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The solution of non-linear problem arising in infiltration phenomenon in unsaturated soil by optimal homotopy analysis method

Research Article

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Abstract: This paper discusses the non linear problem arising in infiltration phenomenon in unsaturated soil. The infiltration phenomenon in unsaturated soil described by second ordered partial differential equation, which gives moisture contain of soil, with suitable conditions. The level of moisture contain has been observed and it can be analysed by optimal homotopy analysis method. The analysis between OHAM and HAM is also discussed and its numerical as well as graphical representation also mentioned.

MSC: 35K25 • 35Q35

Keywords: Infiltration • OHAM • Convergence control parameter

1. Introduction

A very large portion of the water falling as rain on the land surfaces moves into the soil through processes of infiltration, evaporation, drainage and absorption of soil water by plant roots. Hydrologists have tended, nevertheless, to pay relatively little attention to the phenomenon of water movement in unsaturated soils. Most research on this topic has been done by soil scientist, hydrologist and soil physicist. The unsteady and unsaturated flow of water through soils is due to content changes as a function of time and entire pore spaces are not completely filled with flowing liquid respectively. The water infiltrations system and the underground disposal of seepage and waste water are encountered by these flows, which are described by nonlinear partial differential equation. Knowledge concerning such flows helps some workers like hydrologist, agriculturalists, many fields of science and engineering. The uses of analytical techniques for groundwater flow and mass transport in the unsaturated porous media has significant increasingly form last few years and it has a great important for hydrologist, agriculturists and people related with water resources sciences. Salinity in rivers, lakes, and the ocean is conceptually simple, but technically challenging to define and measure precisely. Conceptually the salinity is the quantity of dissolved salt content of the water. Salts are compounds like sodium chloride, magnesium sulfate, potassium nitrate, and sodium bicarbonate which dissolve into ions. The concentration of dissolved chloride ions is sometimes referred to as chlorinity. Operationally, dissolved matter is defined as that which can pass through a very fine filter (historically a filter with a pore size of $0.45\mu m$, but nowadays usually $0.45\mu m$). Salinity can be expressed in the form of a mass fraction, i.e. the mass of the dissolved material in a unit mass of solution. Seawater typically has a salinity of around $35g/kg$, although lower values are typical near coasts where rivers enter the ocean. Rivers and lakes can have a wide range of salinities, from less than $0.01g/kg$ to a few g/kg, although there are many places where higher salinities are found. The Dead Sea has a salinity of more than $200g/kg$. Soil moisture is more generally considered within the context of hydrology, where it represents the immediate store

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of infiltrating rainfall, before it either evapotranspiration or contributes to groundwater recharge. This term is used in hydrogeology, soil sciences and soil mechanics. In saturated ground water aquifers, all available pore spaces are filled with water. Above a capillary fringe pore spaces have air in them too. When the porous medium in question is soil, water content is synonymous with soil moisture. A common example is the mixing of viscous fluids in chemical engineering applications: Inducing turbulence would require a prohibitively large amount of energy and might in addition destroy the internal structure of many complex fluids. Similar processes are encountered in environmental problems such as the spreading of aqueous or no aqueous pollutants following an accidental discharge; the mixing of these pollutants with surrounding water flows generally includes underground porous layers. Whatever pore size is used in the definition, the resulting salinity value of a given sample of natural water will not vary by more than a few percent. Physical oceanographers working in the abyssal ocean, however, are often concerned with precision and intercomparability of measurements by different researchers, at different times, to almost five significant digits. A bottled seawater product known as IAPSO Standard Seawater is used by oceanographers to standardize their measurements with enough precision to meet this requirement. Analytical solution provide better insight into the physics behind the transport phenomenon and efficient to use. Analytical approaches are for the most limited to situations of simple geometry domains, linear governing equation and homogeneous porous media. Analytical solutions of the partial differential equation for unsaturated flow under various boundary and initial condition are difficult to obtain because of the nonlinearity in soil hydraulic parameters. Exact analytical solution typically requires specialized forms of the hydraulic conductivity and diffusivity functions for nonlinear diffusion-advection equation. Several investigators have described different relation between the diffusivity coefficient and volumetric water content problems in unsaturated porous media. The exponential hydraulic conductivity function has been widely used, but it is known to have a limited range of application to many real soils. The Brooks-Corey model [1] is a relationship between the reduced water content θ^* and the soil suction.

