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# On the monatic number of a graph

Research Article

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**Abstract:** In a graph $G = (V, E)$, a set $M \subseteq V(G)$ is said to be a monopoly set of $G$ if every vertex $v \in V - M$ has, at least, $\frac{d(v)}{2}$ neighbors in $M$. The monopoly size $mo(G)$ is the minimum cardinality of a monopoly set. An M-partition of a graph $G$ is the partition of the vertex set $V(G)$ of $G$ into $k$ disjoint monopoly sets. The monatic number of $G$, denoted by $\mu(G)$, is the maximum number of sets in M-partition of $G$. In this paper, we establish Nordhaus-Gaddum inequalities for monatic number of a graph. It is shown that, for any connected graph $G$ with at least two edges, $\mu(L(G)) = 3$ if and only if $G = C_{3k}$, where $L(G)$ is the line graph of $G$ and $C_{3k}$ is a cycle with $3k$ vertices, for $k \geq 1$. The monatic numbers of the join $G_1 + G_2$ and corona $G_1 \circ G_2$ of any two graph $G_1$ and $G_2$ are found.

**MSC:** 05C70 • 05C69

**Keywords:** Monopoly set • Monotic number • Monopoly partition • Nordhaus-Gaddum inequalities



## 1. Introduction

The concept of monopoly in a graph was introduced by Khoshkhak K. et al. [1]. Some mathematical properties of monopoly in graphs have studied in [2], Other types of monopoly in graphs have been subsequently proposed by the authors [3–6]. In particular, the monopoly in graphs is a dynamic monopoly (dynamos) that, when colored black at a certain time step, will cause the entire graph to be colored black in the next time step under an irreversible majority conversion process. Dynamos were first introduced by Peleg D. [7]. For more details in monopoly and dynamos in graphs, we refer the reader to [8–12].

Partitioning is a fundamental operation on graphs. It is the problem of classifying the graph vertices into many disjoint sets satisfying certain condition. Partitioning big graph is an important task to reduce complexity or for parallelization. Graph partitioning is most important, demanding in application problems like social network, road network, scientific simulation, air traffic control, image analysis etc. In general, graph partitioning is NP-complete [13]. Motivated by these, the authors in [14], studied partitioning a graph into monopoly sets and introduced the monatic number, $\mu(G)$, of a graph. The monopoly partition (shortly, $M$-partition) of a graph $G$ is a partition of $V(G)$ into $k$ disjoint monopoly sets. The monatic number of $G$, denoted by $\mu(G)$, is the maximum number of sets in M-partition of $G$.

We begin by stating the terminology and notations used through this article.

A graph $G = (V, E)$ is a simple graph, that is finite, having no loops no multiple and directed edges. As usual, we denote by $n = |V|$ and $m = |E|$ to the number of vertices and edges in a graph $G$, respectively. For a vertex $v$ the open neighborhood of $v$ in a graph $G$, denoted $N_G(v)$, (or $N(v)$ if no confuse), is the set of all vertices that are adjacent to $v$ and the closed neighborhood of $v$ is $N_G[v] = N_G(v) \cup \{v\}$. The degree of vertex a $v$ in $G$ is $d_G(v) = |N_G(v)|$, and the

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degree of a vertex $v$ with respect to a subset $S \subset V(G)$ is $d_S(v) = |N(v) \cap S|$. We denote by $\Delta(G)$ and $\delta(G)$ to maximum and minimum degree among the vertices of $G$, respectively. An isolated vertex in $G$ is a vertex with degree zero. A graph $K_1$ is called a trivial graph. As usual, $\overline{G}$ denotes the complement of $G$, for a subset $S \subseteq V$, $\overline{S} = V - S$ and $kG$ denotes the $k$ disjoint copies of $G$. The Friendship graph $F_n$ for $n \geq 2$, is the graph constructed by joining $n$ copies of $K_3$ graph with a common vertex. A set $I \subseteq V$ is independent if no two vertices in $I$ are adjacent. The independent sets of maximum cardinality are called maximum independent sets. The number of vertices in a maximum independent set is the independence number (or vertex independence number) of $G$ and is denoted by $\alpha(G)$. A clique of a graph $G$ is a complete induced subgraph of $G$ and the clique of largest possible size is referred to as a maximum clique. The clique number of a graph $G$, denoted $\omega(G)$, is the number of vertices in a maximum clique of G. For more terminologies and notations in graph theory, we refer the reader to the books [15, 16].

