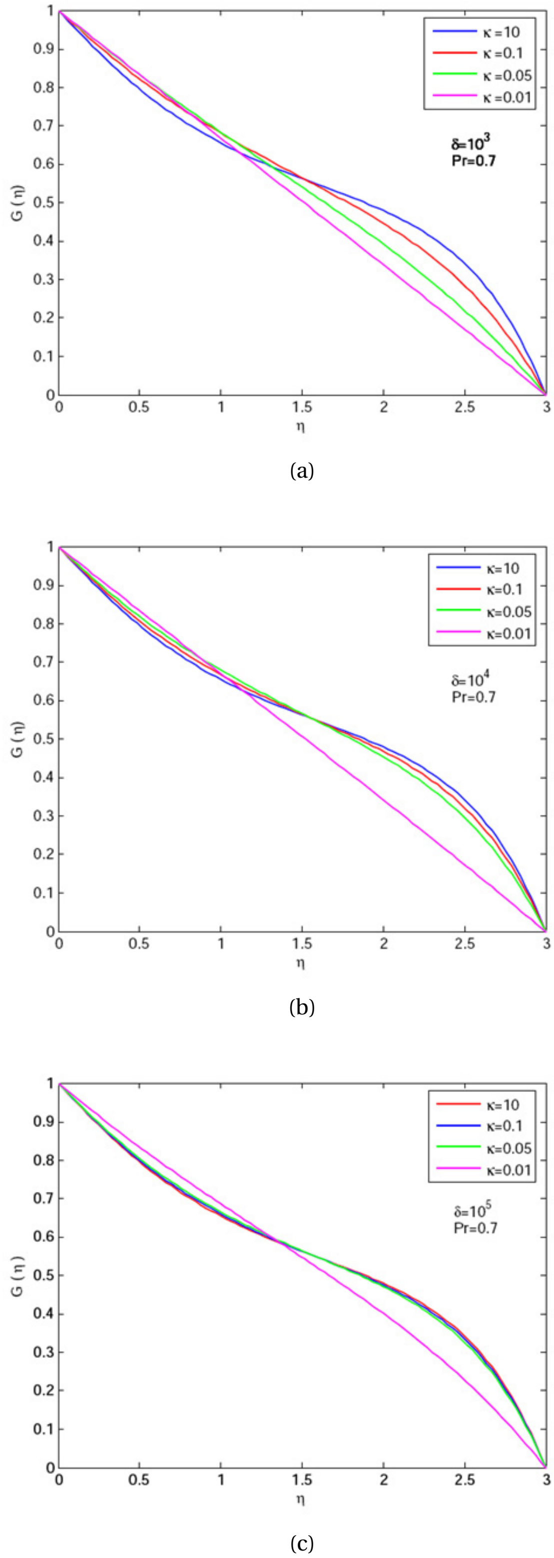
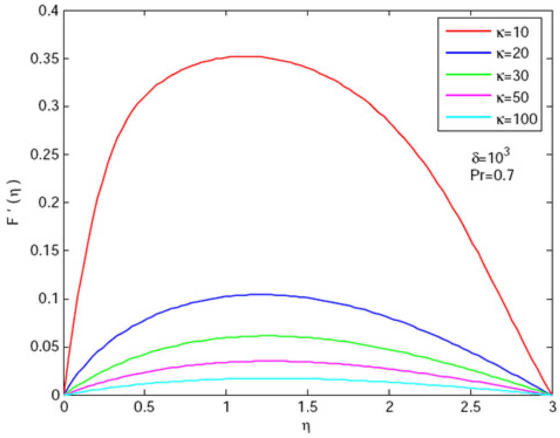


Fig. 4. Influence of κ and δ on similaratity temperature function of Powell-Eyring fluid.

ematical simplicity. On the other hand fluid models other than Power-law model presented in Table 1 are mathematically more complex and the natures of partial differential equations governing these flows are too non-liner boundary value type and hence their analytical or numerical solution is bit difficult. For the present study the partial differential equation model, although mathematically more complex, is chosen mainly due to two reasons. Firstly, it can be deduced from kinetic theory of liquids rather than the empirical relation as in power-law model. Secondly, it correctly reduces to Newtonian behavior for both low and high shear rate. This reason is somewhat opposite to pseudo plastic system whereas the power-law model has infinite effective viscosity for low shear rate and thus limiting its range of applicability.