The reduced water content θ^* is defined as a function of two values of moisture: the saturation of the moisture θ_s and the residual of the moisture θ_r ; $\theta^* = \frac{\theta-\theta_r}{\theta_s-\theta_r}$ where $\theta_r < \theta < \theta_s, 0 < \theta^* < 1$. The Gardner model [2] provides a relationship between the saturation degree and the soil suction using three empirical constant α, n and m as,

$$\theta^* = S^* = \begin{cases} \left(1+(\alpha\psi)^n\right)^{-m} & if\ \psi > 0 \\ 1 & if\ \psi \le 0 \end{cases}$$

where, parameter m and n are related by the relation: $m = 1 - \frac{1}{n}$. Other functions developed by Van Genuchten [3] are firmly established for practical applications, which provide a relationship between the saturation degree and the soil suction using three empirical constant. Such special forms of the hydraulic functions make it possible to linearize the governing flow equations, and hence solve them analytically. Solutions to the linearized unsaturated flow equations are generally limited to the steady flow in semi-infinite, homogenous soils [4, 5] and to transient flow in homogeneous and layered soils [6]. The Broadbridge and White Model (1988) have adopted a function which allows for the transformation of the soil water diffusivity $D(\theta)$ as function has the form:

$$D(\theta) = \frac{a}{(b-\theta)^2} \tag{1}$$

where a and b are constant. As a second step in the solution of the nonlinear flow problem, Broadbridge and White (1988) developed an expression for $K(\theta)$ which in conjunction with the assumed function for $D(\theta)$ transforms equation (8) to the weakly nonlinear Burger's equation. This expression for $K(\theta)$ is given as

$$K(\theta) = \beta + \gamma(b-\theta) + \frac{\lambda}{2(b-\theta)} \tag{2}$$

where β, γ and λ are constants. Ground water recharge problem has been discussed by many researchers with different viewpoints. Swartzendruber uses [7] method to get graphical illustration of the mathematical solution for horizontal water function. Verma & Mishra [8] have obtained solution by similarity transformation of a one-dimensional vertical ground water recharges through porous media. Parikh et al. [9] have obtained transcental solution of Fokker-Planck equation of vertical groundwater recharge in dry region. Hari Prasad et al. [10] had provided a numerical model to simulate water flow through unsaturated zones and studied the effect of unsaturated soil parameters on water movement during different processes such as gravity drainage and infiltration, using singular perturbation technique, Meher & Mehta [11] have provided an approximate solution where the change in average diffusivity coefficient being very small; it has been treated as constant. Also there are many techniques to solve the governing non linear partial differential equations, for example Patel and Meher [12] use Laplace Adomian DecompositionMethod for the soliton solutions of Boussinesq-Burger equations, Pirzada and Vakaskar [13] had gave the comparative solutions of heat equation, Darji and Timol [14] gave a deductive group symmetry approach to Generalized Non-Newtonian fluid flow analysis of heat transfer in natural convection and analysis for unsteady natural convective boundary layer flow of Sisko fluid [15].

