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A short note on Floor van Lamoen's Cyclic Hexagon

Research Article

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Abstract: In this article we study the metric properties of Floor van Lemoen's cyclic Hexagon.

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1. Introduction

Let G be the centroid of triangle ABC and AD, BE, CF are the medians of the triangle (where D, E and F lies on the sides BC, CA and AB) which divides the triangle ABC into six sub triangles. Several years ago, Floor van Lamoen discovered that the six circumcenters of these sub triangles are concylic. That is the six circumcenters of these sub triangles form a cyclic hexagon see Fig. 1 (for the recognition sake let us call this cyclic hexagon as a **Floor van Lamoen's Cyclic Hexagon**). This was posed as a problem in the American Mathematical Monthly [1] and was solved in [2–5]. In this note we study the metric properties of Floor van Lamoen's cyclic hexagon such as lengths of sides, diagonals and their related properties.

2. Notation and background

Let ABC be a non equilateral triangle. We denote its side-lengths by a, b, c angles A, B, C its area by Δ and its classical center centroid as G. Let AD, BE, CF are the medians whose lengths are m_a, m_b and m_c respectively and G divides each median in the ration 2:1. Let S_1, S_2, S_3, S_4, S_5, and S_6 are the circumcenters of the sub triangles $\Delta AFG, \Delta GFB, \Delta BGD, \Delta DGC, \Delta CGE$ and ΔEGA respectively (see Fig. 2). Clearly $S_1S_2S_3S_4S_5S_6$ is a Floor van Lamoen's Cyclic Hexagon. Let$\mathbb{R}, \Psi$ and Λ are the circumradius, perimeter and area of Floor van Lamoen's Cyclic Hexagon respectively.

We adopt the following notations for the various angles in the triangle

$\angle BAD = A_1$, $\angle DAC = A_2$, $\angle ACF = C_1$, $\angle FCB = C_2$, $\angle CBE = B_1$ and $\angle FBA = B_2$

$\angle ADB = D_1$, $\angle ADC = D_2$, $\angle CEB = E_1$, $\angle BEA = E_2$, $\angle AFC = F_1$ and $\angle BFC = F_2$

And it is clear $A = A_1 + A_2$, $B = B_1 + B_2$, $C = C_1 + C_2$ and $D_1 + D_2 = E_1 + E_2 = F_1 + F_2 = 180^0$

And we use

$$V_{hexagon} = \sqrt{5\left({m_a}^2{m_b}^2 + {m_b}^2{m_c}^2 + {m_c}^2{m_a}^2\right) - 2\left({m_a}^4 + {m_b}^4 + {m_c}^4\right)} = \frac{3}{4}\sqrt{5\left(a^2b^2 + b^2c^2 + c^2a^2\right) - 2\left(a^4 + b^4 + c^4\right)}$$

The following formulas are well known [6]

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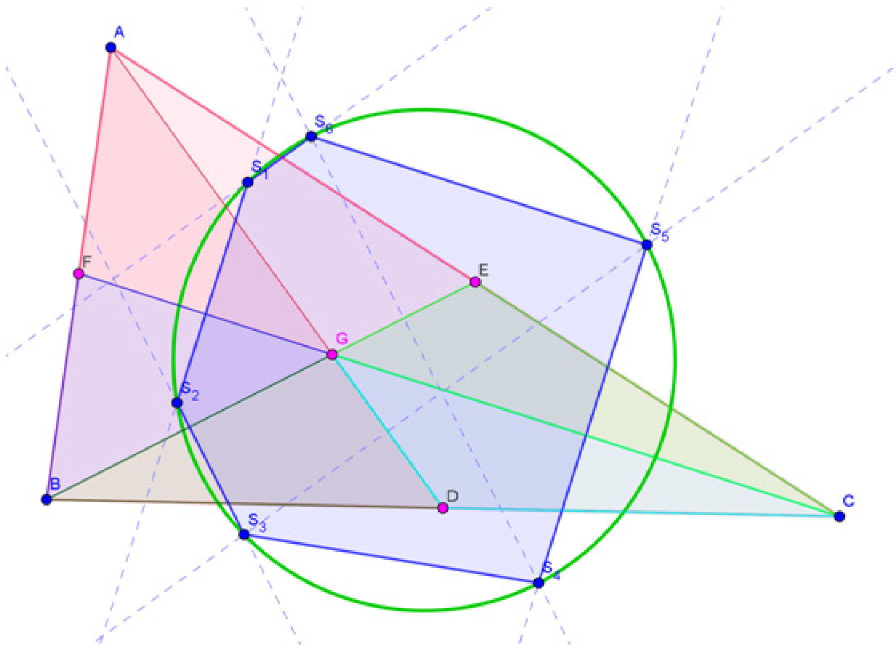

Fig. 1.

