Int. J. Adv. Appl. Math. and Mech. 4(2) (2016) 12 – 20 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Thermoelastic theories on elliptical profile objects: An overview & prospective

Review Article

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Received 12 September 2016; accepted (in revised version) 05 October 2016

Abstract: The article investigates the present review of thermoelastic theories on elliptic objects in elliptical coordinate system published on paper after 1950. The main theme of the thermoelastic problems on elliptic structures is to establish operational methods to solve the governing differential equations. In the thermoelastic problems, we have considered few practical problems of technical interest taking into account the elliptical shape geometry by using numerical procedures. Furthermore, by considering a circle as a special kind of ellipse, it is seen that the temperature distribution and comparative study of a circular solution can be derived as a special case from the present mathematical solution.

MSC: 35B07 • 35G30 • 35K05 • 44A10

Keywords: Elliptical objects • Temperature distribution • Thermal stresses • Vibration • Deflection • Integral transform

1. Introduction

In modern engineering applications, elliptical structures are being used extensively due to the added advantage of combining physical, mechanical, as well as thermal properties of different materials compared to other shape objects. Many of these applications require a detailed knowledge of transient temperature and heat-flux distribution within the elliptical objects. Things get further complicated when internal heat generation persists on the object under consideration and further becomes unpredictable when sectional heat supply is impacted on the body. Both analytical and numerical techniques are the best methodology to solve such problems. Nonetheless, numerical solutions are preferred and prevalent in practice, due to either unavailability or higher mathematical complexity of the corresponding exact solutions. Rather, limited use of analytical solutions should not diminish their merit over numerical ones; since exact solutions, if available, provide an insight into the governing physics of the problem, which is normally missing in any numerical solution. Moreover, analyzing closed-form solutions to obtain optimal design options for any particular application of interest is relatively simpler. In addition, exact solutions find their applications in validating and comparing various numerical algorithms to help improve computational efficiency of computer codes that currently rely on numerical techniques. Although heat conduction problems for elliptical profile shape have been studied in great detail and various solution methods have been arrived at; including orthogonal and quasi-orthogonal expansion technique, Laplace transform method, Green's function approach, finite integral transform technique are readily available; there is a continued need to develop and explore novel methods to solve problems for which exact solutions still do not exist. One such problem is to determine exact temperature distribution and thermoelastic behaviour in elliptical coordinates with objects subjected to the generation of heat. Numerous applications involving

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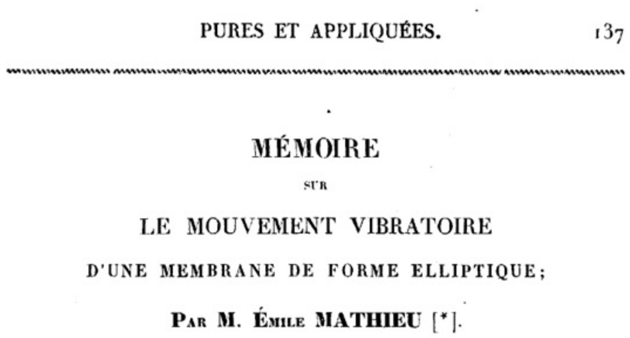

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MÉMOIRE

SUR

LE MOUVEMENT VIBRATOIRE

D'UNE MEMBRANE DE FORME ELLIPTIQUE;

PAR M. ÉMILE MATHIEU [*].

Fig. 1. The first paper ever published on Mathieu function [1].

elliptical geometry require evaluation of temperature distribution and its thermal effect on it. One distinctive example is an elliptical nuclear fuel rod, which is a new type of nuclear fuel rod included in nuclear Pressurised Water Reactors (PWR) and Gas-Cooled Fast Reactors (GCFR), called 4^{th} generation reactors. Elliptical structured composite materials continue to experience increased adoption in aerospace, marine, automobile, and civil structures due to their high specific strength, high stiffness, and light weight. Moreover, several other applications including multilayer insulation materials, double heat-flux conductimeter, typical laser absorption calorimetry experiments, cryogenic systems, and other cylindrical building structures would benefit from such analytical solutions. However, most papers which treat only constitutive equations using numerical procedures, that is, finite element methods, finite difference methods etc. will be excluded initially from the review, but will be considered during practical problem section. Thermoelasticity details on elliptical objects will be reviewed in the following sequence: (a) historical review at a glance, (b) temperature fields, (c) thermal deflection, (d) thermoelastic induced vibrations, (e) time-reversal thermoelasticity, (f) thermoelasticity, (g) transition to circular objects, (h) practical problem of technical interest, and (i) discussion and remarks.