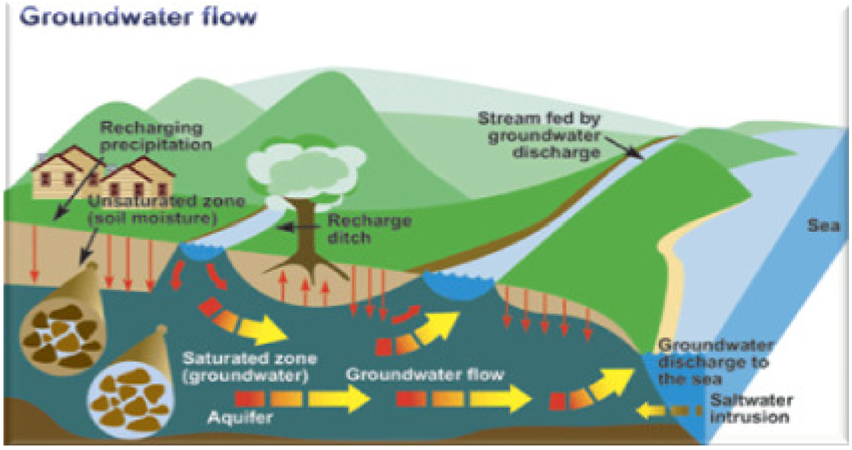

Fig. 1. Representation groundwater phenomenon by spreading moisture content in the pore space of the unsaturated porous media.

2. Assumption and statement of the problem

In the present model, Darcian-based unsaturated flow equation described over the length of a large basin L in homogeneous porous media, and there is no air resistance to the flow (i.e. the porous medium contains only the flowing liquid water and empty voids of air). The air in the void space is approximately at atmospheric pressure. The moisture content at the soil surface was considered as time dependent function. In this case consider the flow in vertical downward direction up to length L by neglecting spreading in other directions (Fig. 1) . We obtain one dimensional nonlinear partial differential equation for diffusion-convection processes by combining Darcy's law (Darcy, 1856) for unsaturated flow with the continuity equation.

The purpose of this investigated model is to discuss the approximate analytical solution of non-linear partial differential equation arising in ground water recharge phenomenon, to examine the moisture content in homogeneous porous media. The change in moisture content in porous media and distribution of pore pressure can be calculated using optimal homotopy analysis method. Its solution provides the moisture content of the porous media (soil) at any depth Z at time $T > 0$.

3. Mathematical formulation

When water flow through unsaturated porous media in vertically downward direction, the hydraulic conductivity varies nonlinearly with the volumetric water content: $K = K(\theta)$. The variation of the hydraulic conductivity with the volumetric water content in unsaturated homogeneous porous media for small Reynolds number the volume of flow of water described by Darcy's law as [16],

$$\vec{V} = -K(\theta)\nabla H, \tag{3}$$

where $\vec{V}$ = volume flux of moisture, $K(\theta)$ = coefficient of the volumetric water content and ∇H = gradient of the whole moisture potential.
Such ground water flow satisfies the equation of continuity as follows,

$$\frac{\partial}{\partial t}\left(\rho_s \phi S\right) = -\nabla M \tag{4}$$

where ρ_s is the bulk density of the soil on dry weight basis M is the mass of flux of the water at any time $t \geq 0$. Considering that water is incompressible, and $M = \rho\vec{V}$ and also considering the fact that the water content of the soil is given by standard relation with saturation of soil S as $\theta = \phi S$ [17], where ϕ porosity and S is a saturation of the soil.
Eq. (4) reduces to,

$$\frac{\partial}{\partial t}\left(\rho_s \theta\right) = -\nabla\left(\rho\vec{V}\right), \tag{5}$$

where ρ is the flux density.
Using Eq. (3) in (5), we get

$$\frac{\partial}{\partial t}\left(\rho_s \theta\right) = -\nabla\left(\rho\left(-K(\theta)\nabla H\right)\right) \tag{6}$$

It is also considered here as that the flow takes place only in vertical downward direction [18], Eq. (6) reduced to,

$$\rho_s \frac{\partial \theta}{\partial t} = \rho \frac{\partial}{\partial z}\left(K(\theta)\frac{\partial H}{\partial z}\right) \tag{7}$$