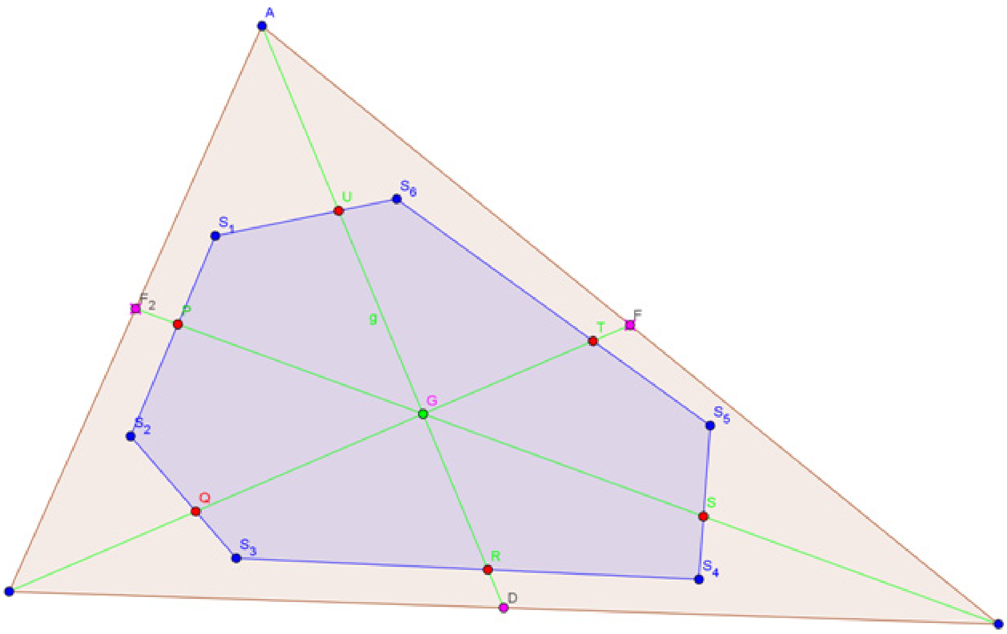

Fig. 2.

(a) $\Delta = \frac{abc}{4R}$ where R is circumradius of ΔABC

(b) $4{m_a}^2 = 2b^2 + 2c^2 - a^2, 4{m_b}^2 = 2a^2 + 2c^2 - b^2$, and $4{m_c}^2 = 2a^2 + 2b^2 - c^2$

(c) $4\left({m_a}^2 + {m_b}^2 + {m_c}^2\right) = 3\left(a^2 + b^2 + c^2\right)$

(d) $16\left({m_a}^4 + {m_b}^4 + {m_c}^4\right) = 9\left(a^4 + b^4 + c^4\right)$

(e) $16\left({m_a}^2{m_b}^2 + {m_b}^2{m_c}^2 + {m_c}^2{m_a}^2\right) = 9\left(a^2b^2 + b^2c^2 + c^2a^2\right)$

(f) $9a^2 = 8{m_b}^2 + 8{m_c}^2 - 4{m_a}^2, 9b^2 = 8{m_a}^2 + 8{m_c}^2 - 4{m_b}^2$ and $9c^2 = 8{m_a}^2 + 8{m_b}^2 - 4{m_c}^2$
$81a^2b^2 = 80{m_a}^2{m_b}^2 + 32{m_b}^2{m_c}^2 + 32{m_c}^2{m_a}^2 - 32{m_a}^4 - 32{m_b}^4 + 64{m_c}^4$
$81b^2c^2 = 80{m_b}^2{m_c}^2 + 32{m_c}^2{m_a}^2 + 32{m_a}^2{m_b}^2 - 32{m_b}^4 - 32{m_c}^4 + 64{m_a}^4$
$81c^2a^2 = 80{m_c}^2{m_a}^2 + 32{m_a}^2{m_b}^2 + 32{m_b}^2{m_c}^2 - 32{m_c}^4 - 32{m_a}^4 + 64{m_b}^4$

(g) $\Delta^2 = \frac{1}{16}\left(2a^2b^2 + 2b^2c^2 + 2c^2a^2 - a^4 - b^4 - c^4\right)$
$\Delta^2 = \frac{1}{9}\left(2{m_a}^2{m_b}^2 + 2{m_b}^2{m_c}^2 + 2{m_c}^2{m_a}^2 - {m_a}^4 - {m_b}^4 - {m_c}^4\right)$
$\left({m_a}^2{m_b}^2 + {m_b}^2{m_c}^2 + {m_c}^2{m_a}^2\right) = {V_{hexagon}}^2 - 18\Delta^2$
$\left({m_a}^4 + {m_b}^4 + {m_c}^4\right) = 2{V_{hexagon}}^2 - 45\Delta^2$
$\left(a^2b^2 + b^2c^2 + c^2a^2\right) = \frac{16}{9}{V_{hexagon}}^2 - 32\Delta^2$
$\left(a^4 + b^4 + c^4\right) = \frac{32}{9}{V_{hexagon}}^2 - 80\Delta^2$

(h) In any triangle ABC, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$ (sine rule)

(i) In any triangle ABC, $a^2 = b^2 + c^2 - 2bc\cos A, b^2 = c^2 + a^2 - 2ca\cos B, c^2 = a^2 + b^2 - 2ab\cos C$ (cosine rule)

(j) $2bc\cos A = b^2 + c^2 - a^2 = \frac{4}{9}\left(5{m_a}^2 - {m_b}^2 - {m_c}^2\right)$
$2ca\cos B = c^2 + a^2 - b^2 = \frac{4}{9}\left(5{m_b}^2 - {m_c}^2 - {m_a}^2\right)$
$2ab\cos C = a^2 + b^2 - c^2 = \frac{4}{9}\left(5{m_c}^2 - {m_a}^2 - {m_b}^2\right)$

3. Preliminaries

We use the following lemmas in proving the results related to Floor van Lamoen's Cyclic Hexagon.