2. Historical review at a glance

When PDEs such as Laplace's, Poisson's, and the wave equation are solved with cylindrical or spherical boundary conditions by separating variables in a coordinate system appropriate to the problem, we find radial solutions, which are generally the Bessel functions. When in cylindrical problems the circular boundary condition is changed to elliptical, we are led to what are now known as Mathieu functions, which might alternatively have been called elliptic cylinder functions. In fact, in 1868, Mathieu [1] developed the leading terms of series solutions of the vibrating elliptical drumhead of a stretched membrane having elliptical boundary condition (refer Fig. 1). Mathieu functions, like many other functions, which are useful in technical fields and applied Mathematics, have originated as a result of the investigations on practical problems.

Most highly cited literature reviews on theory of Mathieu function is given by [7]. McLachlan [6] obtained mathematical solution of the heat conduction problem for elliptical cylinder in the form of an infinite Mathieu function series considering special case by neglecting surface resistance. Gupta [17] introduced a finite transform involving Mathieu functions used for obtaining the solutions of boundary value problem involving elliptic cylinders. Choubey [31] also introduced a finite Mathieu transform whose kernel is given by Mathieu function; to solve heat conduction in a hollow elliptic cylinder with radiation. Kirkpatic and Stokey [12] extended McLachlan's solution involving numerical calculation. Sugano et al. [40] dealt with transient thermal stress on a confocal hollow elliptical structure with both face surfaces insulated perfectly and obtained the analytical solution with couple-stresses. Sato [52, 53] subsequently obtained heat conduction problem of an infinite elliptical cylinder during heating and cooling considering the effect of the surface resistance. There are few reports, [8, 10, 11, 23–25, 33], on the solution of large deflection problems subjected to uniform pressure based on the calculus of variation, Hamilton's principle or any other. Many solutions [27–29] have been reported for large deflection problems for different objects that were subjected to thermal loading. Sato [48, 49] used Mathieu functions to study the bending of a clamped as well as simply supported elliptical plate undergoing the combined action of uniform lateral load over its entire surface and uniform in-plane force distributed at its middle plane. Few theoretical studies concerned with the problems of elastic elliptical membrane vibrations have also been reported so far [14, 56, 59]. Choubey [21] has considered a time reversal heat conduction problem of heat conduction for a solid elliptical cylinder by applying a finite Mathieu transform. Recently, Bagde [57] investigated the time reversal inverse heat conduction problem of an elliptical plate to determine the temperature distribution and unknown temperature gradient at a particular point for all time with the help of Mathieu transform and finite Marchi-Fasulo transform. Taking Green's function in elliptic coordinates, a study was conducted by few researchers [54, 56]. Recently, El Dhaba [47] used boundary integral method to solve the problem of plane, uncoupled

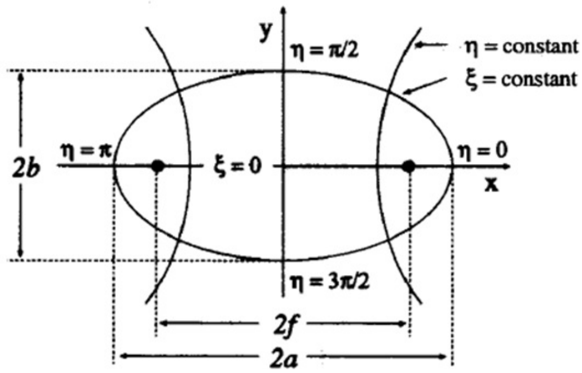

Fig. 2. Elliptical configuration profile.

linear thermoelasticity with heat sources for an infinite cylinder with elliptical cross section, subjected to a uniform pressure having thermal radiation condition at its boundary. Hasheminejad [55] obtained the exact solution for dynamic response of an elastic elliptical membrane by employing eigenfunction expansion in terms of transcendental and modified Mathieu functions. Just recently few literatures dealing with internal heat generation were found. For example, Kulkarni et al. [60] investigated the heat transfer and thermal stress analysis of cylinder with internal heat generation under steady temperature conditions using integral transform methods. Gaikwad [61] investigated the thermoelastic problem in a circular sector disk subject to heat generation by using finite Hankel transform and the generalized finite Fourier transform technique. Pandit [62] studied the effect of variable thermal conductivity and internal heat generation in thermal stress analysis of rectangular plate subjected to temperature variation using finite difference method.

