

# A dynamics of a Prey-predator model with defence mechanism for Prey

Research Article

Bachchu Sk\*, S. Alam

*Department of Mathematics, Indian Institute Engineering Science and Technology, Shibpur, Botanic Garden, Howrah - 711 103, West Bengal, India*

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**Abstract:** In this paper, we dedicate to ourselves to the investigation of the dynamics of the model by establishing the conditions of the nonexistence of periodic orbits, and the existence and uniqueness of limit cycles. The existence, boundedness, uniqueness of solutions of the model are established. We find the impact of herd behavior mechanism of prey population to the model system analytically. Here we also analyze how herd behavior of prey controls the dynamics of the model system near origin in a ecologically meaningful way.

**MSC:** 92D25

**Keywords:** Predator prey model • Limit cycle • Defense mechanism • Square root functional responses • Extinction domain

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## 1. Introduction

The dynamic relationship between predators and the prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to its universal existence and importance. Predator-prey interaction is an important area in ecology and mathematical ecology for which many problems still remain open [1]. Lotka [2]-Volterra [3] model was the first in this context to describe the interaction of species. After that many complex models are developed to study prey-predator systems. In the last few decades a number of prey-predator models have been studied extensively, where several possible dynamics have been considered, starting from the famous work of Lotka and Volterra to the recent works [4–18] which depart from classical assumptions. Several researcher has shown that interaction between prey-predator and corresponding response function plays an important role in the dynamics of prey-predator model. Till now several classical response function namely mass action Holling, type-II, type-III, and type-IV, Tanner ratio dependence [19], Beddington-DeAngelis [20] has been introduced. Cosner et al. [21] have nicely shown how different types of response function can be introduced when prey-predator forms different kind of spatial pattern. In a recent work of Ajraldi et al. [22] took a different view of interaction among the species. They argued the interactions not just of individuals of two populations that intermingle on a common ground, but consider a more elaborated social model, in which the individuals of one population gather together in herds, to wander about in search of food sources and for defensive purposes. The concept of group defense has already been considered, via suitable assumptions on the form and type of functional responses of the prey modeled in very general terms. Specifically, there is a threshold on the size of herd of the prey beyond which the predator's hunting capabilities begin to fall. In other words, the larger the prey population is, the smaller the success of hunting and the corresponding return rate are for predators. A similar reasoning was given by Chottapadhyay et al. [23] in to the formulation of a plankton model in which toxic phytoplankton releases poison through the surface of a three-dimensional patch. More recently, Braza [24] considered a predator-prey model in which the prey exhibits herd behavior, so that the predator interacts with the prey along the outer corridor of the herd of prey. As a mathematical consequence of the herd behavior, they

\* Corresponding author. Tel.: +91 8622 02 2344.

E-mail address(es): [bachchusk143@gmail.com](mailto:bachchusk143@gmail.com) (Bachchu Sk).

considered competition models and predator-prey systems in which interaction terms use the square root of the prey population rather than simply the prey population. Braza shown that the origin to be either locally stable or unstable, depending on the location of the values of the predator and prey populations in the phase plane which is ecologically more sensible. On the other hand, several researcher have formalized different mathematical models taking account of spread of disease in prey population [25–28], but not much intensive attention has been given to the disease in predator which itself can give huge advantage to the prey population. We have analyzed the behaviors of the model system near the origin, and in other equilibrium points. The stability near the origin has been studied from a new angle with the help of non-linear analysis (using appropriate powers of the variable) which allows more realistic results. Finally, we find the impact of herd behavior mechanism of prey population to the model system analytically. We find out the condition of the non-existence of periodic orbits and the existence and uniqueness of limit cycles by a short discussion.