A set $M \subseteq V(G)$ is called a monopoly set of $G$ if for every vertex $v \in V(G) - M$ has at least $\frac{d(v)}{2}$ neighbors in $M$. The monopoly size of $G$, denoted by $mo(G)$, is the minimum cardinality of a monopoly set in $G$.

Ahmed Naji and Soner N. D. in [14], have studied partitioning a graph into monopoly sets and introduced the monatic number $\mu(G)$ of a graph $G$. Where the monopoly partition (shortly, $M$-partition) of a graph $G$ is a partition of $V(G)$ into $k$ disjoint monopoly sets. The monatic number of $G$, denoted by $\mu(G)$, is the maximum positive integer $k$ such that $V(G)$ can be partitioned into $k$ pairwise disjoint monopoly sets. They have discussed the basic properties and found some bounds for the monatic number of graphs. They have found that for any graph $G$, $1 \leq \mu(G) \leq 3$ and they showed the relationships between the monatic number of a graph $G$ and some others parameters of $G$ as chormatic number, domitic number, independence number, clique number and etc. also, they have showed that any graph with $\mu(G) = 3$ is an eulerian graph. For more results and details the reader referred to [14]. In this paper, we continue in study of the monatic number of graphs. We establish Nordhaus-Gaddum inequalities for monatic number of a graph. Also, It is shown that, for any connected graph $G$ with at least two edges, $\mu(L(G)) = 3$ if and only if $G = C_3 k$, Where$L(G)$ isthelinegraphof$G$ and$C_{3k}$ isacyclewith3$k$ vertices,for$k \geq 1$. The monatic numbers of the join $G_1 + G_2$ and corona $G_1 \circ G_2$ of any two graph $G_1$ and $G_2$ are found.

The following are some fundamental results which will be required for many of our arguments in this paper and which are finding in [14]:

**Theorem 1.1.**
*For any graph G without isolated vertices,*

$$2 \leq \mu(G) \leq 3.$$

**Proposition 1.1.**
*Let G be a graph of order $n$. Then $\mu(G) = 1$, if and only if G having an isolated vertex.*

**Theorem 1.2.**
*For any graph G without isolated vertices. If G has a vertex of odd degree, then $\mu(G) = 2$.*

**Corollary 1.1.**
*For any graph G. If G has $\mu(G) = 3$, then every vertex of G is of even degree.*

**Theorem 1.3.**
*For any graph G. If $\mu(G) = 3$, then every partite set in $M$-partition of G is an independent set.*

**Proposition 1.2.**
*Let $\{M_1, M_2, M_3\}$ be an $M$-partition of a graph G. Then*

$$d_{M_i}(v) = d_{M_j}(v) = \frac{d(v)}{2}$$

*for every $v \in M_k$, where $i, j$ and $k \in \{1, 2, 3\}$ and $k \neq i \neq j$.*

**Theorem 1.4.**
*Let G be a graph with a clique number $\omega(G)$. If $\mu(G) = 3$, then $\omega(G) \leq 3$.*

**Theorem 1.5.**
*For any graph G of order $n$. If $\mu(G) = 3$, then $\alpha(G) \geq \lceil \frac{n}{3} \rceil$.*

**Theorem 1.6.**
*Let G be a graph of order n and maximum degree $\Delta(G)$ = nâĹŠ1. Then $\mu(G) = 3$, if and only if $G = K_3$ or$G = F_n$.*

**Theorem 1.7.**
*Let $\{M_1, M_2, M_3\}$ be an M-partition of a graph G such that $|M_1| \le |M_2| \le |M_3|$. Then*

1. $mo(G) \le |M_1| \le \lfloor \frac{n}{3} \rfloor$;
2. $|M_1| \le |M_2| \le \frac{n-mo(G)}{2}$;
3. $\lceil \frac{n}{3} \rceil \le |M_3| \le |M_1||M_2|$.