Lemma 3.1.
If R_1, R_2, R_3, R_4, R_5 and R_6 are the circumradii of the triangles $\Delta AFG, \Delta GFB, \Delta BGD, \Delta DGC, \Delta CGE$ and ΔEGA whose circumcenters are S_1, S_2, S_3, S_4, S_5 and S_6 respectively then

$$GS_1 = AS_1 = FS_1 = R_1 = \frac{cm_am_c}{6\Delta} \tag{1}$$
$$GS_2 = BS_2 = FS_2 = R_2 = \frac{cm_bm_c}{6\Delta} \tag{2}$$
$$GS_3 = BS_3 = DS_3 = R_3 = \frac{am_am_b}{6\Delta} \tag{3}$$
$$GS_4 = CS_4 = DS_4 = R_4 = \frac{am_cm_a}{6\Delta} \tag{4}$$
$$GS_5 = CS_5 = ES_5 = R_5 = \frac{bm_bm_c}{6\Delta} \tag{5}$$
$$GS_6 = AS_6 = ES_6 = R_6 = \frac{bm_am_b}{6\Delta} \tag{6}$$

Proof. We are familiar with the fact about centroid G is $[\Delta AGB] = [\Delta BGC] = [\Delta CGA] = \dfrac{\Delta}{3}$.
It implies $[\Delta AGF] = [\Delta FGB] = [\Delta BGD] = [\Delta DGC] = [\Delta CGE] = [\Delta EGA] = \dfrac{\Delta}{6}$
Now using (a), for ΔAGF, $R_1 = \dfrac{AG.GF.AF}{4\frac{\Delta}{6}} = \dfrac{\frac{2}{3}m_a\frac{1}{3}m_c\frac{c}{2}}{\frac{2\Delta}{3}} = \dfrac{cm_am_c}{6\Delta}$
Which completes the proof of (1). Similarly we can prove (2) to (6). □

Lemma 3.2.
If A_1, A_2, B_1, B_2, C_2, C_3, D_1, D_2, E_1, E_2, F_1 and F_2 are the angles in the triangle as prescribed in section 2 then

$$\cos A_1 = \frac{9c^2 + 16{m_a}^2 - 4{m_c}^2}{24cm_a} = \frac{3{m_a}^2 + {m_b}^2 - {m_c}^2}{3cm_a} \tag{7}$$
$$\cos A_2 = \frac{9b^2 + 16{m_a}^2 - 4{m_b}^2}{24bm_a} = \frac{3{m_a}^2 + {m_c}^2 - {m_b}^2}{3bm_a} \tag{8}$$
$$\cos B_1 = \frac{9a^2 + 16{m_b}^2 - 4{m_a}^2}{24am_b} = \frac{3{m_b}^2 + {m_c}^2 - {m_a}^2}{3am_b} \tag{9}$$
$$\cos B_2 = \frac{9c^2 + 16{m_b}^2 - 4{m_c}^2}{24cm_b} = \frac{3{m_b}^2 + {m_a}^2 - {m_c}^2}{3cm_b} \tag{10}$$
$$\cos C_1 = \frac{9b^2 + 16{m_c}^2 - 4{m_b}^2}{24bm_c} = \frac{3{m_c}^2 + {m_a}^2 - {m_b}^2}{3bm_c} \tag{11}$$
$$\cos C_2 = \frac{9a^2 + 16{m_c}^2 - 4{m_a}^2}{24am_c} = \frac{3{m_c}^2 + {m_b}^2 - {m_a}^2}{3am_c} \tag{12}$$

$$\cos D_1 = -\cos D_2 = \frac{9a^2 + 4m_a{}^2 - 16m_b{}^2}{12am_a} = -\left(\frac{9a^2 + 4m_a{}^2 - 16m_c{}^2}{12am_a}\right) = \frac{2\left(m_c{}^2 - m_b{}^2\right)}{3am_a} \tag{13}$$

$$\cos E_1 = -\cos E_2 = \frac{9b^2 + 4m_b{}^2 - 16m_c{}^2}{12bm_b} = -\left(\frac{9b^2 + 4m_b{}^2 - 16m_c{}^2}{12bm_b}\right) = \frac{2\left(m_a{}^2 - m_c{}^2\right)}{3bm_b} \tag{14}$$

$$\cos F_1 = -\cos F_2 = \frac{9c^2 + 4m_c{}^2 - 16m_a{}^2}{12cm_c} = -\left(\frac{9c^2 + 4m_c{}^2 - 16m_b{}^2}{12cm_c}\right) = \frac{2\left(m_b{}^2 - m_a{}^2\right)}{3cm_c} \tag{15}$$

$$\sin A_1 = \frac{\Delta}{cm_a}, \sin A_2 = \frac{\Delta}{bm_a}, \sin B_1 = \frac{\Delta}{am_b}, \sin B_2 = \frac{\Delta}{cm_b}, \sin C_1 = \frac{\Delta}{bm_c}, \sin C_2 = \frac{\Delta}{am_c} \tag{16}$$

$$\sin D_1 = \sin D_2 = \frac{2\Delta}{am_a}, \sin E_1 = \sin E_2 = \frac{2\Delta}{bm_b}, \sin F_1 = \sin F_2 = \frac{2\Delta}{cm_c} \tag{17}$$