In unsaturated soil instead of pressure head h one introduces, the soil suction $\psi = |h|$ by negative forces of capillary and pressure head is negative . For reduced water content only the soil suction really matter. The water thus moves inside the unsaturated soil from a point having a greater pressure head (or a lower suction value $\psi = |h|$) to another point by a smaller pressure head (or a greater suction), until these values become equal. For unsaturated porous media, H is total soil moisture potential: $H = \psi - gz$, where ψ is the pressure potential (soil matric suction), z is the elevation in the vertical downward direction of flow, and g is gravitational constant. Hence Eq. (7) will be,

$$\frac{\partial \theta}{\partial t} = \frac{\rho}{\rho_s}\frac{\partial}{\partial z}\left(K(\theta)\frac{\partial \psi}{\partial z}\right) - \frac{\rho}{\rho_s} g \frac{\partial K(\theta)}{\partial z} \tag{8}$$

The Eq. (8) can be written as,

$$\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) - \frac{\rho g}{\rho_s} K'(\theta)\frac{\partial \theta}{\partial z}, \tag{9}$$

where, z is depth in vertical downward direction, t is time, $\theta(z,t)$ is volumetric soil water content, $D(\theta) = \frac{\rho K}{\rho_s}\frac{\partial \psi}{\partial \theta}$ is called the diffusivity coefficient, $K(\theta)$ is the coefficient of the volumetric water content and $K(\theta)$. The expression in Eq. (9) is written as θ dependent equation. Generally θ dependent equation is called one dimension Fokker-Planck equation. This Eq. (9) is model based on Darcy-Buckingham approach in vertical downward direction flow of water in unsaturated porous media. For the sake of convenience, we consider that the moisture content in uniform soil take place in positively downward direction from $z = 0$ top of the bottom $z = L$, where water table is saturated as shown in figure 1. We consider following new dimensionless variables $Z = \frac{z}{L}$ and $T = \frac{\rho g}{\rho_s L} t$ has been introduced to simplify the Eq. (9) as,

$$\frac{\partial \theta}{\partial T} = \varepsilon \frac{\partial}{\partial Z}\left(D(\theta)\frac{\partial \theta}{\partial Z}\right) - K'(\theta)\frac{\partial \theta}{\partial Z} \tag{10}$$

where, $\varepsilon = \frac{\rho_s L}{\rho g}$ is an auxiliary parameter. As given by (Broadridge and White model [4] the soil water diffusivity $D(\theta)$ and hydraulic conductivity $K(\theta)$ from Eq. (1) and (2) are written as

$$D(\theta) = \frac{a}{b^2}\left(1 - \frac{\theta}{b}\right)^{-2} = \frac{a}{b^2}\left(1 + \frac{2\theta}{b}\right) \tag{11}$$

since θ is very small [19] and

$$K(\theta) = \beta + \gamma b + \frac{\delta}{2b} + \left(\frac{\delta}{2b^2} - \gamma\right)\theta \tag{12}$$

where a, b, β, γ and λ are constants. Using Eq. (11) and (12) in (10), we get,

$$\frac{\partial \theta}{\partial T} = \varepsilon \frac{a}{b^3}\frac{\partial}{\partial Z}\left((b + 2\theta)\frac{\partial \theta}{\partial Z}\right) - \left(\frac{\delta}{2b^2} - \gamma\right)\frac{\partial \theta}{\partial Z} \tag{13}$$

The Eq. (13) is nonlinear second order partial differential equation which governs moisture content of soils for the one-dimensional unsteady flow in unsaturated porous medium in downward direction. Eq. (13) rewritten as,

$$\frac{\partial \theta}{\partial T} = \varepsilon \left[A\left(\frac{\partial \theta}{\partial Z}\right)^2 + B\frac{\partial^2 \theta}{\partial Z^2} + A\theta \frac{\partial^2 \theta}{\partial Z^2}\right] - C\frac{\partial \theta}{\partial Z} \tag{14}$$

where $A = \frac{2a}{b^3}, B = \frac{a}{b^2}$ and $C = \left(\frac{\delta}{2b^2} - \gamma\right)$.
Subject to appropriate conditions,

$$\begin{aligned} &\theta(Z,0) = \theta_c e^Z \;\; ; T > 0 \\ &\theta(0,T) = \theta_c \;\;\;\;\; ; Z > 0 \end{aligned} \tag{15}$$