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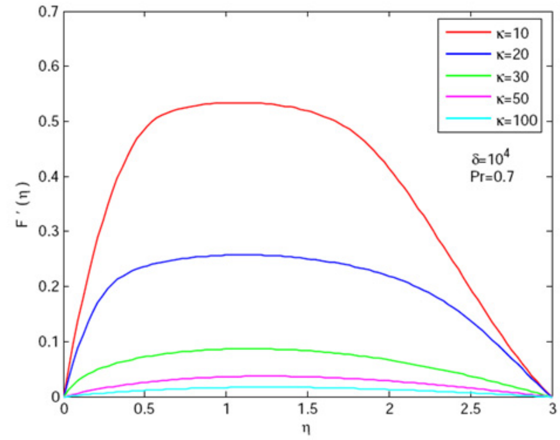

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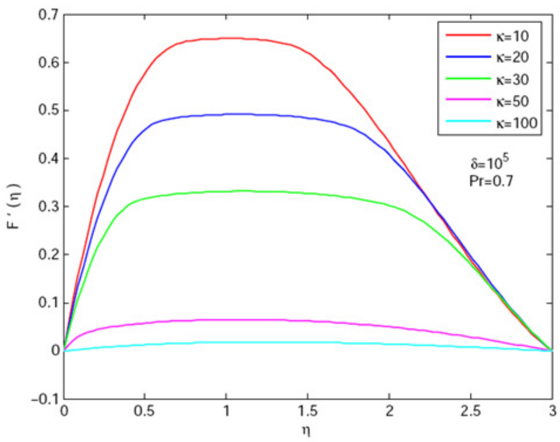

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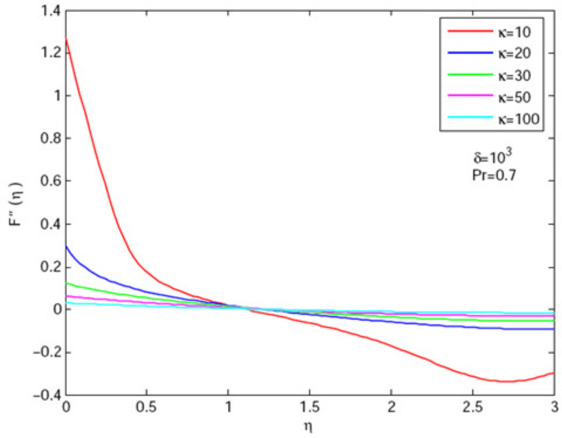
Fig. 5. Influence of κ and δ on similaratity velocity function of Williamson fluid.



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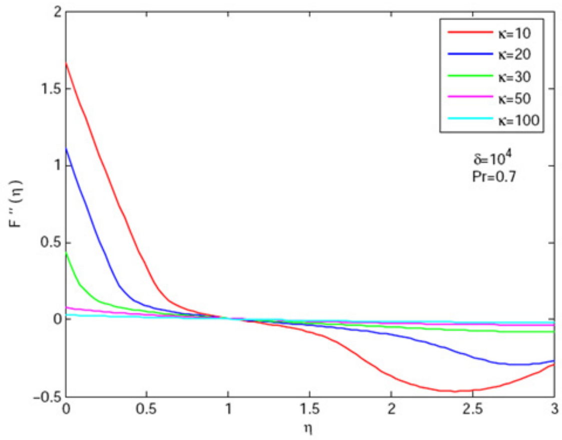

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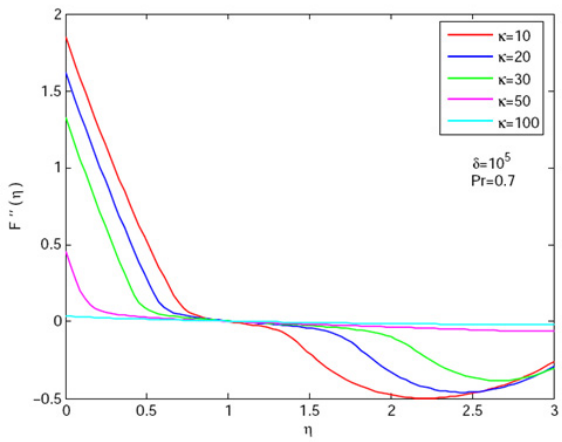


(c)