3. Temperature fields

The energy equation [46] in the Cartesian coordinates (x, y, z) applicable to elliptical objects is given by

$$\rho\, c_v T,_t = \{\lambda\, T,_x\},_x + \left\{\lambda\, T,_y\right\},_y + \left\{\lambda\, T,_y\right\},_y + \hat{q} \tag{1}$$

where, $\hat{q}$ is the energy generation, thermal diffusivity as $\kappa = \lambda/\rho C$, in which λ being the thermal conductivity of the material, ρ is the density and c_v is the calorific capacity.

The elliptical geometry requires the use of a boundary-fitted coordinate system. A three-dimensional general transformation of the following type should be considered:

$$x = c\cosh\xi\cos\eta,\; y = c\sinh\xi\sin\eta,\; z = z \tag{2}$$

The curves η = constant represent a family of confocal hyperbolas while the curves ξ = constant represent a family of confocal ellipses. The length *2d* is the distance between their common foci. Both sets of curves intersect each other orthogonally at every point in space as shown in Fig. 2. If we use the variable mapping given in Eq. (2), the heat conduction Eq. (1) is reduced to

$$T,_t/\kappa = h^2\{T,_{\xi\xi} + T,_{\eta\eta}\} + \{T,_{zz}\},_y + \hat{q}/\lambda \tag{3}$$

where the notation h^2 should be referred as $h^{-2} = (c^2/2)(\cosh 2\xi - \cos 2\eta)$.

In this section, only papers dealing with heat conduction problems have been reviewed in brief. While reviewing previous literatures, it was observed that most of the papers which come under the preview of solid mechanics have extensively studied conductivity equation with corresponding initial and boundary conditions as required. For example, Rubin [50] investigated the problem of heat conduction between confocal elliptical surfaces solved within the context of the theory of a Cosserat shell. Erdogdu [42, 43] developed finite difference procedure for heat conduction in elliptical cylinders using heat flow lines. Dicker [15] obtained solution by means of the Galerkin method for the transient temperature distribution in an elliptical domain and a cylindrical shell. Kogan [22] applied direct analytical methods to solve the third boundary problem of heat conduction in an elliptical cylinder. Tranter [9] has given a formal solution for the flow of heat in the region bounded internally by an elliptical cylinder. Sato [53] performed theoretical analysis for the transient heat conduction of a very long hollow confocal elliptical cylinder in comparison with its cross-section, i.e. an infinite hollow confocal elliptical cylinder, is given by the use of the method of separation of variables. Recently, corresponding authors [63, 64] have obtained solution for the governing equation considering internal heat generation within the object using few new transform suitable for boundary conditions under consideration.

4. Thermal deflection

The bending problems of elliptical plates under the action of various external forces for the determination of stresses and deflections have got wide consideration for practical applications in aircraft structures during the past years [2, 8, 13, 44]. Miskioglu [32] furnished an experimental solution to the quasi-static problem of transient thermal stresses around an elliptical hole in a plate with the photothermoelastic model and they further extended their work [34] towards the elliptical boundaries of defects that are located near a free edge. Solution for elliptical thin plates bent by a concentrated moment placed at the center of the plate is sought for built-in edges and for simply supported edges has been investigated by Cheng [16] with the aid of tensor calculus in which the expressions for moments, shearing forces and the biharmonic equation for deflection were transformed into elliptic coordinates. Parnes [37] formulated a higher-order boundary perturbation method (B.P.M.) to treat a class of problems defined in an elliptical domain with associated boundary conditions expressed in terms of second-order derivatives which was applied to study a simply-supported elliptic plate subjected to a central lateral point load. In a series of studies, Sato [48, 49] used Mathieu functions to study the bending of a clamped as well as simply supported elliptical plate undergoing the combined action of uniform lateral load over its entire surface and uniform in-plane force distributed in its middle plane. Most of the studies considered by aforementioned authors have reflected the bending of an elliptical structural element subjected to the combined action of uniform lateral load and in-plane force treated in terms of the exact analytical method in case of being supported simply so as to rotate freely. As yet, it is observed that nobody has studied any thermoelastic problem for elliptical plates with boundary conditions as Dirichlet type, in which sources are generated according to the linear function of the temperatures, which will also, satisfy the time-dependent heat conduction equation. As quoted in section two, many authors [27–29] have obtained large deflection based on total strain energy concept. The author has modified the strain energy method devised by Berger as

