

A theory of exactly integrable quadratic Liénard type equations

Research Article

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Abstract: The dynamics of quadratic Liénard type equations is usually investigated in the only context of periodic solutions. The problem of interest in this work is then to demonstrate the existence of a simple variable transformation generating a class of exactly integrable quadratic Liénard type equations that preserves the three distinct damped dynamical operating regimes of nonlinear oscillators. Specific examples of equation belonging to this class and their exact solutions in terms of the periodic solution to linear harmonic oscillator are provided for illustrating the developed theory.

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1. Introduction

Many dynamical systems in fields of physics, engineering and quantitative biology applications are represented in terms of nonlinear differential equations [1–6]. Recently, it is shown in [3–9] that the mixed and quadratic Liénard type equations [10–12] may be used to mathematically represent the deformation under relaxation process and oscillations in a variety of viscoelastic nonlinear dynamical systems. The quadratic Liénard type differential equation constitutes an attractive theoretical subject since it may model periodic motion of nonlinear oscillators in the context of dissipative nonlinearity properties. The quadratically dissipative Liénard type equation is usually constructed from an analytical approach consisting of transforming the linear harmonic oscillator through a nonlocal transformation. In doing so, its dynamics is often restricted to investigating the periodic solutions. In other words, this class of differential equations includes in general the only term of natural frequency of the associated linear harmonic oscillator. So, many of these nonlinear dissipative differential equations could not be used following an exact analytical approach to investigate adequately all the three damped parametric operating regimes, to say, the well known overdamped, critically damped and underdamped oscillations for dynamical sys-

tems. It is then justified, in this context, for overcoming such a situation, to design a general class of exactly integrable quadratically dissipative Liénard type equations related to damped linear harmonic oscillator equation. The fundamental question to be solved is then how to choose appropriately the variable transformation mapping the damped linear harmonic oscillator equation into the desired class of exactly integrable quadratic Liénard type equations. In this work such a transformation is proposed and used to map the damped linear harmonic oscillator equation into a general class of mixed Liénard type equations, which is reduced to the proposed class of exactly integrable quadratic Liénard type equations under some restriction (section 2). An application of the developed theory is performed (section 3) and the results are discussed (section 4) and finally some conclusions of the work are given in the last section.

2. Nonlinear general theory

This section is devoted to formulate the mathematical problem and to develop the solving process.

2.1. Statement of theoretical problem

This part consists of setting the mathematical problem under solving process. To do so, suppose that the damped linear harmonic oscillator equation is of the form

$$\ddot{y} + \lambda \dot{y} + \omega_0^2 y = 0 \quad (1)$$

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where overdot denotes a differentiation with respect to time t , ω_0 designates the natural frequency and λ the viscous damping factor. The problem consists here of finding, given the Eq. (1), the general class of mixed Liénard type equations, that can be reduced to the general class of exactly integrable quadratic Liénard type equations having the ability to exhibit the three distinct damped dynamical operating regimes.

2.2. Theory of mixed Liénard type equations

To generate the general class of mixed Liénard type equations, consider the following transformation

$$\frac{\dot{y}}{y} = a\dot{x}e^{f(x)} + b \quad (2)$$

where $f(x)$ is an arbitrary function of x , a and b are constants. Using the identity (2), Eq. (1) turns after some mathematical manipulations, into the general class of nonlinear damped oscillator

$$\ddot{x} + \left[f'(x) + ae^{f(x)} \right] \dot{x}^2 + (2b + \lambda)\dot{x} + \left(\frac{\lambda b}{a} + \frac{b^2}{a} + \frac{\omega_0^2}{a} \right) e^{-f(x)} = 0 \quad (3)$$

where prime denotes differentiation with respect to variable $x(t)$. Eq. (3) is known as a mixed Liénard type equation since it contains together the standard linear viscous damping term and a quadratic damping term. It may be observed that the used change of variable modifies the linear viscous damping term, λ , to $\lambda + 2b$ in the mixed Liénard type equation. This fact will be used for determining the desired class of quadratic Liénard type equations that may exhibit all the three damped parametric regimes (subsection 2.3).