4. Solution by OHAM

According to OHAM [20, 21] first we construct the zeroth order deformation equation as

$$(1-q)\mathscr{L}\left[\theta(Z,T;q)-\theta_0(Z,T)\right]=c_0 q\mathscr{N}\left[\theta(Z,T;q)\right] \tag{16}$$

where, $q\in[0,1]$ is the embedding parameter, c_0 is an auxiliary parameter also know as convergence control parameter. We choose auxiliary linear operator $\mathscr{L}=\frac{\partial}{\partial T}$, [22] and $\theta_0(Z,T)=0.1e^{Z}+0.25*TZ$ [18] is an initial guess of $\theta(Z,T)$ According to OHAM expanding $\theta(Z,\mathrm{T};q)$ in maclaurin series with respect to q, then the corresponding m^{th}-order deformation equation is given by

$$\mathscr{L}\left[\theta_m(Z,T)-\chi_m\theta_{m-1}(Z,T)\right]=c_0\delta_m\left[\theta_{m-1}(Z,T)\right], \tag{17}$$

where

$$\delta_m\left[S_{m-1}(Z,T)\right]=\frac{\partial}{\partial t}(\theta_{m-1})-0.8\sum_{r=0}^{m-1}(\theta_r)_Z(\theta_{m-1-r})_Z+0.4(\theta_{m-1})_{ZZ}-0.8(\theta_{m-1})_{ZZ}+1.5*(\theta_{m-1})_Z$$

and

$$\chi_m=\begin{cases}0, & m\le 1\\ 1, & m>1\end{cases}$$

Applying inverse operator both side on Eq. (17)

$$\theta_m(Z,T)=\chi_m\theta_{m-1}(Z,T)+c_0\mathscr{L}^{-1}\delta_m\left[\theta_{m-1}(Z,T)\right], \tag{18}$$

The Convergence control parameter c_0 play an important role in the OHAM. One can gain convergent series solution simply by choosing a proper auxiliary parameter c_0. This is the reason why we call c_0 as the convergence-control parameter. In 2007, Yabushita et al. [23] applied the HAM to solve two coupled non-linear ODEs. They suggested the so-called "optimization method" to find out the optimal convergence-control parameters by means of the minimum of the squared residual of governing equation as follow

$$E_m(c_0)=\frac{1}{(M+1)(N+1)}\sum_{i=0}^{M}\sum_{j=0}^{N}\left\{\mathscr{N}\left[\sum_{n=0}^{m}\theta_n\left(\frac{i}{M},\frac{j}{N}\right)\right]^2\right\}. \tag{19}$$

Eq. (19) gives the square residual error at mth-order approximation. It is obvious that we can find the optimal value of convergence-control parameter c_0 at the any order of approximation. If there exists convergence-control parameter, c_0 for which we get the minimum of the squared residua E_m, is so called the optimal convergence-control parameter c_0.

The optimal value of convergence control parameter $c_0=-0.011$ with minimum square residual error $E_6=3.0E-02$ at sixth order approximation, which can be notice by Fig. 2. Substituting the optimal value of convergence control parameter and solve Eq. (18) for different values of m we have approximations at different order. Adding all the order including initial guess we get the approximate solution as follows

$$\begin{aligned}\theta(Z,T)&=\theta_0(Z,T)+\theta_1(Z,T)+\ldots+\theta_6(Z,T)\\ &=T(0.00001\mathrm{e}^{3Z}T+(-0.002+0.0002T)T+\mathrm{e}^{2Z}(0.0007\\ &\quad+T(-0.00001+T(0.000006+0.00002Z)))-0.003Z\\ &\quad+\mathrm{e}^{Z}(-0.001+T(0.0002+0.0007Z+T(-0.000009-0.000009Z\\ &\quad+T(0.000001+(0.000007+0.000003Z)Z)))))\end{aligned} \tag{20}$$