Proof. Consider ΔAFG, using cosine rule we have,

$$\cos A_1 = \frac{AF^2 + AG^2 - FG^2}{2AF.AG} = \frac{\frac{c^2}{4} + \frac{4m_a{}^2}{9} - \frac{m_c{}^2}{9}}{2\left(\frac{c}{2}\right)\left(\frac{2}{3}m_a\right)}$$

Further simplification gives

$$cos A_1 = \frac{9c^2 + 16m_a{}^2 - 4m_c{}^2}{24cm_a} \tag{18}$$

Now by replacing $9c^2$ using **(f)**, (18) can be further simplified to get (7)

In the similar manner we can prove (8) to (12).

Now for (13), we apply cosine rule for the angles F_1, F_2 of the triangles ΔAFG and ΔBFG respectively.

Using **(f)**, and using the fact $F_1 + F_2 = 180°$ (straight angle) and with little algebra we can prove conclusion (13). Similiarly we can prove (14), (15).

Now for (16), (17) we use Lemma 3.1 and sine rule for the triangles $\Delta AFG, \Delta GFB, \Delta BGD, \Delta DGC, \Delta CGE$ and ΔEGA. □

Lemma 3.3.

$$V^2_{hexagon} \geqslant 27\Delta^2$$

Proof. We know that, for any real x, y and z, $x^2 + y^2 + z^2 \geqslant xy + yz + zx$.

If we consider $x = m_a{}^2, y = m_b{}^2$ and $z = m_c{}^2$ then $\left(m_a{}^4 + m_b{}^4 + m_c{}^4\right) \geqslant \left(m_a{}^2 m_b{}^2 + m_b{}^2 m_c{}^2 + m_c{}^2 m_a{}^2\right)$.

Using **(g)**, the above inequality rewritten as

$$2V^2_{hexagon} - 45\Delta^2 \geqslant V^2_{hexagon} - 18\Delta^2$$

It implies

$$V^2_{hexagon} \geqslant 27\Delta^2$$

Hence proved. □

4. Main theorems

Theorem 4.1.
If $S_1S_2S_3S_4S_5S_6$ is a Floor van Lamoen's Cyclic Hexagon then

$$S_1S_2 = \frac{c^2 m_c}{4\Delta} = \frac{m_c\left(2m_a{}^2 + 2m_b{}^2 - m_c{}^2\right)}{9\Delta} \tag{19}$$

$$S_3S_4 = \frac{a^2 m_a}{4\Delta} = \frac{m_a\left(2m_b{}^2 + 2m_c{}^2 - m_a{}^2\right)}{9\Delta} \tag{20}$$

$$S_5S_6 = \frac{b^2 m_b}{4\Delta} = \frac{m_b\left(2m_c{}^2 + 2m_a{}^2 - m_b{}^2\right)}{9\Delta} \tag{21}$$

Proof. Clearly from ΔAGB, the line S_1S_2 acts as the perpendicular bisector of FG at P (See Fig. 2), So

$$S_1S_2 = S_1P + S_2P = S_1G\sin(\angle S_1GP) + S_2G\sin(\angle S_2GP) \tag{22}$$

Since S_1, S_2 are circumcenters of the triangles $\Delta AGF, \Delta BGF$. We have $\angle S_1GP = 90 - \angle GAF = 90 - A_1$ and $\angle S_2GP = 90 - \angle GBF = 90 - B_2$. By replacing these in (22), we get

$$S_1S_2 = S_1G\sin(90 - A_1) + S_2G\sin(90 - B_2) = S_1G\cos(A_1) + S_2G\cos(B_2) \tag{23}$$

Further simplification of (23) using Lemma 3.1, Lemma 3.2 and **(f)**, gives conclusion (19). In the similar manner we can prove (20), (21). □

Theorem 4.2.
If $S_1S_2S_3S_4S_5S_6$ is a Floor van Lamoen's Cyclic Hexagon then

$$S_2S_3 = \frac{m_b\left(2b^2 - c^2 - a^2\right)}{12\Delta} = \frac{m_b\left({m_c}^2 + {m_a}^2 - 2{m_b}^2\right)}{9\Delta} \tag{24}$$
$$S_4S_5 = \frac{m_c\left(2c^2 - a^2 - b^2\right)}{12\Delta} = \frac{m_c\left({m_a}^2 + {m_b}^2 - 2{m_c}^2\right)}{9\Delta} \tag{25}$$
$$S_6S_1 = \frac{m_a\left(2a^2 - b^2 - c^2\right)}{12\Delta} = \frac{m_a\left({m_b}^2 + {m_c}^2 - 2{m_a}^2\right)}{9\Delta} \tag{26}$$

Proof. Clearly from the quadrilateral $FGDB$, the line S_2S_3 is acts as perpendicular bisector of BG at Q (Fig. 2), So

$$S_2S_3 = S_2Q + S_3Q = S_2G\sin(\angle S_2GQ) + S_3G\sin(\angle S_3GQ) \tag{27}$$

