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Application of fractional sub-equation method to the space-time fractional differential equations

Research Article

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Abstract: In this paper we discuss the efficiency of the sub-equation method to construct the exact analytical solutions of the nonlinear time-fractional equations. The fractional sub-equation method is considered for application to the space-time fractional Telegraph and Burgers-Huxley equations. These solutions include the generalized trigonometric function solutions, generalized hyperbolic function solutions, and rational function solutions, which they are benefit to further understand the concepts of the complicated nonlinear physical phenomena and fractional differential equations. In this work we use of Mathematica for computations and programming.

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1. Introduction

The ordinary and partial fractional differential equations(FDEs) have been applied in many fields such as signal processing, engineering, chemistry, control theory, biology, etc in recent years [1–3]. Many physical phenomena can be modeled using fractional differential equations. Investigation and Searching of the exact travelling wave solutions for nonlinear FDEs plays an important role in the study of nonlinear physical phenomena. Finding exact solutions of most of the fractional PDEs is not easy, but constructing and searching exact solutions for nonlinear fractional partial differential equations is a continuing investigation. Many powerful methods for obtaining exact solutions of nonlinear FDEs have been presented such as, Hirotas bilinear method [4], Fourier transform, Laplace transform[5], Adomian decomposition method [6], homotopy perturbation method [7], variational iteration method [8], Bäcklund transformation [9],Fractional Lie group method [10] , application of eigenfunctions method [11] and so on. In this paper the fractional sub-equation method will be employed to find the exact solutions for nonlinear FDEs. The fractional sub-equation method is a very powerful mathematical technique for finding exact solutions of nonlinear ordinary differential equations. In this method the exact solutions of the nonlinear FDEs can be expressed as a polynomial and the degree of the this polynomial can be determined by considering the homogeneous balance between the highest order derivatives and nonlinear terms in the considered equation. The aim of this paper is to find exact solutions of the space-time fractional Telegraph and Burgers-Huxley equations using the fractional sub-equation method. In this letter we use this method to solve the following two FDEs :
(I) The form of space-time fractional Telegraph equation [12]

$$D_t^{2\alpha} u - D_x^{2\alpha} u + D_t^{\alpha} u + mu + nu^3 = 0. \tag{1}$$

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(II) The form of space-time fractional of Burgers-Huxley equation[13]

$$D_t^\alpha u + D_x^{2\alpha} u + 3uD_x^\alpha u + ru + u^2 + u^3 = 0. \tag{2}$$

The present Letter is motivated by the desire to propose a fractional sub-equation method to construct exact analytical solutions of nonlinear fractional differential equations with the Caputo fractional derivative of order α is defined by the expression [1]

$$f_-^{(\alpha)}(x) = \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{f'(\xi)}{(x-\xi)^\alpha} d\xi, \tag{3}$$

for $f : \mathbb{R} \longrightarrow \mathbb{R}$. The rest of this Letter is organized as follows. In Section 2, we describe the fractional sub-equation method for solving fractional differential equations. In Section 3, we give two applications of the proposed method to nonlinear equations. In Section 4, some conclusions are given.

2. Fractional sub-equation method for finding the exact solutions of nonlinear FDEs

In this section, we outline the main steps of this method for solving fractional differential equations. For a given fractional differential equation in two variables x and t we have

$$p\left(u, u_x, u_t, D_t^\alpha u, D_x^\alpha u, \ldots\right) = 0, \quad 0 < \alpha < 1, \tag{4}$$

where $D_t^\alpha u$ and $D_x^\alpha u$ are Caputo fractional derivatives of u, $u = u(x,t)$ is an unknown function, P is a polynomial in u and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved.
To determine u explicitly, we take the following four steps [14]:
Step1. By using the traveling wave transformation:

$$u(x,t) = u(\xi), \qquad \xi = kx + ct, \tag{5}$$

where c and k are constants to be determined later, the FDE (4) is reduced to the following nonlinear fractional ordinary differential equation (ODE) for $u = u(\xi)$:

$$p\left(u, ku', cu', k^\alpha D_\xi^\alpha u, c^\alpha D_\xi^\alpha u, \ldots\right) = 0, \quad 0 < \alpha < 1, \tag{6}$$

because

$$\begin{aligned}
[v(ax+b)]^{(\alpha)} = u^{(\alpha)}(x) &= \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{u'(\xi)}{(x-\xi)^\alpha} d\xi \\
&= \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{x} \frac{v'(a\xi+b)}{(x-\xi)^\alpha} a d\xi \\
&= a^\alpha \frac{1}{\Gamma(1-\alpha)} \int_{-\infty}^{ax+b} \frac{v'(z)}{(ax+b-z)^\alpha} dz = a^\alpha v_-^{(\alpha)}(ax+b).
\end{aligned} \tag{7}$$

Where $a > 0$, $b \in \mathbb{R}$.