## 2. Mathematical model formation

We consider a prey-predator model where the prey population is gathered in a group so it exhibits a herd behavior. Let  $X$  is prey population density,  $Y$  is susceptible predator population density. We consider that the prey population grows logistically in the absence of predator with intrinsic growth rate  $r$  and  $K$  be the carrying capacity of the prey population. We consider  $m$  be the mortality rate of predator species,  $a$  is search efficiency of predator for prey and  $e$  is the biomass conversion coefficient. Here we also assume that the susceptible predators consumes the prey and in the corresponding response function the square root of the prey population was taken due to herd structure of prey as suggested by Braza by [24]. With these assumptions we can write the model system as follows:

$$\begin{aligned}\frac{dX}{dT} &= rX\left(1 - \frac{X}{K}\right) - a\sqrt{X}Y \\ \frac{dY}{dT} &= ae\sqrt{X}Y - mY\end{aligned}\quad (1)$$

The model (1) takes the following dimensionless form:

$$\begin{aligned}\frac{dx}{dt} &= x(1 - x) - \sqrt{x}y \\ \frac{dy}{dt} &= c\sqrt{x}y - sy\end{aligned}\quad (2)$$

With the re-scaling variables  $x = \frac{X}{K}$ ,  $y = \frac{aY}{r\sqrt{X}}$ ; where  $t = Tr$ ,  $c = \frac{ae}{r}\sqrt{K}$  and  $s = \frac{m}{r}$ .

## 3. Some preliminary results

For model (2) the positive invariant set is the first quadrant  $R_+^2$ , since  $x = 0$  and  $y = 0$  are its solutions. The boundedness of the solutions can be obtained by using the method in [23]. In fact, consider  $z(t) = cx(t) + y(t)$ , we calculate that  $\frac{dz}{dt} + sz = cx(1 + s - x) \leq c\frac{(1+s)^2}{4}$ . Then  $\lim_{t \rightarrow \infty} z(t) = c\frac{(1+s)^2}{4s}$ , which implies the boundedness of the solutions.

The trivial equilibrium point  $E_0(0,0)$  and the boundary equilibrium  $E_1(0,0)$  always exist, and there exists a unique positive equilibrium  $E^*(x^*, y^*)$  if and only if  $x^* < 1$ , where  $x^* = \frac{s^2}{c^2}$  and  $y^* = (1 - x^*)\sqrt{x^*}$ .

It is easy to see that the local stabilities of  $E_1(0,0)$  and  $E^*(x^*, y^*)$  can be determined by analyzing eigen values of the following jacobian matrix:

$$J = \begin{pmatrix} 1 - 2x - \frac{y}{2\sqrt{x}} & \sqrt{x} \\ \frac{cy}{2\sqrt{x}} & -s + c\sqrt{x} \end{pmatrix}$$

However, due to the singularity of the matrix  $J$ , the stability of  $E_0(0,0)$  cannot be determined by by using jacobian matrix  $J$ . Instead, using the idea of Ref. [24], we can obtained the singular dynamics of the model (2) near the origin as follows:

- For any initial value  $(x_0, y_0) \in \{(x, y) \in R_+^2\}$ :  $x \ll 1, y \ll 1$  and  $y = 2\sqrt{x}(s + \frac{1}{2})$ , then the orbit of model (2) goes into the origin equilibrium along the parabola  $y = 2\sqrt{x}(s + \frac{1}{2})$ .
- For any initial value  $(x_0, y_0) \in \{(x, y) \in R_+^2\}$ :  $x \ll 1, y \ll 1$  and  $y > 2\sqrt{x}(s + \frac{1}{2})$ , then the orbit of model (2) terminates at  $x = 0$  and some positive value of  $y$ , after which  $y$  decreases to zero.
- For any initial value  $(x_0, y_0) \in \{(x, y) \in R_+^2\}$ :  $x \ll 1, y \ll 1$  and  $y < 2\sqrt{x}(s + \frac{1}{2})$ , then the orbit of model (2) gives saddle behavior.

(d) For boundary equilibrium point  $E_1(0, 0)$ , the eigen values of Jacobian matrix  $J$  are  $-1$  and  $c-s$ , which implies that  $E_1(0, 0)$  is stable node point when  $s > c$  and a saddle point when  $s < c$ . For positive equilibrium point  $E^*(x^*, y^*)$ ,

$$J(E^*) = \begin{pmatrix} \frac{1-3x^*}{2} & -\sqrt{x^*} \\ \frac{cy^*}{2\sqrt{x^*}} & 0 \end{pmatrix}$$

It is easy to see that the determinant of  $J(E^*)$  is always positive and its trace is  $tr J(E^*) = \frac{1-3x^*}{2}$ . This immediate shows that  $E^*$  is locally asymptotically stable when  $\frac{1}{3} < x^*$  and unstable when  $\frac{1}{3} > x^*$ . Thus  $\frac{1}{3} = x^*$  is the Hopf-bifurcation critical value. That is the model (2) undergoes a Hopf-bifurcation at  $x^* = \frac{1}{3}$ .