Since S_2, S_3 are circumcenters of the triangles $\Delta BGF, \Delta BGD$.
We have $\angle S_2GQ = 90 - \angle BFG = 90 - F_2$ and $\angle S_3GQ = 90 - \angle GDB = 90 - D_1$.
By replacing these in (27), we get

$$S_2S_3 = S_2G\sin(90 - F_2) + S_3G\sin(90 - D_1) = S_2G\cos(F_2) + S_3G\cos(D_1) \tag{28}$$

Further simplification of (28) using Lemma 3.1, Lemma 3.2 and **(f)**, gives conclusion (24). In the similar manner we can prove (25), (26). □

Theorem 4.3.
If $S_1S_2S_3S_4S_5S_6$ is a Floor van Lamoen's Cyclic Hexagon then

$$S_1S_3 = S_4S_6 = \frac{m_a}{9\Delta}V_{hexagon} \tag{29}$$
$$S_3S_5 = S_6S_2 = \frac{m_b}{9\Delta}V_{hexagon} \tag{30}$$
$$S_5S_1 = S_2S_4 = \frac{m_c}{9\Delta}V_{hexagon} \tag{31}$$

Proof. By angle chasing we can show that $\angle S_1GS_3 = 180 - (\angle S_1GA + \angle S_3GD) = F_1 + B_2$ and $\angle S_1S_2S_3 = \angle S_1S_2G + \angle S_3S_2G = F_2 + B_1$.
Hence $\angle S_1S_2S_3 + \angle S_1GS_3 = F_2 + B_1 + F_1 + B_2 = 180 + B$.
Consider the quadrilateral $S_1S_2S_3G$. We apply Bretschneider's theorem for the quadrilateral $S_1S_2S_3G$, we get

$$S_1{S_3}^2.G{S_2}^2 = S_1{S_2}^2.G{S_3}^2 + S_2{S_3}^2.G{S_1}^2 - 2S_1S_2.GS_3.S_2S_3.GS_1\cos(\angle S_1S_2S_3 + \angle S_1GS_3)$$

By replacing $GS_2, S_1S_2, GS_3, S_2S_3, GS_1$ using Lemma 3.1, Theorem 4.1 and Theorem 4.2, the above equation can be rewritten as

$$S_1{S_3}^2\left(\frac{c^2{m_b}^2{m_c}^2}{36\Delta^2}\right) = \left(\frac{{m_c}^2\left(2{m_a}^2 + 2{m_b}^2 - {m_c}^2\right)^2}{81\Delta^2}\right)\left(\frac{a^2{m_a}^2{m_b}^2}{36\Delta^2}\right) + \left(\frac{{m_b}^2\left({m_c}^2 + {m_a}^2 - 2{m_b}^2\right)^2}{81\Delta^2}\right)\left(\frac{c^2{m_a}^2{m_c}^2}{36\Delta^2}\right)$$
$$-2\left(\frac{m_c\left(2{m_a}^2 + 2{m_b}^2 - {m_c}^2\right)}{9\Delta}\right)\left(\frac{am_am_b}{6\Delta}\right)\left(\frac{m_b\left({m_c}^2 + {m_a}^2 - 2{m_b}^2\right)}{9\Delta}\right)\left(\frac{cm_am_c}{6\Delta}\right)\cos(180 + B)$$

By replacing $\cos(180 + B) = -\cos B$ and using **(j)**, the above equation can be further simplified as

$$S_1S_3{}^2\left(\frac{729c^2\Delta^2}{m_a{}^2}\right) = 9a^2(2m_a{}^2+2m_b{}^2-m_c{}^2)^2$$
$$+9c^2(m_c{}^2+m_a{}^2-2m_b{}^2)^2+4(2m_a{}^2+2m_b{}^2-m_c{}^2)(m_c{}^2+m_a{}^2-2m_b{}^2)(5m_b{}^2-m_c{}^2-m_a{}^2)$$

By replacing $9c^2, 9a^2$ and using (f), the above equation can be rewritten as

$$S_1S_3{}^2 = \frac{m_a{}^2}{81\Delta^2}\left(5\left(m_a{}^2m_b{}^2+m_b{}^2m_c{}^2+m_c{}^2m_a{}^2\right)-2\left(m_a{}^4+m_b{}^4+m_c{}^4\right)\right)$$

Further simplification gives $S_1S_3 = \frac{m_a}{9\Delta}V_{hexagon}$

Similarly by considering the quadrilateral $S_4S_5S_6G$ we can prove $S_4S_6 = \frac{m_a}{9\Delta}V_{hexagon}$.

So $S_1S_3 = S_4S_6 = \frac{m_a}{9\Delta}V_{hexagon}$

Which completes the proof of (29). In the similar manner we can prove (30) and (31). □

Theorem 4.4.