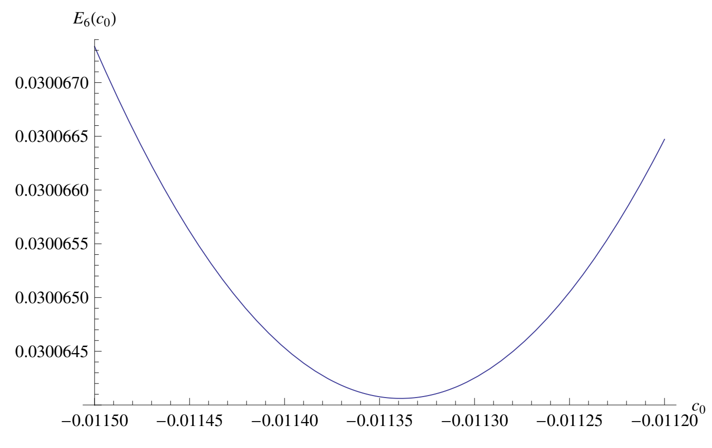

Fig. 2. Square residual errors

5. Numerical and graphical representation

The Eq. (20) represent the moisture content $\theta(Z,T)$. Table 1 shows the tabular values of moisture contained.

Table 1. Moisture content $\theta(Z,T)$ for different depth z for fixed time $T = 0.0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0$.

Z	$T=0$	$T=0.1$	$T=0.2$	$T=0.3$	$T=0.4$	$T=0.5$	$T=0.6$	$T=0.7$	$T=0.8$	$T=0.9$	$T=1.0$
0	0.1	0.0996	0.0989	0.0981	0.097	0.0958	0.0944	0.0928	0.091	0.0891	0.087
0.1	0.1105	0.1124	0.1142	0.1157	0.117	0.1182	0.1192	0.1201	0.1207	0.1213	0.1217
0.2	0.1221	0.1264	0.1305	0.1344	0.1382	0.1418	0.1453	0.1486	0.1517	0.1548	0.1577
0.3	0.135	0.1416	0.1481	0.1545	0.1607	0.1667	0.1727	0.1785	0.1842	0.1897	0.1952
0.4	0.1492	0.1582	0.1672	0.176	0.1846	0.1932	0.2016	0.21	0.2182	0.2264	0.2345
0.5	0.1649	0.1764	0.1877	0.199	0.2102	0.2213	0.2323	0.2432	0.2541	0.265	0.2758
0.6	0.1822	0.1962	0.21	0.2238	0.2376	0.2513	0.2649	0.2785	0.2921	0.3057	0.3193
0.7	0.2014	0.2178	0.2343	0.2506	0.267	0.2833	0.2997	0.3161	0.3325	0.3489	0.3654
0.8	0.2226	0.2416	0.2606	0.2796	0.2987	0.3178	0.3369	0.3562	0.3755	0.395	0.4145
0.9	0.246	0.2676	0.2893	0.3111	0.3329	0.3549	0.377	0.3992	0.4216	0.4442	0.4671
1	0.2718	0.2962	0.3207	0.3453	0.37	0.395	0.4201	0.4455	0.4712	0.4972	0.5235

Table 2. Comparison between OHAM and HAM [19] results of Moisture content $\theta(Z,T)$

Z	T=0.1		T=0.2		T=0.3		T=0.4		T=0.5	
	OHAM	HAM	OHAM	HAM	OHAM	HAM	OHAM	HAM	OHAM	HAM
0.1	0.1124	0.1196	0.1142	0.1302	0.1157	0.1422	0.117	0.1557	0.1182	0.1706
0.2	0.1264	0.1347	0.1305	0.1488	0.1344	0.1643	0.1382	0.1813	0.1418	0.1997
0.3	0.1416	0.1511	0.1481	0.1687	0.1545	0.1878	0.1607	0.2083	0.1667	0.2303
0.4	0.1582	0.1689	0.1672	0.1901	0.176	0.2127	0.1846	0.2369	0.1932	0.2624
0.5	0.1764	0.1883	0.1877	0.2131	0.199	0.2394	0.2102	0.2672	0.2213	0.2964
0.6	0.1962	0.2093	0.21	0.2379	0.2238	0.2679	0.2376	0.2994	0.2513	0.3323
0.7	0.2178	0.2323	0.2343	0.2647	0.2506	0.2985	0.267	0.3337	0.2833	0.3704
0.8	0.2416	0.2574	0.2606	0.2936	0.2796	0.3312	0.2987	0.3703	0.3178	0.4108
0.9	0.2676	0.2847	0.2893	0.3249	0.3111	0.3665	0.3329	0.4095	0.3549	0.4539
1.0	0.2962	0.3146	0.3207	0.3588	0.3453	0.4044	0.37	0.4515	0.395	0.4999