## 2. Nordhaus-Gaddum type result for the monatic number of graphs

In this section, we establish a Nordhaus-Gaddum type result for the monatic number of a non-trivial graph $G$ and characterize all graph $G$ with $\mu(G) = \mu(\overline{G}) = 3$. The following result is immediately consequences of the results in above.

**Proposition 2.1.**
*For any non-trivial graph $G$,*

$$3 \le \mu(G) + \mu(\overline{G}) \le 6.$$

**Theorem 2.1.**
*Let $G$ be a non-trivial graph with $\mu(G) = 3$. Then $\mu(\overline{G}) = 1$, if and only if $G = F_n$ or $G = K_3$.*

*Proof.* The result is consequences of Proposition 1.1 and Theorem 1.6. □

**Theorem 2.2.**
*For any graph $G$ of order $n \ge 10$,*

$$\mu(G) + \mu(\overline{G}) \le 5.$$

*Proof.* Let $G$ be a graph of order $n \ge 10$. Clearly, If $\mu(G) \le 2$, then the result is holding. Hence, we assume that $\mu(G) = 3$. Then by Theorem 1.5, $\alpha(G) \ge \lceil \frac{n}{3} \rceil$ and by the well-known result $\omega(\overline{G}) = \alpha(G)$, we get $\omega(\overline{G}) \ge 4$. Hence, by Theorem 1.4, $\mu(\overline{G}) \le 2$. Therefore, $\mu(G) + \mu(\overline{G}) \le 5$. □

**Proposition 2.2.**
*Let $G$ be a graph with at least ten vertices and $\mu(G) = 3$. If $G \ne F_n$, then $\mu(\overline{G}) = 2$.*

**Theorem 2.3.**
*Let $G$ be a graph with even order $n$ and $\mu(G) = 3$. Then $\mu(\overline{G}) = 2$.*

*Proof.* Let $d(v)$ and $\overline{d}(v)$ denote the degree of a vertex $v$ in $G$ and $\overline{G}$, respectively. Since $mu(G) = 3$ then by Corollary 1.1, $d(v)$ is even for every $v \in V(G)$ and since $n$ is also even it follows that $\overline{d}(v) = n - d(v) - 1$ is odd for every $v \in V(\overline{G})$. Thus by Theorem 1.2, $\mu(\overline{G}) = 2$. □

The bounds of the inequality, $3 \le \mu(G) + \mu(\overline{G}) \le 6$, in Proposition 2.1, are sharp. There are many non-trivial graphs satisfies the lower bound, for example every complete graph $K_n$ with $n \ge 4$ vertices and there are only few graphs satisfies the upper bound. In the following result we characterize all graphs satisfies the sharpness of upper bound.

**Remark 2.1.**
In next $\mathscr{G}$ denote to the family of forbidden graphs in Fig. 1. It is clear, by the easily chick, that for every $G \in \mathscr{G}$:

1. $G$ is tri-partite of order $n = 9$.
2. The number of edges between any two partite sets of $G$ is equal to $\frac{2m}{3}$.
3. Every partite set is a monopoly set of $G$.
4. $\overline{G} \in \mathscr{G}$.
5. $mo(G) = \alpha(G) = \omega(G) = 3$.
6. $\mu(G) = 3$.