2.3. Theory of quadratic Liénard type equations

This subsection aims to carry out the general class of exactly integrable quadratic Liénard type equations that preserves the information about of all the three damped dynamical operating regimes and to perform an analysis of the obtained results. For this, the restriction $\lambda + 2b = 0$, gives the following general class of exactly integrable quadratic Liénard type equations under question

$$\ddot{x} + \left[f'(x) + ae^{f(x)} \right] \dot{x}^2 + \left(\frac{4\omega_0^2 - \lambda^2}{4a} \right) e^{-f(x)} = 0 \quad (4)$$

It may be clearly seen that the general solution of Eq. (4) will depend once the sign of parameter a is fixed, on the sign of term $4\omega_0^2 - \lambda^2$, as exactly known for the damped linear harmonic oscillator equation. In other words, this equation provides the ability to investigate analytically the effects of viscous damping factor λ on the dynamics of nonlinear oscillators belonging to this general class of quadratic Liénard type equations. An interesting subclass is obtained by restricting $f(x) = -\ln[h(x)]$, in Eq. (4), that is

$$\ddot{x} + \left[-\frac{h'(x)}{h(x)} + \frac{a}{h(x)} \right] \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} h(x) = 0 \quad (5)$$

where $h(x)$ is an arbitrary function of x . Eq. (5) is of the general form

$$\ddot{x} + \left[-\frac{h'(x)}{h(x)} + \frac{a}{h(x)} \right] \dot{x}^2 + c h(x) = 0 \quad (6)$$

where c is an arbitrary constant. Eq. (6) belongs to the more general class of quadratic Liénard type equations which has been studied by Sabatini [13]. In [13] the period function is analyzed and first integral and hamiltonian are provided for this class of quadratic Liénard type equations from a relevant approach of dynamical systems theory. The isochronicity property of Eq. (6) requires $c = 1$. That being so, the transformation

$$\frac{\dot{u}}{u} = a \frac{\dot{x}}{h(x)} \quad (7)$$

where $h(x) \neq 0$, deduced from Eq. (2), can map the Eq. (6), into the second order linear harmonic oscillator equation

$$\ddot{u} + ac u = 0 \quad (8)$$

where $u(t) = A \sin(\omega t + \varphi)$, $\omega = \sqrt{ac}$, with $ac > 0$, the constants A and φ are determined using the initial conditions, so that the general exact analytical solution to Eq. (6) as a function of the harmonic solution to Eq. (8) can be deduced following the first order differential equation

$$\dot{x} - \frac{\omega}{a} h(x) \cot(\omega t + \varphi) = 0 \quad (9)$$

The quantity $\omega^2 = ac = \frac{4\omega_0^2 - \lambda^2}{4} > 0$ in the case of Eq. (5) defines the underdamped parametric operating regime. Eq. (8) may also provide non-oscillatory solutions to Eq. (6), restricting $ac < 0$, to say $\frac{4\omega_0^2 - \lambda^2}{4} < 0$, for Eq. (5), which defines the overdamped oscillations regime. The case $ac = 0$, that is to say, $\frac{4\omega_0^2 - \lambda^2}{4} = 0$, for Eq. (5), which corresponds to the critically damped oscillations regime, leads to a free particle dynamics. It is worth to note, on the other hand, that the linearizing transformation (7) can map the mixed Liénard type equation

$$\ddot{x} + \left[-\frac{h'(x)}{h(x)} + \frac{a}{h(x)} \right] \dot{x}^2 + d \dot{x} + c h(x) = 0 \quad (10)$$

into

$$\ddot{u} + d \dot{u} + ac u = 0 \quad (11)$$

which may be used to deduce the general exact explicit solution to Eq. (10). The coefficient d is an arbitrary constant. In this perspective, it is convenient to show the ability of the proposed theory to be used for generating and solving some interesting specific generalized quadratic Liénard type equations in the case $\frac{4\omega_0^2 - \lambda^2}{4} > 0$ (section 3).

3. Application of theory

3.1. Generalized quadratic Morse type equation

The perturbed Morse type equation has gained in the field of nonlinear dynamics a particular interest due to its applications in quantum mechanics [11]. It is intended in this part to perform a generalization of this type of nonlinear oscillator applying the current theory. So, the restriction

$$h(x) = \frac{1 - e^{-\mu x}}{\mu}$$

in Eq. (5) leads to the generalized quadratic Morse type equation

$$\ddot{x} + \mu \left(\frac{a - e^{-\mu x}}{1 - e^{-\mu x}} \right) \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a\mu} (1 - e^{-\mu x}) = 0 \quad (12)$$

Restricting $\frac{4\omega_0^2 - \lambda^2}{4} > 0$, and using Eq. (9), the general exact analytical solution for Eq. (12) as a function of the periodic solution to Eq. (8) may be written as

$$x(t) = \frac{1}{\mu} \ln \left[1 + \left(\frac{A}{K} \sin(\omega t + \varphi) \right)^{\frac{1}{a}} \right] \quad (13)$$

where $\omega = \sqrt{\frac{4\omega_0^2 - \lambda^2}{4}}$, and K is an arbitrary constant of integration.