Z	T=0.6		T=0.7		T=0.8		T=0.9		T=1.0	
	OHAM	HAM	OHAM	HAM	OHAM	HAM	OHAM	HAM	OHAM	HAM
0.1	0.1192	0.1871	0.1201	0.2049	0.1207	0.2242	0.1213	0.2449	0.1217	0.2669
0.2	0.1453	0.2196	0.1486	0.2409	0.1517	0.2637	0.1548	0.2878	0.1577	0.3134
0.3	0.1727	0.2537	0.1785	0.2785	0.1842	0.3048	0.1897	0.3325	0.1952	0.3615
0.4	0.2016	0.2894	0.21	0.3178	0.2182	0.3477	0.2264	0.3789	0.2345	0.4115
0.5	0.2323	0.3269	0.2432	0.3591	0.2541	0.3925	0.265	0.4273	0.2758	0.4636
0.6	0.2649	0.3666	0.2785	0.4023	0.2921	0.4395	0.3057	0.4779	0.3193	0.5178
0.7	0.2997	0.4085	0.3161	0.4479	0.3325	0.4888	0.3489	0.5309	0.3654	0.5746
0.8	0.3369	0.4527	0.3562	0.496	0.3755	0.5407	0.395	0.5867	0.4145	0.6341
0.9	0.377	0.4997	0.3992	0.5469	0.4216	0.5954	0.4442	0.6453	0.4671	0.6965
1	0.4201	0.5496	0.4455	0.6007	0.4712	0.6532	0.4972	0.7069	0.5235	0.7621

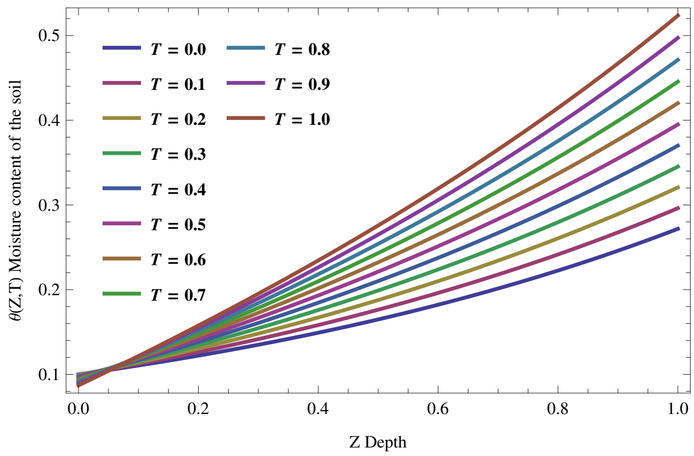

Fig. 3. Represents moisture content $\theta(Z,T)$ vs. depth Z for depth $0 \leq Z \leq 1.0$ and time $0 \leq T \leq 1.0$ are fixed

6. Conclusion

The Eq. (20) represents moisture content of the soil for any depth Z and for any time T. The optimal value of convergence control parameter c_0 is successfully obtained by reducing the square residual error. The numerical values of moisture contain is cited in the Table 1 and from the Fig. 3 it can be conclude that the moisture content of soil is increasing parabolic as depth and time increasing. Table 2 represents the numerical values which are analysis between optimal homotopy analysis method and homotopy analysis method and it is observed that at the depth the left-hand boundary condition is not satisfied in HAM while it is fully satisfies by OHAM.

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