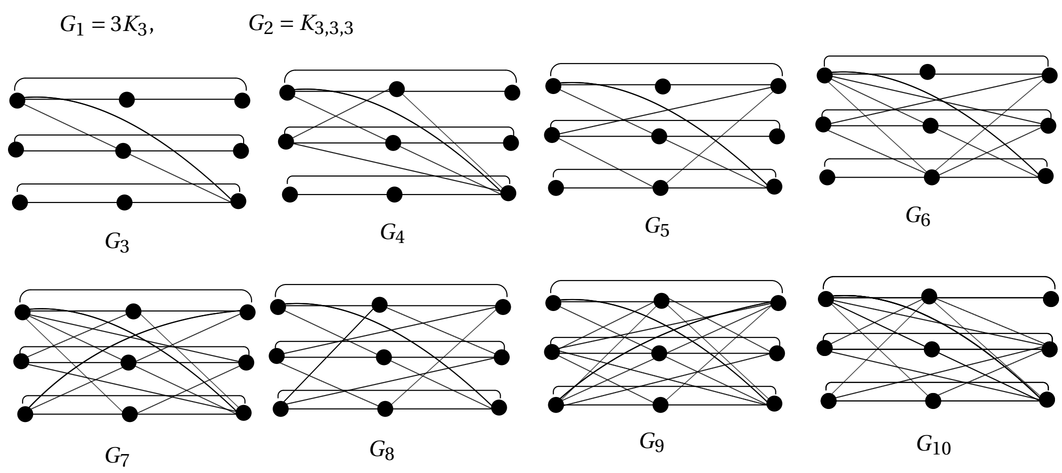


**Fig. 1.** Family of forbidden graphs with $\mu(G) = \mu(\overline{G}) = 3$.

**Theorem 2.4.**
*Let G be a graph of order n. Then $\mu(G) + \mu(\overline{G}) = 6$, if and only if $G \in \mathcal{G}$.*

*Proof.* Let $G \in \mathcal{G}$. Then by Remark 2.1, $\mu(G) = \mu(\overline{G}) = 3$. Hence, the result is holding.

Conversely, It is clear that $\mu(G)+\mu(\overline{G}) = 6$, if and only if $\mu(G) = \mu(\overline{G}) = 3$. Then for the simplicity of proof we suppose that $\mu(G) = 3$. Now, if $G$ with $n$ vertices such that $n \geq 10$, then by Theorem 2.2, $\mu(G) + \mu(\overline{G}) \leq 5$. Hence, we may assume that $n \leq 9$ and we consider the following cases:

**Case 1:** If $n \leq 9$ is even, then by Theorem 2.3, $\mu(\overline{G}) = 2$ and hence, $\mu(G) + \mu(\overline{G}) = 5$.

**Case 2:** If $n \leq 9$ is odd, let $\{M_1, M_2, M_3\}$ be M-partition of a graph $G$. we consider the following subcases:

**Subcases 2.1:** If $n = 3$, then $\mu(G) = 3$ if and only if $G = K_3$ and hence $\mu(\overline{G} = 1$.

**Subcases 2.2:** If $n = 5$, then by using Theorem 1.7, we can distributed five vertices among three sets $M_1, M_2, M_3$, such that every partite set is an independent monopoly set, only by one way as $|M_1| : |M_2| : |M_3| = 1 : 2 : 2$, otherwise, and hance $\mu(G) = 3$, if and only if $G = F_2$. Since, $G$ has a vertex of $n-1$ degree, then $\overline{G}$ has an isolated vertex and hence $\mu(\overline{G}) = 1$.

**Subcases 2.3:** If $n = 7$, again by Theorem 1.7, we can distributed nine vertices among three sets, every set is an independent monopoly set, by two ways, as $|M_1| : |M_2| : |M_3|$. These ways are $1 : 3 : 3$ and $2 : 2 : 3$. Clearly, in the way $1 : 3 : 3$ $G = F_3$ and hence $\mu(\overline{G}) = 1$. In the way $2 : 2 : 3$, by normal chick, we get only two graphs $G_1$ and $G_2$ with $\mu(G_1) = \mu(G_2) = 3$. Fig. 2 shows $G_1$ and $G_2$. By study the M-partition of $\overline{G_1}$ and $\overline{G_2}$, for $G_1$ and

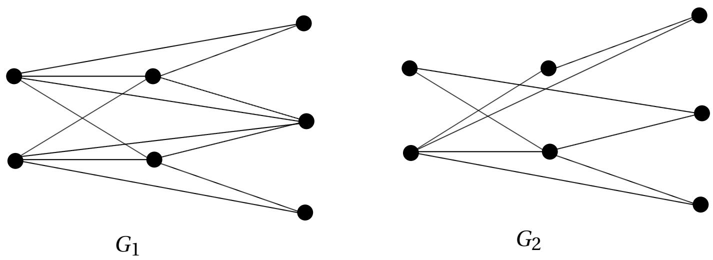


**Fig. 2.** Graphs with 7 vertices and $\mu(G) = 3$.

$G_2$ in Fig. 2 we get $\mu(\overline{G_1}) = 2$ and $\mu(\overline{G_2}) = 2$.