#### 4. Nonexistence of periodic orbits

##### Theorem 4.1.

Assume that  $\frac{1}{3} < x^*$ , then model(2) has no nontrivial periodic orbits in  $R_+^2$ .

*Proof.* We first claim that model (2) has no nontrivial periodic orbits in  $R_+^2$  if  $x^* \leq 1$  due to the fact that when  $x^* \geq 1$  model (2) has no positive equilibrium. Therefore we must assume  $x^* < 1$  in the rest of this section. Next we claim that model(2) has no nontrivial periodic orbit in  $R_+^2$  when  $\frac{1}{3} < x^* < 1$ .

If the statement is not true, then there exists a nontrivial periodic orbit  $\Gamma(t) = (x(t), y(t))$  with period  $T > 0$  in  $R_+^2$ . Noticing that

$$\int_0^T \left(1 - x - \frac{y}{\sqrt{x}}\right) dt = \int_0^T \frac{x'(t)}{x(t)} dt = 0,$$

$$\int_0^T (c\sqrt{x} - s) dt = \int_0^T \frac{y'(t)}{y(t)} dt = 0$$

$$\int_0^T (c\sqrt{x} - s) dt = \int_0^T \frac{x'(t)}{x(t)} dt = 0, \text{ and}$$

$$tr(J_\tau) = 1 - 2x - s + c\sqrt{x} - \frac{y}{2\sqrt{x}} = \frac{x'}{2x} + \frac{y'}{y} + tr(J_E^*) + \frac{3(x^*-x)}{2}$$

$$\int_0^T tr(J_\tau) dt = tr(J_E^*)T + \int_0^T \frac{3(x^*-x)}{2} dt$$

Let  $\phi_1(x) = \frac{x^*-x}{\sqrt{x^*-\sqrt{x}}}$  for  $x \neq x^*$ . We can prove that  $\phi_1(x) > 0$  and  $\phi_1'(x) > 0$ . In fact, noting

$$\phi_1'(x) = \frac{-\sqrt{x^*} + \frac{x^*}{2\sqrt{x}} + \frac{\sqrt{x}}{2}}{(\sqrt{x}-\sqrt{x^*})^2},$$

and defining  $\phi_2(x) = -\sqrt{x^*} + \frac{x^*}{2\sqrt{x}} + \frac{\sqrt{x}}{2}$ , we have  $\phi_2'(x) = \frac{x-x^*}{4x^{\frac{3}{2}}}$  which implies  $\phi_2(x) > \min \phi_2(x) = \phi_2(x^*) = 0$  for  $x \neq x^*$ . Then we know the conclusion  $\phi_1'(x) > 0$  is true. It follows from the Green's theorem and  $x - x^* = (\sqrt{x^*} - \sqrt{x})\phi_1(x) = -\frac{y'}{cy}\phi_1(x)$

$$\int_0^T (x - x^*) dt = -\frac{1}{c} \int_0^T \frac{y'}{y} \phi_1(x) dt = -\frac{1}{c} \int \int \frac{\phi_1'(x)}{y(t)} dx dy < 0 \text{ over } \Omega. \text{ Where } \Omega \text{ is the bounded region enclosed by } \Gamma.$$

Then we have from the condition  $\frac{1}{3} < x^* < 1$ ,  $\int_0^T tr(J_\tau) dt < 0$ . Then the divergent criterion [33] implies that all the periodic solutions are orbitally stable, which is contradictory with the stability of  $E^*$ . Therefore, we can conclude that model (2) has no nontrivial periodic orbits in  $R_+^2$  when  $\frac{1}{3} < x^*$ .  $\square$

#### 5. Existence and uniqueness of limit cycles

##### Theorem 5.1.

Assume that  $\frac{1}{3} > x^*$ , then model (2) has precisely one limit cycle and it is stable.