Imposing $a = 1$, and $\lambda = 0$, Eq. (12) reduces to the perturbed Morse type equation analyzed in [11], and Eq. (13) for $K = \frac{-1}{\mu}$, provides the same solution as in [11]. On the other hand, it is worth mentioning that Eq. (12) may exhibit a rich diversity of bifurcations and chaos behaviors under forcing function. It is also possible to appropriately choose other expressions for the function $h(x)$ for generating other interesting generalized quadratic Liénard type equations (subsection 3.2).

3.2. Other generalized quadratic Liénard type equations

Case 1: $h(x) = \alpha x^l$

The corresponding equation, after the relation (5), is of the form

$$\ddot{x} + \left(\frac{-l}{x} + \frac{a}{\alpha x^l} \right) \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} \alpha x^l = 0 \quad (14)$$

The general exact explicit solution to Eq. (14) may be written as

$$x(t) = \left[\frac{\alpha(1-l)}{a} \right]^{\frac{1}{1-l}} \left[\ln \left| \frac{A}{K} \sin(\omega t + \varphi) \right| \right]^{\frac{1}{1-l}} \quad (15)$$

Eq. (14) for $\alpha = 1$, and $\lambda = 0$, reduces to

$$\ddot{x} + \left(\frac{-l}{x} + \frac{a}{x^l} \right) \dot{x}^2 + \frac{\omega_0^2}{a} x^l = 0 \quad (16)$$

which is studied in [14], where a first integral approach was proposed for solving. The dynamics of Eq. (16) for $l = 1$ is investigated in [4, 6].

Case 2: $h(x) = x^{l+1} + \alpha^l x$

The application of Eq. (5) gives, as quadratic Liénard type differential equation

$$\ddot{x} + \left[\frac{-(l+1)x^l - \alpha^l + a}{x(x^l + \alpha^l)} \right] \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} x(x^l + \alpha^l) = 0 \quad (17)$$

with the general exact explicit solution given by

$$x(t) = \frac{\alpha \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{\alpha}{a}}}{\left[1 - \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{\alpha}{a}} \right]^{\frac{1}{l}}} \quad (18)$$

An interesting case of (17) is obtained by restricting $l = 2$, and $\alpha^l = a$, that is

$$\ddot{x} - \frac{3x}{a+x^2} \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} (ax + x^3) = 0 \quad (19)$$

since Eq. (19) is a generalized form of an equation which has been studied by Sabatini [13]. The equation which is mentioned by Sabatini [13] corresponds to $a = 1$, and $\frac{4\omega_0^2 - \lambda^2}{4a} = 1$. A more generalized form of Eq. (19) may also be written as

$$\ddot{x} - \frac{3\alpha x}{a + \alpha x^2} \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} (ax + \alpha x^3) = 0 \quad (20)$$

choosing $h(x) = ax + \alpha x^3$.

On the other hand, the parametric choice $l = 1$, $\frac{\alpha}{a} = 1$, and $\frac{a}{K} = 1$, for Eq. (17) gives also an interesting result

$$x(t) = \frac{A \sin(\omega t + \varphi)}{1 - \frac{A}{K} \sin(\omega t + \varphi)} \quad (21)$$

since the solution (21) may exhibit frequency independent amplitude of oscillations, that is to say, isochronicity property as in the case of the harmonic oscillator.