If $S_1S_2S_3S_4S_5S_6$ is a Floor van Lamoen's Cyclic Hexagon then

$$S_1S_4 = S_2S_5 = S_3S_6 = \frac{m_am_bm_c}{3\Delta} \tag{32}$$

Proof. By angle chasing we can show that

$$\angle S_1GS_4 = 180-\angle S_4GD+\angle S_1GA = 180-(90-C_2)+(90-F_1) = 180-F_1+C_2 = F_2+C_2 = 180-B$$

Now by applying cosine rule for the triangle S_1GS_4, we get

$$S_1S_4{}^2 = S_1G^2+S_4G^2-2S_1G.S_4G\cos(\angle S_1GS_4) \tag{33}$$

By replacing GS_1 and GS_4 in (33), using Lemma 3.1 and further simplification gives

$$S_1S_4{}^2 = \frac{m_a{}^2m_c{}^2}{36\Delta^2}\left(c^2+a^2+2ca\cos B\right) \tag{34}$$

Using (f) and (j), (34) can be further simplified to get $S_1S_4 = \frac{m_am_bm_c}{3\Delta}$.

Similarly we can prove $S_2S_5 = S_3S_6 = \frac{m_am_bm_c}{3\Delta}$.

Hence, $S_1S_4 = S_2S_5 = S_3S_6 = \frac{m_am_bm_c}{3\Delta}$.

That is the three diagonals of Floor van Lamoen's Cyclic Hexagon are equal. □

Theorem 4.5.

The three diagonals S_1S_4, S_2S_5, and S_3S_6 of Floor van Cyclic Hexagon are never concurrent.

Proof. Let us prove this Theorem using indirect method of proof. If such Floor van Lamoen's Cyclic Hexagon exists then it follows $S_1S_2 . S_3S_4 . S_5$ to $S_6 = S_2S_3 . S_4S_5 . S_6S_1$.

That is if we want prove the three diagonals are concurrent ([7, 8]) it is enough to prove

$$S_1S_2 . S_3S_4 . S_5S_6 = S_2S_3 . S_4S_5 . S_6S_1$$

By replacing the values of $S_1S_2, S_2S_3, S_4S_5, S_5S_6$ and S_6S_1, using Theorem 4.1 and Theorem 4.2, we get

$$\left[\frac{m_c\left(2m_a{}^2+2m_b{}^2-m_c{}^2\right)}{9\Delta}\right]\left[\frac{m_a\left(2m_b{}^2+2m_c{}^2-m_a{}^2\right)}{9\Delta}\right]\left[\frac{m_b\left(2m_c{}^2+2m_a{}^2-m_b{}^2\right)}{9\Delta}\right]$$
$$=\left[\frac{m_b\left(m_c{}^2+m_a{}^2-2m_b{}^2\right)}{9\Delta}\right]\left[\frac{m_c\left(m_a{}^2+m_b{}^2-2m_c{}^2\right)}{9\Delta}\right]\left[\frac{m_a\left(m_b{}^2+m_c{}^2-2m_a{}^2\right)}{9\Delta}\right]$$

It implies

$$\left(m_a{}^2+2m_b{}^2-\frac{m_c{}^2}{2}\right)\left(m_b{}^2+m_c{}^2-\frac{m_a{}^2}{2}\right)\left(m_b{}^2+m_c{}^2-\frac{m_a{}^2}{2}\right)$$
$$=\left(m_c{}^2+m_a{}^2-2m_b{}^2\right)\left(m_a{}^2+m_b{}^2-2m_c{}^2\right)\left(m_b{}^2+m_c{}^2-2m_a{}^2\right) \tag{35}$$

Clearly there dosen't exist such m_a, m_b and m_c which which can satisfies (35) Which concludes that the three diagonals are never concurrent. □

Remark 4.1.
It is clear that from Theorem 4.1 to Theorem 4.4, there exist such a reference triangle which two of circumcenters of 6 subtriangles coincide with each other.

In particular from Theorem 4.2, if the pair of circumcenters (S_2, S_3) or (S_4, S_5) or (S_6, S_1) are coincide with each other, it could happen only when the set of quadruple points $(B, F, G, D), (C, D, G, E), (E, G, F, A)$ are concyclic. From that we have $2b^2 = a^2 + c^2$ or $2c^2 = a^2 + b^2$ or $2a^2 = c^2 + b^2$. There are many interesting properties of such reference triangles. We simply mention that it is similar to its own triangle of medians. Specifically, $m_a = \frac{\sqrt{3}}{2}a, m_b = \frac{\sqrt{3}}{2}b$ and $m_c = \frac{\sqrt{3}}{2}c$.

Theorem 4.6.
If $S_1S_2S_3S_4S_5S_6$ is a Floor van Lamoen's Cyclic Hexagon then

Perimeter of Floor van Lamoen's Cyclic Hexagon:

$$\Psi = \frac{1}{3\Delta}\left[(m_a + m_b + m_c)(m_a m_b + m_b m_c + m_c m_a) - \left({m_a}^3 + {m_b}^3 + {m_c}^3 + 3m_a m_b m_c\right)\right] \tag{36}$$

Area of Floor van Lamoen's Cyclic Hexagon:

$$\left.\begin{aligned} \Lambda &= \frac{36\Delta^2 - \left({m_a}^4 + {m_b}^4 + {m_c}^4\right)}{108\Delta} \\ &= \frac{45\Delta^2 - 2\left({m_a}^2{m_b}^2 + {m_b}^2{m_c}^2 + {m_c}^2{m_a}^2\right)}{108\Delta} \\ &= \frac{64\Delta^2 - \left(a^4 + b^4 + c^4\right)}{192\Delta} \\ &= \frac{40\Delta^2 - \left(a^2b^2 + b^2c^2 + c^2a^2\right)}{96\Delta} \\ &= \frac{81\Delta^2 - 2{V_{hexagon}}^2}{108\Delta} \end{aligned}\right\} \tag{37}$$