**Subcases 2.4** If $n = 9$, once again by Theorem 1.7, we can distributed nine vertices among three sets, every set is an independent monopoly set, by three ways, as $|M_1| : |M_2| : |M_3|$. These ways are $1:4:4$, $2:3:4$ and $3:3:3$. In the ways $1:4:4$ and $2:3:4$, $G$ has $\alpha \geq 4$ and hence $\omega(\overline{G}) \geq 4$. Then by Theorem 1.4, $\mu(\overline{G}) = 2$. By normal chick, in way 3:3:3 of distribution, we get only ten graphs as in Fig. 1 such that $\mu(G) = \mu(\overline{G}) = 3$.

Therefore, by the above cases, if $\mu(G) = 3$, then $\mu(\overline{G}) = 3$, if and only if $G \in \mathscr{G}$.

□

## 3. The monatic number of the union, join and corona product of graphs

**Definition 3.1.**
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then

**(i)** The union of $G_1$ and $G_2$, denoted $G_1 \cup G_2$, is a graph whose vertex set is $V_1 \cup V_2$ and the edge set is $E_1 \cup E_2$.

**(ii)** The join of $G_1$ and $G_2$, denoted $G_1 + G_2 = (V, E)$, is a graph having vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2 \cup \{e = uv : u \in V_1, v \in V_2\}$.

**(iii)** The corona product $G_1 \circ G_2$ of $G_1$ and $G_2$ is obtained by taking one copy of $G_1$ and $|V(G_1)|$ copies of $G_2$; and by joining each vertex of the i-th copy of $G_2$ to the i-th vertex of $G_1$, where 1âĽď$i$âĽď$|V(G_1)|$.

It is clear that $G_1 \cup G_2 = G_2 \cup G_1$ and $G_1 + G_2 = G_2 + G_1$ but $G_1 \circ G_2 \neq G_2 \circ G_1$.

### 3.1. The monatic number of the union of graphs

**Theorem 3.1.**
*Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be any tow graphs. Then*

$$\mu(G_1 \cup G_2) = \min\{\mu(G_1), \mu(G_2)\}.$$

*Proof.* Since for any two graphs $G_1$ and $G_2$, a set $M \subseteq V_1 \cup V_2$ is a monopoly set of $G_1 \cup G_2$ if and only if $M$ is a union of a monopoly of $G_1$ and a monopoly set of $G_2$, then the result follows. □

### 3.2. The monatic number of the join of graphs

**Theorem 3.2.**
*Let $G_1$ and $G_2$ be connected graphs. Then $\mu(G_1 + G_2) = 3$, if and only if $\{G_1, G_2\} = \{K_1, K_2\}$.*

*Proof.* Since $G_1 + G_2$ is a connected graph for any two graphs, it follows that $\mu(G_1 + G_2) \in \{2, 3\}$. Let $\{G_1, G_2\} = \{K_1, K_2\}$. Then $G_1 + G_2 = K_3$ and hence $\mu(G_1 + G_2) = 3$.

Conversely, let $\{G_1, G_2\} \neq \{K_1, K_2\}$. We consider the following cases

**Case 1:** If $G_1 = K_1$ and $G_2 \neq K_2$, then $G_1 + G_2$ having a vertex with $n-1$ degree and since $G_2$ is connected and $G_2 \neq K_2$. Thus $G_1 + G_2 \notin \{K_3, F_n\}$ and hence by Theorem 1.6, $\mu(G_1 + G_2) = 2$.

**Case 2:** If $G_1$ and $G_2$ non-trivial graphs, $G_i \neq K_1$, for $i = 1, 2$, then by the definition of $G_1 + G_2$, $\omega(G_1 + G_2) \geq 4$. Hence by Theorem 1.4, $\mu(G_1 + G_2) = 2$.

□

**Corollary 3.1.**
*For any non-trivial connected graphs $G_1$ and $G_2$, $\mu(G_1 + G_2) = 2$.*

**Corollary 3.2.**
*If $\{G_1, G_2\} = \{K_1, sK_2\}$, for $s \geq 1$, then $\mu(G_1 + G_2) = 3$.*

### 3.3. The monatic number of the corona product of graphs

**Theorem 3.3.**
*Let $G$ be a graph with at least two vertices. Then*

1. $\mu(K_1 \circ G) = 3$, *if and only if* $G = sK_2$, *for* $s \geq 1$.
2. $\mu(G \circ K_1) = 2$.