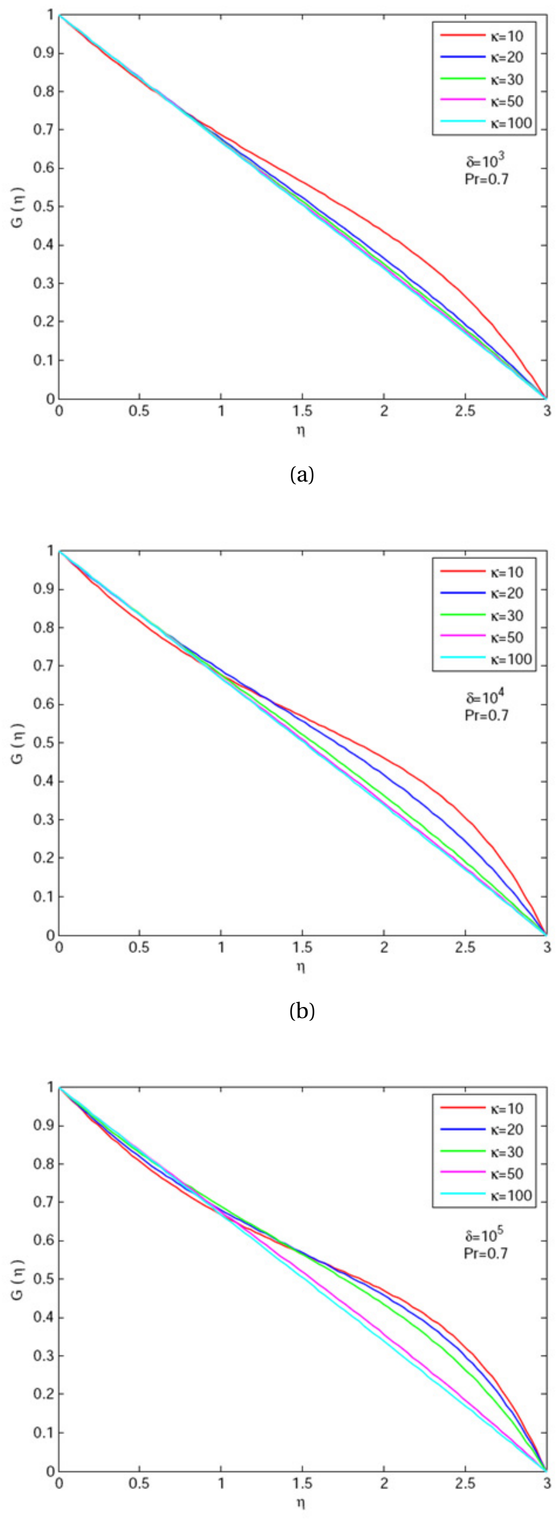
Fig. 6. Influence of κ and δ on similaratity skin coefficient function of Williamson fluid.

5.1. Powell-Eyring fluid model

The study of PowellâĂŞEyring fluid is preferred because it can be deduced from the kinetic theory of liquids rather than empirical relation. Moreover, it correctly reduces to Newtonian behavior for low and high shear rates though this fluid model is mathematically more complex. Mathematically, the Powell-Eyring model can written as, (Skelland [12], Bird et. al. [20])

$$\tau_{yx} = \mu \frac{\partial u}{\partial y} + \frac{1}{B} \sinh^{-1}\left(\frac{1}{C}\frac{\partial u}{\partial y}\right) \tag{39}$$

Introducing the dimensionless quantities into Eq. (39) and differentiating with respect to η (for simplicity dropping the



(a)

(b)

(c)

Fig. 7. Influence of κ and δ on similaratity temperature function of Williamson fluid.

asterisk), we get

$$H'(\eta) = F''' + \frac{\kappa}{\left\{1+\delta(F'')^2\right\}^{1/2}} F'''$$

where $\delta = \frac{\rho U_\infty^3 \mathrm{Gr}}{\mu L C^2}$ and $\kappa = \frac{1}{\mu B C}$ are dimensionless numbers and can be referred as flow parameters.

Substituting the value in Eq. (29), we get

$$F''' = \frac{\frac{1}{3}\left\{(F')^2 - 2FF'' - 3G\right\}\left\{1+\delta(F'')^2\right\}^{1/2}}{\kappa + \left\{1+\delta(F'')^2\right\}^{1/2}} \tag{40}$$

Eq. (40) along with energy equation (30) and boundary conditions (31) represent the similar system of governing equations for Powell-Eyring fluid model.

The skin friction coefficient for the Powell-Eyring fluid model, in terms of flow parameters is,

$$\frac{1}{2} C_{f_x} \sqrt{\mathrm{Re}_x/\mathrm{Gr}_x} \equiv \tau_w = F''(0) + \frac{\kappa}{\sqrt{\delta}} \sinh^{-1}\left\{\sqrt{\delta} F''(0)\right\} \tag{41}$$

5.2. Williamson fluid Model

The purely viscous Non-Newtonian behavior in the shear rate range could not accurately described by the Power-Law model, which has one constant. But can be described by the Williamson model, which has two constants. Mathematically, the strain-stress relationship for the Williamson fluid is given by, (Skelland [12], Bird et. al. [20])

$$\tau_{yx} = \left(\frac{A}{B + \frac{\partial u}{\partial y}} + \mu_\infty\right) \frac{\partial u}{\partial y} \tag{42}$$