Proof. From Theorem 4.1, Theorem 4.2 and further simplification gives conclusion (36).
Now for (37) we proceed as follows:

Let P, Q, R, S, T and U are the foot of perpendiculars drawn from G to the sides of hexagon(Fig. 2). Clearly area of cyclic hexagon $S_1S_2S_3S_4S_5S_6$ is

$$\Lambda = [S_1GS_2] + [S_2GS_3] + [S_3GS_4] + [S_4GS_5] + [S_5GS_6] + [S_6GS_1] \tag{38}$$

Clearjy the points P, Q, R, S, T and U are the foot of perpendiculars drawn from G to the sides of hexagon $S_1S_2S_3S_4S_5S_6$ and they are the mid points of the lines GF, GB, GD, GC, GE and GA respectively. So (38) equivalently rewritten as

$$\Lambda = \frac{1}{2}(S_1S_2.GP + S_2S_3.GQ + S_3S_4.GR + S_4S_5.GS + S_5S_6.GT + S_6S_1.GU) \tag{39}$$

The further simplification of (39) Using (19)-(21), (24)-(26), **(d)**-**(h)** gives the conclusion (37). □

Theorem 4.7.
If $\mathbb{R}$ is the circum radius of Floor van Lamoen's Cyclic Hexagon then $\mathbb{R} = \frac{2m_a m_b m_c}{27\Delta^2}V_{hexagon}$

Proof. Consider the triangle $S_1S_2S_3$. By angle chasing we can prove $\angle S_1S_2S_3 = \angle S_1S_2G + \angle GS_2S_3 = B_2 + F_2$.
Now using Lemma 3.1, Theorem 4.1, Theorem 4.2 and **(f)**, we have

$$\begin{aligned} \sin(B_2 + F_2) &= \sin B_2 \cos F_2 + \cos B_2 \sin F_2 \\ &= \left[\frac{\Delta}{cm_b}\right]\left[\frac{2\left({m_a}^2 - {m_b}^2\right)}{3cm_c}\right] + \left[\frac{3{m_b}^2 + {m_a}^2 - {m_c}^2}{3cm_b}\right]\left[\frac{2\Delta}{cm_c}\right] \\ &= \frac{2\Delta}{3c^2m_cm_b}\left[2{m_a}^2 + 2{m_b}^2 - {m_c}^2\right] \\ &= \frac{3\Delta}{2m_cm_b} \end{aligned} \tag{40}$$

Now by applying the sine rule for the triangle $S_1S_2S_3$, we have

$$\mathbb{R} = \frac{S_1S_3}{2\sin(B_2+F_2)} \tag{41}$$

Now using (29) and (40), (41) can be further simplified so as to prove $\mathbb{R} = \frac{2m_am_bm_c}{27\Delta^2}V_{hexagon}$. □

Theorem 4.8.
If Ψ is the Perimeter of Floor van Lamoen's Cyclic Hexagon then

$$\frac{\Psi}{3} \leqslant S_1S_4 = S_2S_5 = S_3S_6 = \frac{m_am_bm_c}{3\Delta}$$

and equality holds when the reference triangle ABC is a equilateral.

Proof. We know by Schur's inequality, For some non negative real x, y, z and for positive k, we have

$$x^k(x-y)(x-z) + y^k(y-z)(y-x) + z^k(z-x)(z-y) \geqslant 0 \tag{42}$$

and the equality holds when $x = y = z$.
If $k = 1$ then (42) can be written as

$$x^3 + x^3 + x^3 + 6xyz \geqslant (x+y+z)(xy+yz+zx) \tag{43}$$

If we replace x, y, z with m_a, m_b, m_c in (43), we get

$$m_a{}^3 + m_b{}^3 + m_c{}^3 + 6m_am_bm_c \geqslant (m_a+m_b+m_c)(m_am_b+m_bm_c+m_cm_a) \tag{44}$$

And equality holds when $m_a = m_b = m_c$. It implies the triangle ABC is an equilateral.
By combining (44) and (36), we can prove that $m_am_bm_c \geqslant \Psi\Delta$, this implies $\frac{\Psi}{3} \leqslant S_1S_4 = S_2S_5 = S_3S_6 = \frac{m_am_bm_c}{3\Delta}$ □

Theorem 4.9.
If Λ is the Area of Floor van Lamoen's Cyclic Hexagon then $\Lambda \leqslant \frac{\Delta}{4}$ and equality holds when the reference triangle ABC is a equilateral.