Step2. Suppose the reduced equation obtained in Step 1 has a solution in the form

$$u(\xi) = \sum_{i=0}^{n} a_i \varphi^i, \tag{8}$$

where a_i $(i = 0, 1, 2, \cdots, n)$ are constants to be determined later, n is a positive integer determined by balancing the highest order derivatives and nonlinear terms in Eq. (4) or Eq. (6) (see Ref. [15] for details), and $\varphi = \varphi(\xi)$ satisfies the following fractional Riccati equation:

$$D_\xi^\alpha \varphi = \sigma + \varphi^2, \qquad 0 < \alpha \le 1. \tag{9}$$

The solution of this equation is [16]:

$$\varphi(\xi) = \begin{cases}
-\sqrt{-\sigma}\ tanh_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0, \\
-\sqrt{-\sigma}\ coth_\alpha(\sqrt{-\sigma}\xi), & \sigma < 0, \\
\sqrt{\sigma}\ tan_\alpha(\sqrt{\sigma}\xi), & \sigma > 0, \\
-\sqrt{\sigma}\ cot_\alpha(\sqrt{\sigma}\xi), & \sigma > 0, \\
-\frac{\Gamma(1+\alpha)}{\xi^\alpha + \omega}, \quad \omega = const., & \sigma = 0.
\end{cases} \tag{10}$$

The generalized hyperbolic and trigonometric functions are defined by the Mittag-Leffler function

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(1+k\alpha)}. \tag{11}$$

Step3. Substituting (8) along with Eq. (9) into Eq. (6), we can get a polynomial in $\varphi(\xi)$. Setting all the coefficients of φ^k $(k = 0, 1, 2, \ldots)$ to zero, yields a set of overdetermined nonlinear algebraic equations for $c, k, a_i (i = 0, 1, 2, \ldots, n)$.
Step4. Assuming that the constants $c, k, a_i (i = 0, 1, 2, \ldots, n)$ can be obtained by solving the algebraic equations in Step 3, substituting these constants and the solutions of Eq. (9) into (8), we can obtain the explicit solutions of Eq. (4) immediately.

3. Applications

In this section, we apply the fractional sub-equation method for solving the FDEs (1) and (2).

Example 3.1.
We consider the space-time fractional Telegraph equation in the form

$$D_t^{2\alpha}u - D_x^{2\alpha}u + D_t^{\alpha}u + mu + nu^3 = 0. \tag{12}$$

This equation is a differential equation which describes the voltage and current on an electrical transmission line with distance and time. The equations come from Oliver Heaviside who in the 1880s developed the transmission line model. The model demonstrates that the electromagnetic waves can be reflected on the wire, and that wave patterns can appear along the line. We take the traveling wave transformation

$$u = u(\xi), \quad \xi = ct + kx, \tag{13}$$

then Eq. (12) is reduced into a nonlinear fractional ODE easy to solve

$$c^{2\alpha}D_\xi^{2\alpha}u - k^{2\alpha}D_\xi^{2\alpha}u + c^{\alpha}D_\xi^{\alpha}u + mu + nu^3 = 0. \tag{14}$$

We next suppose Eq. (14) has a solution in the form

$$u = \sum_{i=0}^{n} a_i\varphi^i, \tag{15}$$

where φ satisfies the subequation (9). By balancing the highest order derivative terms and nonlinear terms in Eq. (14), We then substitute Eq. (15) given the value of n = 1, along with Eq. (9), into Eq. (14) and collect the coefficients of φ^j and set them to be zero, a set of algebraic equations are obtained as follows

$$\begin{aligned}
&\sigma a_1 c^{\alpha} + a_0^3 + ma_0 = 0,\\
&2\sigma a_1 c^{2\alpha} + 3a_0^2 a_1 + ma_1 - 2k^{2\alpha}\sigma a_1 = 0,\\
&a_1 c^{\alpha} + 3a_0 a_1^2 = 0,\\
&2a_1 c^{2\alpha} + a_1^3 - 2k^{2\alpha}a_1 = 0.
\end{aligned}$$

Solving the set of algebraic equations yields

$$a_0 = \pm\frac{i\sqrt{m}}{2}, \quad a_1 = \pm\frac{2i\sqrt{m}}{3m}c^{\alpha}, \quad c^{\alpha} = -\frac{3\sqrt{c^{2\alpha}m - k^{2\alpha}m}}{\sqrt{2}}, \quad k^{\alpha} \neq 0, \quad \sigma \neq 0. \tag{16}$$