Case 3: $h(x) = x^{l+1} - \alpha^l x$

Applying Eq. (5), one can get the quadratic Liénard type equation

$$\ddot{x} + \left[\frac{-(l+1)x^l + \alpha^l + a}{x(x^l - \alpha^l)} \right] \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} (x^{l+1} - \alpha^l x) = 0 \quad (22)$$

The corresponding general exact solution becomes

$$x(t) = \frac{\alpha}{\left[1 - \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{\alpha}{a}} \right]^{\frac{1}{l}}} \quad (23)$$

Case 4: $h(x) = \sqrt{x^2 + \alpha^2}$

The corresponding quadratic Liénard type equation, after Eq. (5), can be expressed in the form

$$\ddot{x} + \left[\frac{a}{\sqrt{x^2 + \alpha^2}} - \frac{x}{x^2 + \alpha^2} \right] \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} \sqrt{x^2 + \alpha^2} = 0 \quad (24)$$

The general exact explicit solution takes the form

$$x(t) = \frac{1}{2} \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{1}{a}} \left[1 - \alpha^2 \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{-2}{a}} \right] \quad (25)$$

An interesting case is obtained for $\alpha = 0$. The choice $\alpha = 0$, and $a = 1$, leads to harmonic solution, but with a frequency of oscillations which is equal to $\frac{4\omega_0^2 - \lambda^2}{4}$.

Case 5: $h(x) = x\sqrt{x^2 + \alpha^2}$

The quadratic Liénard type equation may be written, taking into account Eq. (5) as

$$\ddot{x} + \left[\frac{-(2x^2 + \alpha^2)}{x(x^2 + \alpha^2)} + \frac{a}{x\sqrt{x^2 + \alpha^2}} \right] \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} x\sqrt{x^2 + \alpha^2} = 0 \quad (26)$$

The general exact analytical solution may be formulated in the form

$$x(t) = \frac{2\alpha \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{\alpha}{a}}}{1 - \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{2\alpha}{a}}} \quad (27)$$

Case 6: $h(x) = x + \frac{\alpha^l}{x^{l-1}}$

According to Eq. (5), the corresponding quadratic Liénard type equation may take the form

$$\ddot{x} + \left[\frac{(l-1)\alpha^l - x^l}{x(x^l + \alpha^l)} + \frac{ax^{l-1}}{x^l + \alpha^l} \right] \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} \frac{x^l + \alpha^l}{x^{l-1}} = 0 \quad (28)$$

The general exact explicit solution may be expressed as

$$x(t) = \left[\left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{1}{a}} - \alpha^l \right]^{\frac{1}{l}} \quad (29)$$

Case 7: $h(x) = e^{\alpha x} - e^{-\alpha x}$

The corresponding quadratic Liénard type equation, by application of Eq. (5), may be written as

$$\ddot{x} + \left[\frac{-\alpha(e^{\alpha x} + e^{-\alpha x}) + a}{e^{\alpha x} - e^{-\alpha x}} \right] \dot{x}^2 + \frac{4\omega_0^2 - \lambda^2}{4a} (e^{\alpha x} - e^{-\alpha x}) = 0 \quad (30)$$

which admits as general exact analytical solution

$$x(t) = \frac{1}{\alpha} \ln \left[\frac{1 + \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{2\alpha}{a}}}{1 - \left[\frac{A}{K} \sin(\omega t + \varphi) \right]^{\frac{2\alpha}{a}}} \right] \quad (31)$$

where $\left(\frac{A}{K}\right)^{\frac{2\alpha}{a}} < 1$.

So with that, it is now possible to discuss some implications of the proposed theory (section 4).

4. Discussion

The mathematical problem of interest in this contribution was to find a general class of exactly integrable quadratically dissipative Liénard type equations that may not be only solved in terms of the periodic solution of linear harmonic oscillator by quadratures, but also has the ability to reproduce all the three distinct damped parametric regimes known for oscillator dynamics. The formulation of an appropriate transformation that connects the damped linear harmonic oscillator to the desired class of nonlinear dissipative differential equations has then become a primordial question. In this work a simple variable transformation has been used to solve the stated problem. So, using the defined variable change, the desired class of quadratic Liénard type equations has been constructed. It is clearly observed that the proposed variable change suffices to express the general exact analytical solution to the quadratic Liénard type equation under question, as a function of the periodic solution to the associated linear harmonic oscillator, in the under-damped parametric operating regime, which is defined by the quantity $\frac{4\omega_0^2 - \lambda^2}{4a} > 0$. Thus, it is worth to note that the period of the exact analytical periodic solution to the considered quadratic Liénard type equations will depend on the viscous damping factor λ of the related damped linear harmonic oscillator. More precisely as shown by the illustrative application, the general exact periodic solution of the considered quadratic Liénard type equation, is expressed as a function of the periodic term $u(t) = A \sin(\omega t + \varphi)$, where the period $T = \frac{2\pi}{\omega} = \frac{4\pi}{\sqrt{4\omega_0^2 - \lambda^2}}$, depends strongly on the viscous damping factor λ . So, it can be clearly seen that, as λ approaches $2\omega_0$, the period $T \rightarrow \infty$. As a result, the studied nonlinear dynamical system will behave without oscillations. By contrast, for $\lambda \rightarrow 0$, the period $T = \frac{2\pi}{\omega_0}$, as in the case