$$V = \iint_s \left\langle (D/2)\left\{(\nabla^2\omega)^2 + 12e^2/h^2 - 2(1-\upsilon)\left[\partial_{,xx}\omega\partial_{,yy}\omega - (\partial_{,xy}\omega)^2\right]\right\} - Q\omega\right\rangle dxdy$$
$$- \iint_s \frac{1}{(1-\upsilon)}\left(eN_T - \nabla^2\omega M_T\right)dxdy \quad (4)$$

The deflection results and its associated thermal stresses are obtained in terms of Mathieu function of the first kind of order $2n$.

5. Thermoelastic induced vibrations

Even various theoretical and numerical investigations for the thermally treated membranes of regular shapes (rectangle, circle) have been discussed by many authors from different aspects. In fact, the thermally induced strain due to cyclic changes in temperature within the elastic range of plate produces vibration in objects. Several authors have studied the thermoelastic vibrations due to its practical importance in mechanical, aeronautical and nuclear power industries. Therefore, a number of theoretical studies concerning the problems of elastic elliptical membrane vibrations have been reported so far [14, 54, 56, 59]. Recently, Bera [58] formulated an approximate theory which may appear quite suitable for dealing problems of plates, both for simple and arbitrary shapes. The solution of problems of bending of a plate was based upon the rough idea of the shape of the deflected surface of the plate being physically compatible with the type of fastening at the boundary, the nature of the surface loads, and the geometrical shape of the plate. However, aforementioned researchers haven't considered any thermoelastic problem, particularly in elliptical coordinates system. Though, it has been proved that ample cases of heat production in solids have led to various technical problems in mechanical applications in which heat produced is rapidly sought to be transferred or dissipated. Reviewing the previous studies, it was observed by the author that no analytical procedure has been established, considering internal heat sources generated within the body. The problem of thermally induced vibration of the disc is solved by developing an integral transform for double Laplace differential equation as

$$\bar{f}\left(q_{n,m}\right) = \int_a^b \int_0^{2\pi} (\cosh 2\xi - \cos 2\eta)\widehat{B}e_{2n}\left(\xi, q_{2n,m}\right) ce_{2n}\left(\eta, q_{2n,m}\right) f(\xi,\eta)d\xi d\eta \quad (5)$$

in which

$$\widehat{B}\, e_{2n}(\xi, q_{2n,m}) = A_i\, Ce_{2n}(\xi, q_{2n,m}) - B_i\, fey_{2n}(\xi, q_{2n,m}) + C_i\, Ce_{2n}(\xi, -q_{2n,m}) - D_i\, Fe_{2n}(\xi, -q_{2n,m})$$

where A_i, B_i, C_i, D_i are the coefficients
Its inversion function was derived as

$$f(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \frac{\bar{f}(q_{n,m})\zeta_{2n.m}}{\pi \int_a^b (\cosh 2\xi - \Theta_{2n,m})\ \widehat{B}\ e_{2n}^2(\xi, q_{2n,m})\, d\xi} \quad (6)$$