We, therefore, obtain from Eqs. (10), (13)-(16) exact solutions of Eq. (12), namely, generalized hyperbolic function solutions and generalized trigonometric function solutions as follows

$$u_1(x,t) = \pm\frac{i\sqrt{m}}{2} \pm \frac{2i\sqrt{m}}{3m}c^{\alpha}(-\sqrt{-\sigma}\ tanh_{\alpha}(\sqrt{-\sigma}\xi)), \quad \sigma < 0, \tag{17}$$

$$u_2(x,t) = \pm\frac{i\sqrt{m}}{2} \pm \frac{2i\sqrt{m}}{3m}c^{\alpha}(-\sqrt{-\sigma}\ coth_{\alpha}(\sqrt{-\sigma}\xi)), \quad \sigma < 0, \tag{18}$$

$$u_3(x,t) = \pm\frac{i\sqrt{m}}{2} \pm \frac{2i\sqrt{m}}{3m}c^{\alpha}(\sqrt{\sigma}\ tan_{\alpha}(\sqrt{\sigma}\xi)), \quad \sigma > 0, \tag{19}$$

$$u_4(x,t) = \pm\frac{i\sqrt{m}}{2} \pm \frac{2i\sqrt{m}}{3m}c^{\alpha}(-\sqrt{\sigma}\ cot_{\alpha}(\sqrt{\sigma}\xi)), \quad \sigma > 0. \tag{20}$$

Example 3.2.
We consider the Burgers-Huxley equation in the form

$$D_t^{\alpha}u + D_x^{2\alpha}u + 3uD_x^{\alpha}u + ru + u^2 + u^3 = 0. \tag{21}$$

This equation shows a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transports. The equation was investigated by Satsuma in 1986. Solutions of this equation are of importance in physical problems. To solve Eq.(21), we take the following traveling wave transformation

$$u = u(\xi), \quad \xi = kx + ct, \tag{22}$$

then Eq. (21) is reduced into a nonlinear fractional ODE easy to solve

$$c^{\alpha} D_t^{\alpha} u + k^{2\alpha} D_x^{2\alpha} u + 3k^{\alpha} u D_x^{\alpha} u + ru + u^2 + u^3 = 0. \tag{23}$$

We suppose Eq. (23) has a solution in the form

$$u = \sum_{i=0}^{n} a_i \varphi^i, \tag{24}$$

where φ satisfies the subequation (9).

By balancing the highest order derivative terms and nonlinear terms in Eq. (23), We then substitute Eq. (24) given the value of n = 1, along with Eq. (9), into Eq. (23) and collect the coefficients of φ^j and set them to be zero, a set of algebraic equations are obtained as follows

$$\begin{aligned}
&\sigma a_1 c^{\alpha} + a_0^3 + a_0^2 + r a_0 + 3k^{\alpha} \sigma a_0 a_1 = 0,\\
&3\sigma a_1^2 k^{\alpha} + 2\sigma a_1 k^{2\alpha} + 3a_0^2 a_1 + r a_1 + 2a_0 a_1 = 0,\\
&a_1 c^{\alpha} + 3a_0 a_1^2 + a_1^2 + 3k^{\alpha} a_0 a_1 = 0,\\
&3a_1^2 k^{\alpha} + 2a_1 k^{2\alpha} + a_1^3.
\end{aligned}$$

Solving the set of algebraic equations yields

Case1.1: when $\sigma = 0, r = 0$.

$$\begin{cases} a_0 = 0, & a_1 = -k^{\alpha}, \quad c^{\alpha} = k^{\alpha},\\ a_0 = 0, & a_1 = -2k^{\alpha}, \quad c^{\alpha} = 2k^{\alpha}. \end{cases} \tag{25}$$

Therefore, from (10), (25) we obtain the following rational solutions of Eq. (21)

$$\begin{aligned}
u_1 &= k^{\alpha} \frac{\Gamma(1+\alpha)}{(kx+ct)^{\alpha}+\omega}; \qquad \sigma = 0, \quad r = 0, \quad c^{\alpha} = k^{\alpha},\\
u_2 &= 2k^{\alpha} \frac{\Gamma(1+\alpha)}{(kx+ct)^{\alpha}+\omega}; \qquad \sigma = 0, \quad r = 0, \quad c^{\alpha} = 2k^{\alpha}.
\end{aligned}$$

Case1.2: when $\sigma = 0, r = \frac{1}{4}$.