*Proof.* Let  $\varphi(x) = \sqrt{x}$ ,  $h(x) = \frac{(1-x)}{\sqrt{x}}$  and  $f(x) = -s + c\sqrt{x}$  then model (2) can be rewritten as the following Gause type:

$$\begin{cases} \frac{dx}{dt} = \varphi(x)(h(x) - y), \\ \frac{dy}{dt} = f(x)y. \end{cases}$$

Straightforward calculation gives  $h'(x) = \frac{1}{2\sqrt{x}} - \frac{3}{2\sqrt{x}}$  and

$$\frac{h'(x)\varphi(x)}{f(x)} = \frac{\frac{1}{2}\left(\frac{1}{3} - x\right)}{\frac{c}{3}(\sqrt{x} - \sqrt{x^*})}.$$

Let  $\phi_3(x) = \frac{\frac{1}{3} - x}{\sqrt{x} - \sqrt{x^*}}$  for  $x \neq x^*$  then we have

$$\frac{h'(x)\varphi(x)}{f(x)} = \frac{3}{2c}\phi_3(x) \quad (3)$$

and

$$\phi_3'(x) = \frac{\sqrt{x^*} - \frac{1}{6\sqrt{x}} - \frac{\sqrt{x}}{2}}{(\sqrt{x} - \sqrt{x^*})^2} \quad (4)$$

We conclude that  $\phi_4(x) = -\frac{\sqrt{x}}{2} - \frac{1}{6\sqrt{x}} + \sqrt{x^*} < 0$ .

In fact, it follows from  $\phi_4'(x) = \frac{1}{4(x)^{\frac{3}{2}}}\left(\frac{1}{3} - x\right)$  and the condition  $\frac{1}{3} > x^*$  that  $\max \phi_4(x) = \phi_4\left(\frac{1}{3}\right) = \sqrt{x^*} - \frac{1}{3} < 0$ .

Form (3)-(4) we know that  $\frac{d}{dx} \left\{ \frac{\varphi(x)h'(x)}{f(x)} \right\} < 0$  for  $x \neq x^*$  when  $\frac{1}{3} > x^*$ .

According to the criteria for the uniqueness of limit cycles [12, 13], we conclude that the model (2) has at most one limit cycle and it is stable when it exists. It follows from the existence of the Hopf-bifurcation and ) [Theorem 4.1](#) that model (2) has exactly one limit cycle and it is stable when  $\frac{1}{3} > x^*$ . The proof of the theorem is thus completed.  $\square$

## 6. Global dynamics and discussion:

From the preliminary results, [Theorem 4.1](#) and [Theorem 5.1](#) and boundedness of the solutions, we know that the model (2), when  $x^* \geq 1$  has no positive equilibrium point and boundary equilibrium point is a local attractor; when  $\frac{1}{3} < x^* < 1$  the unique positive equilibrium point is a local attractor and boundary equilibrium point is unstable. Finally when  $\frac{1}{3} > x^*$  the limit cycle is a local attractor and the positive equilibrium and boundary equilibrium are unstable. Thus we can conclude that the positive invariant set of model (2) is divided into two parts which can be defined as  $y = \frac{2s+1}{\sqrt{x}}$  near the origin equilibrium, with one part as the extinction domain in which orbits terminate at positive vertical axis and then decrease to zero along the vertical axis, and the other part as the attraction domain of the corresponding attractor (boundary equilibrium point, positive equilibrium point or limit cycle) in which orbits converge to the attractor. In the present paper, a prey-predator model with defense mechanism for prey species is considered. For this model, we first analyzed the singular dynamics near the origin equilibrium, the local stability of nontrivial equilibrium point and the existence of Hopf-bifurcation. Then we proved the nonexistence of periodic orbits when the positive equilibrium does not exist or is locally asymptotically stable, and the existence and uniqueness of limit cycles when the positive equilibrium point is unstable. Based on these result we obtained the global dynamics of the model. One can see that the dynamics of the predator prey model with defense mechanism is more plentiful than the traditional models and makes much more sense ecologically. The standard models usually predict the origin equilibrium point is a saddle point, i.e. the prey species will recover no matter how small it is relative to the predator species. In contrast, for model (2) solution behavior near the origin equilibrium is singular. If the prey population is suitably smaller than the predator population, then the prey species first goes extinct, causing the predator species to follow suit. This makes perfect ecological sense. The result in the present paper may help us to better understanding the interaction of predator with prey in a real ecosystem, and may explain the field observation in some areas.

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