of the linear harmonic oscillator, so that oscillatory behavior should be expected for the nonlinear oscillator under question. Consequently, the viscous damping factor λ can satisfactorily be used for damping the nonlinear oscillations of the dynamical system represented by the quadratic Liénard type equation under study. It is also worth noting that the current theory has shown the ability to perform the generalization of many quadratic Liénard type equations which are known to be interesting for physical and engineering applications. It should be mentioned that a more general variable transformation may be formulated in the current perspective, that is

$$\frac{\dot{y}}{y} = a\dot{x}^l [g(x)]^m e^{\alpha \int h(x) dx} + b\dot{x}^q [f(x)]^n e^{\beta \int \varphi(x) dx} \quad (32)$$

An interesting specific example of this general variable transformation

$$\frac{\dot{y}}{y} = a\dot{x}g(x) + bf(x) \quad (33)$$

leads to the general class of Liénard equations

$$\ddot{x} + \left[\frac{g'(x) + a[g(x)]^2}{g(x)} \right] \dot{x}^2 + \left[2bf(x) + \frac{b}{a} \frac{f'(x)}{g(x)} + \lambda \right] \dot{x} + \frac{b^2}{a} \frac{[f(x)]^2}{g(x)} + \lambda \frac{b}{a} \frac{f(x)}{g(x)} + \frac{\omega_0^2}{ag(x)} = 0 \quad (34)$$

which reduces for $g(x) = \frac{1}{ax}$, with $ax \neq 0$, to the equation

$$\ddot{x} + [2bf(x) + bx f'(x) + \lambda] \dot{x} + [b^2 f(x) + \lambda b] x f(x) + \omega_0^2 x = 0 \quad (35)$$

which finally gives for the choice $f(x) = x$, the equation

$$\ddot{x} + (3bx + \lambda) \dot{x} + b^2 x^3 + \lambda bx^2 + \omega_0^2 x = 0 \quad (36)$$

known as a generalized modified Emden type equation, and for the choice $f(x) = x^2$, the famous equation

$$\ddot{x} + (4bx^2 + \lambda) \dot{x} + b^2 x^5 + \lambda bx^3 + \omega_0^2 x = 0 \quad (37)$$

known under the name of a generalized Duffing-van der Pol type equation. In [15, 16] an equation of this type has been analytically investigated. An interesting case would be also to choose $g(x) = \frac{1}{ax+k}$ where k is an arbitrary constant, to introduce a quadratic nonlinearity of the form kx^2 in the preceding equation. On the other hand, it should be noted that a theory of three distinct damped dynamical operating regimes of exactly integrable quadratic Liénard type equations could also be carried out using the following identity

$$\frac{\dot{y}}{y} = \frac{a}{x} e^{f(x)} + b \quad (38)$$

5. Conclusion

The quadratically dissipative Liénard type equation is often analytically investigated in the context of periodic oscillations since the equation is derived in many cases by transforming the linear harmonic oscillator equation.

In this work the possibility to analyze the quadratic Liénard type equation in the situation of overdamped, critically damped, underdamped and periodic oscillations is explored. So, using a simple appropriate variable change, a general class of exactly integrable quadratically dissipative Liénard type equations has been constructed by transforming the damped linear harmonic oscillator equation. In doing so, it has been possible to consider all the three damped parametric operating regimes and to solve the quadratic Liénard type differential equation under question in terms of general exact analytical solution. In the case of the underdamped operating regime, it has been found that the general exact solution may be expressed as a function of the periodic solution of the linear harmonic oscillator equation for which the fixed frequency of oscillations depends on the linear viscous damping factor. It has been also observed that the current theory has the ability for generalizing some well known quadratic Liénard type equations arising in the fields of physics and engineering applications. As a noteworthy result, exact analytical periodic solutions for equations belonging to the proposed class of integrable quadratic Liénard type equations in the underdamped parametric regime can be computed with the period of related linear harmonic oscillator.

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