Introducing the dimensionless quantities into equation (43) and differentiating with respect to η (for simplicity dropping the asterisk), we get

$$H'(\eta) = F''' + \frac{\kappa}{\left\{1+\sqrt{\delta}(F'')\right\}^2} F''' \tag{43}$$

The flow parameters be $\delta = \frac{\rho U_\infty^3 \mathrm{Gr}}{\mu_\infty L B^2}$ and $\kappa = \frac{A}{\mu_\infty B}$. It is worth to note that on replacing the constants in new notation A by $1/B$, B by C and the ambient viscosity μ by μ_∞ the flow parameters κ and δ turns in to the same notation as we introduced in the Powell-Eyring fluid model. Substituting the value in Eq. (29), we get

$$F''' = \frac{\frac{1}{3}\left\{(F'')^2 - 2FF'' - 3G\right\}\left\{1+\sqrt{\delta}(F'')\right\}^2}{\kappa + \left\{1+\sqrt{\delta}(F'')\right\}^2} \tag{44}$$

Eq. (44) along with energy equation (30) and boundary conditions (31) represent the similar system of governing equations for Powell-Eyring fluid model.

The skin friction coefficient for the Williamson fluid model, in terms of flow parameters is,

$$\frac{1}{2} C_{f_x} \sqrt{\mathrm{Re}_x/\mathrm{Gr}_x} \equiv \tau_w = F''(0) + \frac{\kappa}{1+\sqrt{\delta} F''(0)} \tag{45}$$

6. Results and discussions

- To analyze the effect of flow parameters on various physical quantities, the numerical solutions have produced for similarity variables like $F'(\eta)$, $F''(\eta)$, $G(\eta)$ and $G'(\eta)$ related to horizontal velocity u, local skin friction coefficient C_{f_x}, temperature θ and local Nusselt number Nu_x respectively, for both under considered Non-Newtonian fluids using algorithm of `bvp4c` in MATLAB.

Table 2. Cmparision of Nusselt number and skin friction coefficient for different Pr of Powell-Eyring and Williamson model.

Pr	Powell-Eyring Midel		Williamson Model	
	$-G'(0)$	$F''(0)$	$-G'(0)$	$F''(0)$
0.2	0.33622739	0.80687361	0.33338772	0.1085166
0.5	0.35176293	0.81970672	0.33367911	0.10913637
0.7	0.36839227	0.82245928	0.3340165	0.10953794
1	0.39854874	0.81962236	0.33474774	0.1101122
3	0.57585065	0.7548464	0.34660802	0.11303697
10	0.85819372	0.64475113	0.44061693	0.11106176

- The numerical profiles are obtained for several sets of values of flow parameters (κ and δ) and Prandtl number (Pr) for Powell-Eyring fluid (Figs. 2-4) and Williamson fluid (Figs. 5-7).
- From the numerical solution of the Powell-Eyring fluid model, it is to be observed that as the value of flow parameter increases, the skin friction at wall increases, Whereas from the numerical solution of Williamson fluid it is to be observed that as the flow parameter increases there is initially a rapid increase in skin friction coefficient and then after slightly increases with given parameters.
- A decrease in flow parameter will decrease the skin friction at wall in Powell-Eyring fluid. On the other hand for Williamson fluid, controlling the value of flow parameter, a decrease in flow parameter will increase the skin friction at wall. This is opposite effect from Powell-Eyring fluid.
- These show the different influence of the flow parameters on skin friction coefficient and local Nusselt number in Powell-Eyring fluid and Williamson fluid.
- Further, it is worth to observe that in both the fluids as the Prandtl number increase; there is sharp fall down in velocity profiles. It shows that only increase in Prandtl number will increase the skin friction coefficients, in other words decrement in Prandtl number will sharply decrease the skin friction coefficient at wall.
- Table 2 depicts the numerical values of local skin friction coefficient and the local Nusselt number expression for different values of Prandtl number in case of Powell-Eyring fluid and Willamson fluid.

7. Conclusion

A rigorous analysis of boundary layer equations for natural convection, laminar incompressible flow of generalized Non-Newtonian fluid past over vertical plate is made using the deductive group symmetry method. The governing system of non-linear partial differential equations is transformed in to the system of non-linear ordinary differential equations by intoducing flow parameters via derived similarity variables. For the particular Non-Newtonian fluids namely Powell-Eyring and Williamson, the reduced non-linear simultaneous equations have solved using `bvp4c` in MATLAB. The numerical solutions for different flow parameters as well as for different Prandtl numbers are obtained and various profiles are generated. A comparative study of influence of flow parameters and Prandtl number on both Non-Newtonian fluids have discussed.

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