6. Time-reversal thermoelasticity

The determination of initial temperature distribution from a known physical distribution of temperature at any instance is known as the time reversal problem. This type of problem is useful in determining the temperature distribution at a prior stage when it is at any position. It is not surprising that considerable amount of work is done in this area on the problem of considerable technical interest on time-reversal heat conduction with internal heat source under different boundary conditions over the past several decades. In recent years, time reversal problems for the circular boundary have been worked out by many authors. Masket [19] considered Green's functions or Influence functions and discussed a class of heat conduction problem which he has termed as 'Time reversal problem'. Sabherwal [20] has considered the time reversal problems in heat conduction for (i) semi-infinite medium, (ii) rectangular plate. Mehta [26] tackled some time reversal heat conduction with the help of integral transform for (i) heat flow on a cylindrical shell of infinite height with heat generation and radiation, (ii) heat flow in a truncated wedge of finite height, (iii) heat flow in a semi-infinite solid containing an exterior plane crack with circular boundary and an infinitely long cylindrical cavity. Under the title of 'time reversal problem' Patel [30] investigated time reversal heat conduction problem in circular cylinder with radiation type boundary conditions using unconventional finite integral transform proposed by Marchi [18]. Choubey [21] has studied a time reversal heat conduction problem of heat conduction for a solid elliptical cylinder by applying a finite Mathieu transform. Recently Bagde [57] investigated the time reversal inverse heat conduction problem of an elliptical plate to determine the temperature distribution and unknown temperature gradient at a particular point for all time ($t > 0$) with the help of Mathieu transform and finite Marchi-Fasulo transform. However, aforementioned researchers have not considered any thermoelastic problem applying the aforesaid principal, particularly in elliptical coordinates system. Though, it has been proved that ample cases of heat production in solids have led to various technical problems in mechanical applications in which heat produced is rapidly sought to be transferred or dissipated. Reviewing the previous studies, it was observed by the author that no analytical procedure has been established, considering internal heat sources generated within the body.

7. Thermoelasticity

An outline of the solution in elliptic co-ordinates is given in Love's [5] 'Elasticity'. Jeffery [3] and Ghosh [4] attempted the problem of equilibrium of an elastic solid under given applied forces and used Airy's stress function which satisfies the linear partial differential equation of the fourth order $\nabla^4\chi = 0$, where $\nabla^4 = \nabla^2\nabla^2$, and ∇^2 is the two-dimensional Laplacian $\partial^2/\partial x^2 + \partial^2/\partial y^2 = 0$. Jeffery obtained complete solution for an infinite plate containing circular hole in bipolar co-ordinates. Ghosh extended the work in elliptic coordinates and applied it to an elliptic cylindrical cavity and shell. It was also observed from the literature review that most researchers only concentrated their studies on heat conduction problems and only few thermoelastic studies have been investigated. For example, in the 1^{st} report for transient thermal stresses in an elliptical plate, Sugano [35] obtained change in the temperature on the elliptical boundary in the form of an infinite series and formulated associated thermal stress in terms of thermoelastic displacement potential and Papkovich-Neuber's stress functions. As another example of heat conduction and plane thermal stress problems in which the solutions [36] are expressed in Mathieu and modified Mathieu functions, analytical solutions in elliptical coordinates are given for both a steady-state temperature field and an associated plane thermal stress problem in an elliptical plate subjected to unaxisymmetric heating on the elliptic boundary with heat transfer on the upper and lower surfaces. Sugano [38] found analytical solutions for both a steady-state heat conduction problem and an associated plane thermal stress problem expressed in elliptical coordinates in a confocal hollow elliptical plate subjected to nonaxisymmetric heatings on the elliptical boundaries, and with heat loss from the upper and lower surfaces into the surrounding media. The temperature function can be expressed in the form of an infinite series of Mathieu and its modified functions. The associated plane thermal stress problem can be formulated in terms of Airy's stress function. Sugano [39] presented analytical solutions for a transient heat conduction problem and an associated plane thermoelastic problem expressed in elliptical coordinates in a confocal hollow elliptical plate subjected to nonaxisymmetric heatings on the inner and outer elliptical boundaries. The transient heat conduction problem is confined to a symmetric one with respect to the x axis, and solved by proposing new integral transforms and their inverse transforms, the kernels of which are expressed in the form of Mathieu and its modified functions. The associated plane thermoelastic problem is formulated in terms of Airy's stress function. Sugano et al. [40] has considered a plane-strain thermoelastic problem expressed in elliptical coordinates in confocal hollow elliptical cylinders with couple-stresses subjected to nonaxisymmetric steady-state heating on the elliptical boundaries. The steady-state heat conduction problem is confined to a symmetric one with respect to the x and y axis. The associated plane-strain thermoelastic couple-stress problem is formulated in terms of Airy's stress function and Mindlin's couple-stress function. Sugano et al. [41] presented a solution for a micropolar plane-strain thermoelastic problem on a confocal hollow elliptical cylinder subjected to symmetric heating with respect to the x and y axis on the outer elliptical boundary. The problem is formulated in elliptical coordinates based on micropolar theory for an elastic solid initiated by Eringen and Suhubi, and extended to a thermoelastic solid with stress functions by Nowacki. Recently, Cheng [45] obtained a new solution in closed form for the thermomechanical deformations of an isotropic linear thermoelastic functionally graded elliptic plate rigidly clamped at the edges. Hsieh [51] determined the solution for inverse problem of a functionally graded material elliptical plate with large deflection and disturbed boundary conditions with uniform load. Based on the classical nonlinear Von Karman plate theory, the governing equations of a thin plate with large deflection were derived using perturbation technique in conjunction with Taylor series expansion of the disturbed boundary conditions. Based on the small-deflection theory, corresponding author [63] proposed the thermal stress components in terms of resultant forces and resultant moments. This is indicated by;