Proof. We know by Theorem 4.6 and (37),

$$\Lambda = \frac{81\Delta^2 - 2V_{hexagon}^2}{108\Delta} \tag{45}$$

And using Lemma 3.3, we have

$$V_{hexagon}^2 \geqslant 27\Delta^2 \tag{46}$$

By combining (45) and (46), we get

$$\Lambda = \frac{81\Delta^2 - 2V_{hexagon}^2}{108\Delta} \leqslant \frac{81\Delta^2 - 54\Delta^2}{108\Delta} = \frac{27\Delta^2}{108\Delta} = \frac{\Delta}{4}$$

It implies $\Lambda \leqslant \frac{\Delta}{4}$ and equality holds when the reference triangle ABC is a equilateral. □

Theorem 4.10.
If $\mathbb{R}, \Psi$ and Λ are the circumradius, perimeter and area of Floor van Lamoen's Cyclic Hexagon then

$$m_am_bm_c \geqslant \Delta\Psi \geqslant 4\Lambda\Psi \tag{47}$$

$$\mathbb{R} \geqslant \frac{2\Psi}{3\sqrt{3}} \text{ and } \mathbb{R} \geqslant \frac{8\Lambda\Psi}{3\sqrt{3}\Delta} \tag{48}$$

and equality holds when the reference triangle ABC is a equilateral.

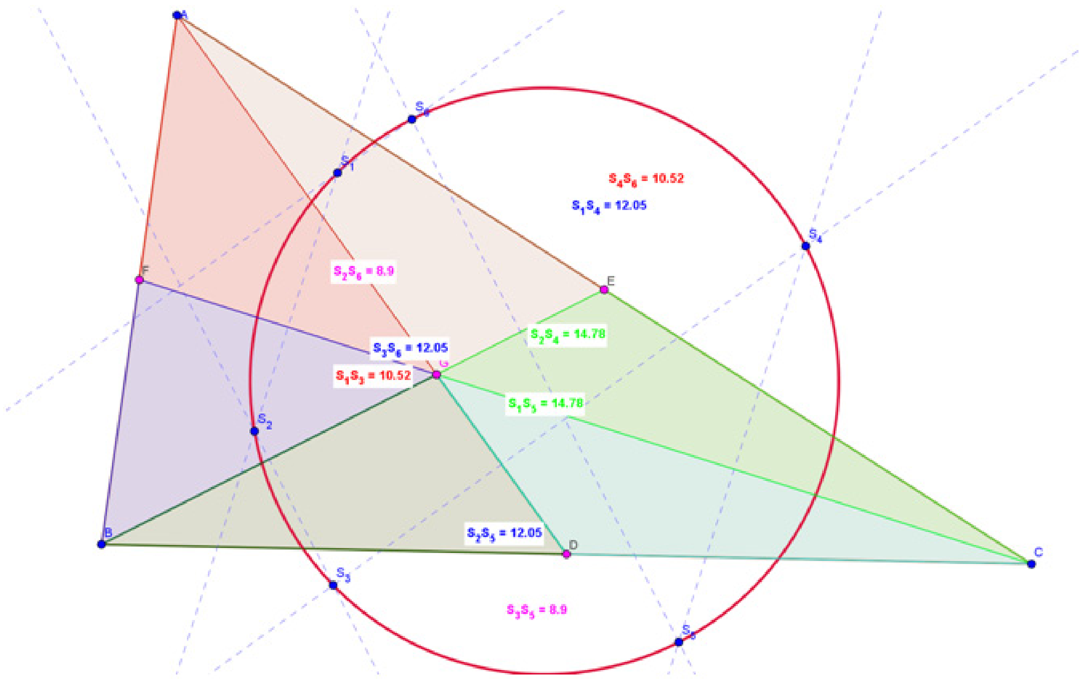

Fig. 3.

Proof. To prove (47), we combine Theorem 4.8 and Theorem 4.9.
For proving (48), we proceed as follows:
From (47), we have $m_a m_b m_c \geqslant \Delta\Psi \geqslant 4\Lambda\Psi$. It implies

$$\frac{2m_a m_b m_c}{27\Delta^2} \geqslant \frac{2\Psi}{27\Delta} \geqslant \frac{8\Lambda\Psi}{27\Delta^2}$$

It further implies

$$\frac{2m_a m_b m_c}{27\Delta^2} V_{hexagon} \geqslant \frac{2\Psi}{27\Delta} V_{hexagon} \geqslant \frac{8\Lambda\Psi}{27\Delta^2} V_{hexagon}$$

Using Theorem 4.6, it can be rewritten as

$$\mathbb{R} \geqslant \frac{2\Psi}{27\Delta} V_{hexagon} \geqslant \frac{8\Lambda\Psi}{27\Delta^2} V_{hexagon}$$

But by Lemma 3.3, we have

$$V_{hexagon}^2 \geqslant 27\Delta^2$$

Hence

$$\mathbb{R} \geqslant \frac{2\Psi}{27\Delta} V_{hexagon} \geqslant \frac{2\Psi}{27\Delta} 3\sqrt{3}\Delta \;\Rightarrow \mathbb{R} \geqslant \frac{2\Psi}{3\sqrt{3}}$$

and

$$\mathbb{R} \geqslant \frac{8\Lambda\Psi}{27\Delta^2} V_{hexagon} \geqslant \frac{8\Lambda\Psi}{27\Delta^2} 3\sqrt{3}\Delta \Rightarrow \mathbb{R} \geqslant \frac{8\Lambda\Psi}{3\sqrt{3}\Delta}$$

Hence proved. □

Remark 4.2.
Clearly Theorem 4.1 to Theorem 4.10 are true if the Floor van Lamoen's Cyclic Hexagon is a convex hexagon and it can be noticed that Theorem 4.1 to Theorem 4.10 are also true even if the Floor van Lamoen's Cyclic Hexagon is a self intersecting hexagon in that order.

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