*Proof.* 1. It is clear that, if $G = sK_2$, for $s \geq 1$, then $K_1 \circ G \in \{K_3, F_s\}$ and hence by Theorem 1.7. $\mu(K_1 \circ G) = 3$.

Conversely, Let $G$ be a graph with at least two vertices. Then by the definition of the corona product of graphs $K_1 \circ G$ having a vertex with $n-1$ degree, where $n = |V(K_1 \circ G)|$. Hence, if $G = sK_2$, then $G \circ G \in \{K_3, F_s\}$ and again by Theorem 1.7, $\mu(K_1 \circ G) = 3$. Otherwise, $\mu(K_1 \circ G) = 2$.

2. Since the vertex of $K_1$ with degree one in $G \circ K_1$ for every copy of $K_1$, then by Corollary 1.1, the result is holding. □

**Corollary 3.3.**
*For every positive integers $s$ and $n$,* $\mu(\overline{K_n} \circ sK_2) = 3$.

**Theorem 3.4.**
*For any connected non-trivial graph $G$,* $\mu(G \circ K_2) = 3$ *if and only if* $\mu(G) = 3$.

*Proof.* Let $v_1, v_2, ..., v_n$ and $x_i, y_i$ be the vertex sets of $G$ and $i-th$ copy of $K_2$, respectively. Suppose that $\mu(G \circ K_2) = 3$ and let $M_1, M_2$ and $M_3$ be the $M$- partition of $G \circ K_2$. Now, assume, on the contrary, that $\mu(G) = 2$. Since $d_{G \circ K_2}(v_i) = d_G(v_i) + 2$, for every $v_i \in V(G)$, it follows that if there exists a vertex $v \in G$ with odd degree then $v$ is also with odd degree in $G \circ K_2$ and hence $\mu(G \circ K_2) = 2$, a contradiction. So in the following we assume that every vertex in $G$ of even degree and set $S_1 = M_1 \cap V(G)$, $S_2 = M_2 \cap V(G)$ and $S_3 = M_3 \cap V(G)$. It is clear that $S_j$ is an independent set, for every $j = 1, 2, 3$. Since, $M_j$ is an independent set, for every $j = 1, 2, 3$, it follows that $M_j \cap \{x_i, y_i\}$ is a singleton set. Thus

$$|N_{G \circ K_2}(v_i) \cap M_j| = \frac{d_{G \circ K_2}(v_i)}{2} = \frac{d_G(v_i) + 2}{2},$$

for every $v_i \notin M_j$ and $i = 1, 2, ..., n$, $j = 1, 2, 3$. Since $N_{G \circ K_2}(v_i) = N_G(v_i) \cup \{x_i, y_i\}$ then

$$\begin{aligned} N_{G \circ K_2}(v_i) \cap M_j &= (N_G(v_i) \cap M_j) \cup \{x_i, y_i\} \\ &= (N_G(v_i) \cap S_j) \cup (\{x_i, y_i\} \cap M_j). \end{aligned}$$

and hence,

$$\begin{aligned} \frac{d_G(v_i) + 2}{2} &= |N_{G \circ K_2}(v_i) \cap M_j| \\ &= |N_G(v_i) \cap S_j| + |\{x_i, y_i\} \cap M_j| \\ &= |N_G(v_i) \cap S_j| + 1. \end{aligned}$$

Hence, $S_i$ is a monopoly set of $G$ for every $j = 1, 2, 3$. Therefore, $S_1, S_2$ and $S_3$ are an $M$-partition of $G$. So $\mu(G) = 3$, which contradicts the assumption.