$$\begin{cases} a_0 = -\frac{1}{2}, & a_1 = -k^{\alpha}, \quad k^{\alpha} = c^{\alpha},\\ a_0 = -\frac{1}{2}, & a_1 = -2k^{\alpha}, \quad k^{\alpha} = 2c^{\alpha}. \end{cases} \tag{26}$$

We obtain the following rational solutions of Eq. (21) using (26)

$$\begin{aligned}
u_3 &= -\frac{1}{2} + k^{\alpha} \frac{\Gamma(1+\alpha)}{(kx+ct)^{\alpha}+\omega}; \qquad \sigma = 0, \quad r = 0, \quad c^{\alpha} = k^{\alpha},\\
u_4 &= -\frac{1}{2} + 2k^{\alpha} \frac{\Gamma(1+\alpha)}{(kx+ct)^{\alpha}+\omega}; \qquad \sigma = 0, \quad r = 0, \quad c^{\alpha} = 2k^{\alpha}.
\end{aligned}$$

Case 2.1: when $\sigma = -\frac{1}{144k^{2\alpha}}, r = \frac{2}{9}$.

$$\begin{cases} a_0 = -\frac{1}{2}, & a_1 = -2k^{\alpha}, \quad k^{\alpha} = 2c^{\alpha},\\ a_0 = -\frac{1}{6}, & a_1 = -2k^{\alpha}, \quad k^{\alpha} = \frac{2}{3}c^{\alpha}. \end{cases} \tag{27}$$

Now from (10), (27) we obtain the following generalized hyperbolic function solutions of Eq. (21)

$$u_5 = -\frac{1}{2} + 2k^{\alpha}(\sqrt{-\sigma}\, tanh_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=2c^{\alpha},$$
$$u_6 = -\frac{1}{2} + 2k^{\alpha}(\sqrt{-\sigma}\, coth_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=2c^{\alpha},$$
$$u_7 = -\frac{1}{6} + 2k^{\alpha}(\sqrt{-\sigma}\, tanh_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=\frac{2}{3}c^{\alpha},$$
$$u_8 = -\frac{1}{6} + 2k^{\alpha}(\sqrt{-\sigma}\, coth_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=\frac{2}{3}c^{\alpha}.$$

Case 2.2: when $\sigma = -\frac{1}{36k^{2\alpha}}, r = \frac{2}{9}$.

$$\begin{cases} a_0 = -\frac{1}{2}, \quad a_1 = -k^{\alpha}, \qquad k^{\alpha} = c^{\alpha}, \\ a_0 = -\frac{1}{3}, \quad a_1 = -2k^{\alpha}, \qquad k^{\alpha} = c^{\alpha}. \end{cases} \tag{28}$$

Using (10), (28) we obtain the following generalized hyperbolic function solutions of Eq. (21)

$$u_9 = -\frac{1}{2} + k^{\alpha}(\sqrt{-\sigma}\, tanh_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=c^{\alpha},$$
$$u_{10} = -\frac{1}{2} + k^{\alpha}(\sqrt{-\sigma}\, coth_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=c^{\alpha},$$
$$u_{11} = -\frac{1}{3} + 2k^{\alpha}(\sqrt{-\sigma}\, tanh_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=c^{\alpha},$$
$$u_{12} = -\frac{1}{3} + 2k^{\alpha}(\sqrt{-\sigma}\, coth_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=c^{\alpha}.$$

Case 3: when $\sigma = -\frac{1}{9k^{2\alpha}}, r = \frac{2}{9}$.

$$a_0 = -\frac{1}{3}, \quad a_1 = -k^{\alpha}, \qquad k^{\alpha} = c^{\alpha}. \tag{29}$$

Finally, from Eqs. (10), (29) we obtain the following generalized hyperbolic function solutions of Eq. (21)

$$u_{13} = -\frac{1}{3} + k^{\alpha}(\sqrt{-\sigma}\, tanh_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=c^{\alpha},$$
$$u_{14} = -\frac{1}{3} + k^{\alpha}(\sqrt{-\sigma}\, coth_{\alpha}(\sqrt{-\sigma}(kx+ct))); \quad \sigma<0, \qquad k^{\alpha}=c^{\alpha}.$$

As $\alpha \to 1$ obtained solutions above become the ones of Eq.(21).

4. Conclusion

We have proposed a fractional sub-equation method to solve fractional differential equations. As applications of the proposed method, some exact analytical solutions of the space-time fractional Telegraph and Burgers-Huxley equations are successfully obtained. These solutions include generalized hyperbolic function solutions, generalized trigonometric function solutions and rational function solutions. Moreover, the proposed method is shown to be a simple, yet powerful algorithm for handling for systems of FDEs. *Mathematica* has been used for computations and programming in this paper.

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