$$\left.\begin{array}{l} \sigma_{\xi\xi} = \frac{1}{\ell} N_{\xi\xi} + \frac{12z}{\ell^3} M_{\xi\xi} + \frac{1}{1-\upsilon}\left(\frac{1}{\ell} N_T + \frac{12z}{\ell^3} M_T - \alpha E T\right) \\ \sigma_{\eta\eta} = \frac{1}{\ell} N_{\eta\eta} + \frac{12z}{\ell^3} M_{\eta\eta} + \frac{1}{1-\upsilon}\left(\frac{1}{\ell} N_T + \frac{12z}{\ell^3} M_T - \alpha E T\right) \\ \sigma_{\xi\eta} = \frac{1}{\ell} N_{\xi\eta} - \frac{12z}{\ell^3} M_{\xi\eta} \end{array}\right\} \tag{7}$$

where the basic equations of resultant forces $(N_{ij},\ i,j=\xi,\eta)$, resultant bending moments per unit width (M_{ij}), transverse shearing forces (Q_{ij}) and the effective shear force intensity (V_{ij}) are defined as

$$\begin{array}{l} N_\xi = N_\eta = N_{\xi\eta} = 0 \\ \left.\begin{array}{l} M_\xi = -2Dh^2\left[\left(\frac{\partial^2\omega}{\partial\xi^2} + \upsilon\frac{\partial^2\omega}{\partial\eta^2}\right) - \frac{(1-\upsilon)\sinh 2\xi}{(\cosh 2\xi - \cos 2\eta)}\frac{\partial\omega}{\partial\xi} + \frac{(1-\upsilon)\sin 2\eta}{(\cosh 2\xi - \cos 2\eta)}\frac{\partial\omega}{\partial\eta}\right] - \frac{M_T}{1-\upsilon} \\ M_\eta = -2Dh^2\left[\left(\upsilon\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2}\right) + \frac{(1-\upsilon)\sinh 2\xi}{(\cosh 2\xi - \cos 2\eta)}\frac{\partial\omega}{\partial\xi} - \frac{(1-\upsilon)\sin 2\eta}{(\cosh 2\xi - \cos 2\eta)}\frac{\partial\omega}{\partial\eta}\right] - \frac{M_T}{1-\upsilon} \\ M_{\xi\eta} = -2D(1-\upsilon)h^2\left[\frac{\partial\omega}{\partial\xi}\sin 2\eta + \frac{\partial\omega}{\partial\eta}\sinh 2\xi - \frac{\partial^2\omega}{\partial\xi\partial\eta}(\cosh 2\xi - \cos 2\eta)\right] \end{array}\right\} \\ \left.\begin{array}{l} Q_\xi = \frac{4\sqrt{2}D}{c^2h^5}\left[2\sinh 2\xi\left(\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2}\right) - (\cosh 2\xi - \cos 2\eta)\frac{\partial}{\partial\xi}\left(\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2}\right)\right] \\ Q_\eta = \frac{4\sqrt{2}D}{c^2h^5}\left[2\sin 2\eta\left(\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2}\right) - (\cosh 2\xi - \cos 2\eta)\frac{\partial}{\partial\eta}\left(\frac{\partial^2\omega}{\partial\xi^2} + \frac{\partial^2\omega}{\partial\eta^2}\right)\right] \end{array}\right\} \\ V_\xi = Q_\xi - \frac{\sqrt{2}}{h}\frac{\partial M_{\xi\eta}}{\partial\eta} \end{array} \tag{8}$$

with α and E denoting coefficient of linear thermal expansion and Young's Modulus of the material of the plate respectively.