Conversely, let $\mu(G) = 3$ and let $M_1, M_2$ and $M_3$ be the $M$-partition of $G$. Then set

$$S_1 = M_1 \cup \{x_i : x_i \in N(v_i), for\ every\ v_i \notin M_1\}$$

$$S_2 = M_2 \cup \{y_i : y_i \in N(v_i), for\ every\ v_i \notin M_2\}$$

$$S_3 = M_3 \cup \{x_i : x_i \in N(v_i), for\ every\ v_i \in M_1\} \cup \{y_i : y_i \in N(v_i), for\ every\ v_i \in M_2\}$$

By chick the sets $S_1, S_2$ and $S_3$, one easily can see it are independent monopoly sets of $G \circ K_2$. Hence, $\mu(G \circ K_2) = 3$. □

**Theorem 3.5.**
*For any connected non-trivial graphs $G_1$ and $G_2$. If* $\Delta(G_2) \geq 2$, *then* $\mu(G_1 \circ G_2) = 2$.

*Proof.* Let $v_1,...,v_{n_1}$ be the vertex set of $G_1$ and let $u$ be the vertex with $d(u) \geq 2$ in $G_2$. Then $d(u) \geq 3$ in $G_1 \circ G_2$. Suppose, on the contrary, that $\mu(G_1 \circ G_2) = 3$ and let $M_1, M_2$ and $M_3$ be the $M$-partition of $G_1 \circ G_2$. Now, let $v_i$ be the vertex of $G_1$ which adjacent to the i-th copy of $G_2$. If, without loss of generality, $v_i \in M_1$, then must be $u \in M_2$ (or $u \in M_3$) and $N_{G_2} \subseteq M_3$ (or resp. $N_{G_2}(u) \subseteq M_3$). Since,

$$d_{G_1 \circ G_2}(u) = d_{G_2}(u) + 1 \geq 2 + 1 > 1 = |N_{G_! \circ G_2}(u) \cap M_1|$$

it follows that $M_1$ is not a monopoly set of $G_1 \circ G_2$, a contradiction. Therefore $\mu(G_1 \circ G_2) = 2$. □

The next main result is an immediately consequence of Theorems 3.3, 3.4 and 3.5.

**Theorem 3.6.**
*Let $G_1$ and $G_2$ be connected non-trivial graphs. Then*

$$\mu(G_1 \circ G_2) = 3, \; if \; and \; only \; if \; \mu(G_1) = 3 \;\; and \; G_2 = K_2.$$

## 4. The monatic number of the line graph of a graph

**Definition 4.1.**
The line graph $L(G)$ of a graph $G = (V, E)$ is a graph with vertex set $E(G)$ in which two vertices are adjacent (joined) if they have an end in common.

It is clear, from the definition, that

$$|V(L(G))| = |E(G)| = m, |E(L(G))| = \frac{1}{2} \sum_{v \in V(G)} d^2(v) - |E(G)|$$

and $\omega(L(G)) = \Delta(G)$.

**Proposition 4.1.**
*Let $G$ be a graph with $K_2$ as a competent. Then $\mu(L(G)) = 1$.*

**Proposition 4.2.**
*For any connected graph $G$ with $\Delta(G) \geq 4$, $\mu(L(G)) = 2$.*

*Proof.* The proof is consequences of the properties of the line graph $L(G)$ of a graph and Theorem 1.4. □

**Theorem 4.1.**
*Let $G$ be a graph with $\mu(G) = 3$. Then $\mu(L(G)) = 3$, if and only if $G = C_{3k}$, for $k \geq 1$.*

*Proof.* Let $G$ be a graph with $\mu(G) = 3$. Since $L(C_k) = C_k$, for any positive integer $k$ and since

$$\mu(C_k) = \begin{cases} 3, & \text{if } k \equiv 0 \pmod 3; \\ 2, & \text{otherwise.} \end{cases}$$

Hence, if $G = C_{3k}$, then $\mu(G) = \mu(L(G)) = 3$.

Conversely, let $G$ be a graph with $\mu(G) = 3$ and $\mu(L(G)) = 3$. Since $\mu(G) = 3$, then by Corollary 1.1, $d_G(v)$ is even for every $v \in V(G)$. Thus, if there exists a vertex $v$ in $G$ with $d_G(v) \geq 4$, then by Proposition 4.2, $\mu(G) = 2$, a contradiction. Hence, for every vertex $v$ in $G$, must $d(v) = 2$. Therefore, $G = C_{3k}$. □

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