8. Transition to Circular objects

We see that during the transition from elliptical to circular objects, the functions and its appropriate boundaries are independent of η.

For example [64]: When the elliptical disc degenerates into an circular disc with the thickness $\ell \to 0$, internal radius ξ_i, and external radius $\xi_0 \to \infty$, occupying the space $D' = \{(x,y,z) \in R^3 : a \le (x^2+y^2)^{1/2} \le b,\ z=\ell\}$, where $r = (x^2+y^2)^{1/2}$ in such a way that $h\ \exp(\xi)/2 \to r$, $h\ \exp(\xi_i)/2 \to a$, and $h\ \exp(\xi_0)/2 \to b$ [7] and taking θ independent of η.

For this we take, $n=0,\ \ q\to 0,\ \ e\to 0,\ \ \cosh 2\xi\, d\xi \to 2rh^2 dr,\ A_2^{(0)} \to 0,\ A_0^{(0)} \to 1/\sqrt{2},\ \lambda_{0,m}^2 \to \alpha_m^2,\ ce_0(\eta, q_{0,m}) \to 1/\sqrt{2},\ ce_0(\xi, q_{0,m}) \to J_0(\alpha_m r),\ Fey_0(\xi_0, q_{0,m}) \to Y_0(\alpha_m r),\ \alpha_m (=\alpha_{0,m})$ are the roots of $J_0(k_1,\alpha_a)\,Y_0(k_2,\alpha_b) - J_0(k_2,\alpha_b)\,Y_0(k_2,\alpha_a) = 0$
where;

$$\left.\begin{array}{l} J_0\left(k_j,\alpha_i r\right) = J_0(\alpha_i r) + k_j J'_0(\alpha_i r) \\ Y_0\left(k_j,\alpha_i r\right) = Y_0(\alpha_i r) + k_j Y'_0(\alpha_i r) \end{array}\right\} \quad i=1,2$$

$$\left.\begin{array}{l} Ce_0\left(k_1,\xi,\eta,q_{0,m}\right) \to Ce_0(k_1, r\alpha_m) \\ Fey_0\left(k_1,\xi,\eta,q_{0,m}\right) \to Fey_0(k_1, r\alpha_m) \\ S_{0,m}\left(k_1,k_2,\xi,\eta,q_{0,m}\right) \to S_{0,m}(k_1,k_2,r\alpha_m) \quad (= S_m(k_1,k_2,r\alpha_m)) \end{array}\right\}$$

9. Practical problems of technical interest

Applicability of elliptical objects holds for wide range of fields, e.g. eddy currents nondestructive testing, wave scattering, offshore dynamics, and minerals prospecting etc. Though it has been proved that ample cases of heat production in solids have led to various technical problems in mechanical applications in which heat produced is rapidly sought to be transferred or dissipated. For instance, gas turbines blades, walls of I.C. engine, outer surface of a space vehicle and other factors all depend on their durability on rapid heat transfer from their surfaces.

10. Discussion and Remarks

For elliptical boundary-value problems, however, there are only a few transforms known, inspite of its potential useful applications as stated in section five. Reviewing all previous literatures, it has been observed that the aforementioned researchers have not taken into consideration any thermoelastic problem expressed in elliptical coordinates with boundary conditions of radiation type, in which sources are generated according to the linear function of the temperatures, which satisfies the time-dependent heat conduction equation. References cited below [35, 36] are some of the closed form solution to steady state thermoelasticity problems. In the case of transient thermoelastic problems, due to mathematical difficulties, closed form solution are, in general, restricted to infinite or semi-infinite domains. Hence it is strongly desired that a precise study of thermoelastic behavior of real homogeneous and nonhomogeneous behaviour of different bodies in elliptical coordinates is needed.

Acknowledgements

We are greatly indebted to Professor N. K. Choubey for bringing the concept of thermoelastic theories on elliptical profile object using analytical method to our attention and for concrete suggestions on improving upon the presentation of the work.

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