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Some sum formulas for products of Pell and Pell-Lucas numbers

Research Article

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Abstract: Let us define $\mathscr{P}$ and $\mathscr{Q}$ to be complex Pell and complex Pell-Lucas numbers. We have determined sum formulas for squares of terms of these complex number sequences. We found some sum formulas for certain products of terms of the Pell and Pell-Lucas sequences.

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1. Introduction

Studies show that there has been an increasing interest in number sequences and their generalizations, such as Pell and Pell-Lucas number sequences. These number sequences and generalizations have very important properties and applications in almost every fields of science and art. Applications of these number sequences provide a wide area for researchers. The features of these number sequences can be seen in [1–5].

The purpose of this paper is to derive some sum formulas for certain products of the terms of these number sequences. We also describe the complex Pell and complex Pell-Lucas numbers, like the complex Fibonacci and complex Lucas numbers [4]. We give formulae of sums for squares of the terms of these complex number sequences.

2. Preliminaries

The complex Pell and complex Pell-Lucas sequences $\mathscr{P}$ and $\mathscr{Q}$ are defined by recurrence relations

$$\mathscr{P}_0 = i, \mathscr{P}_1 = 2, \mathscr{P}_n = 2\mathbf{P}_n + i\mathbf{P}_{n-1} \quad for \quad n \geq 2$$

$$\mathscr{Q}_0 = 4-2i, \mathscr{Q}_1 = 4+2i, \mathscr{Q}_n = 2\mathbf{Q}_n + i\mathbf{Q}_{n-1} \quad for \quad n \geq 2$$

and $i = \sqrt{-1}$. If start from n=0, then the complex Pell and complex Pell-Lucas sequence are given through

n	0	1	2	3	4	5	$\cdots$
P_n	0	1	2	5	12	29	$\cdots$
Q_n	2	2	6	14	34	82	$\cdots$
$\mathscr{P}_n$	i	2	4+i	10+2i	24+5i	58+12i	$\cdots$
$\mathscr{Q}_n$	4-2i	4+2i	12+2i	28+6i	68+14i	164+34i	$\cdots$

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The following sum formulas the Pell and Pell-Lucas numbers are well known [4, 5]:

$$\sum_{k=1}^{n-1} P_k^2 = \frac{P_n P_{n-1}}{2}$$
$$\sum_{k=1}^{n-1} Q_k^2 = \frac{Q_{2n-1} - 2(-1)^n - 4}{2}$$
$$\sum_{k=1}^{n-1} P_k P_{k+1} = \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}$$
$$\sum_{k=1}^{n-1} Q_k Q_{k+1} = \frac{Q_{2n} - 2(-1)^n - 8}{2}$$
$$\sum_{k=0}^{n-1} P_{2k+1} = \frac{P_{2n}}{2}$$
$$\sum_{k=0}^{n-1} Q_{2k+1} = \frac{Q_{2n} - 2}{2}$$
$$\sum_{k=0}^{n-1} Q_{2k} = \frac{Q_{2n-1} + 2}{2}$$

3. Some sum formulas for Pell and Pell-Lucas sequences

Theorem 3.1.
If P_n and Q_n are nth Pell and Pell-Lucas numbers, we have

$$\sum_{k=1}^{n} kP_k P_{k-1} = \frac{(2n+1)(P_{2n+1} - 2P_{n+1}P_n - 1) - P_{2n} + 2n}{8}$$
$$\sum_{k=1}^{n} kQ_k Q_{k-1} = \frac{2nQ_{2n} - Q_{2n-1} + [4n+2](-1)^{n+1}}{4}$$

Proof. Let $A_n = \sum_{k=1}^{n} P_k P_{k-1} = \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}$, then

$$\begin{aligned}
\sum_{k=1}^{n} kP_k P_{k-1} &= P_1P_0 + 2P_2P_1 + 3P_3P_2 + \ldots + nP_nP_{n-1} \\
&= \sum_{k=1}^{n} P_kP_{k-1} + \sum_{k=2}^{n} P_kP_{k-1} + \sum_{k=3}^{n} P_kP_{k-1} + \ldots + \sum_{k=n}^{n} P_kP_{k-1} \\
&= A_n + (A_n - A_1) + (A_n - A_2) + \ldots + (A_n - A_{n-1}) \\
&= nA_n - \sum_{i=1}^{n-1} A_i = n\left(\frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}\right) - \sum_{i=1}^{n-1}\left(\frac{P_{2i+1} - 2P_{i+1}P_i - 1}{4}\right) \\
&= n\left(\frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}\right) - \frac{P_{2n} - 2}{8} + \frac{P_{2n+1} - 2P_{n+1}P_n - 1}{8} + \frac{2n-2}{8} \\
&= \frac{(2n+1)(P_{2n+1} - 2P_{n+1}P_n - 1) - P_{2n} + 2n}{8}
\end{aligned}$$

So, the proof is completed. Accordingly, the value of $\sum_{k=1}^{n} kQ_kQ_{k-1}$ is computed. □

Theorem 3.2.
If $\mathscr{P}_n$ and $\mathscr{Q}_n$ are nth complex Pell and complex Pell-Lucas numbers, we have

$$\sum_{k=1}^{n} \mathscr{P}_k^2 = \left(\frac{4P_{n+1}P_n - P_nP_{n-1}}{2}\right) + i\,(P_{2n+1} - 2P_{n+1}P_n - 1)$$
$$\sum_{k=1}^{n} \mathscr{Q}_k^2 = \left(\frac{4Q_{2n+1} - Q_{2n-1} - 20 + 10(-1)^n}{2}\right) + i\left(2Q_{2n} - 4(-1)^n\right)$$

Proof.

$$\begin{aligned}
\sum_{k=1}^{n} \mathscr{P}_k^2 &= \sum_{k=1}^{n} (2P_k + iP_{k-1})^2 \\
&= 4\sum_{k=1}^{n} P_k^2 + i^2 \sum_{k=1}^{n} P_{k-1}^2 + 4i \sum_{k=1}^{n} P_k P_{k-1} \\
&= 4\left(\frac{P_{n+1}P_n}{2}\right) - \left(\frac{P_n P_{n-1}}{2}\right) + 4i\left(\frac{P_{2n+1} - 2P_{n+1}P_n - 1}{4}\right) \\
&= \left(\frac{4P_{n+1}P_n - P_n P_{n-1}}{2}\right) + i\left(P_{2n+1} - 2P_{n+1}P_n - 1\right)
\end{aligned}$$

So, the proof is completed. Accordingly, the value of $\sum_{k=1}^{n} \mathscr{Q}_k^2$ is computed. □

Theorem 3.3.
If $\mathscr{P}_n$ and $\mathscr{Q}_n$ are nth complex Pell and complex Pell-Lucas numbers, we have

$$\sum_{k=1}^{n} k\mathscr{P}_k^2 = \left[\frac{4n(4P_{n+1}P_n - P_nP_{n-1}) - 4P_{2n+1} + 8P_{n+1}P_n + P_{2n-1} - 2P_nP_{n-1} + 3}{8}\right] + i\left[\frac{2n(P_{2n+1} - 2P_{n+1}P_n) + P_{2n+1} - P_{2n} - 2P_{n+1}P_n - 1}{2}\right]$$

$$\sum_{k=1}^{n} k\mathscr{Q}_k^2 = \begin{cases} \left[\frac{2n(4Q_{2n+1}-Q_{2n-1})-4Q_{2n}+Q_{2n-2}-20n-18}{4}\right] + i\left[2nQ_{2n} - Q_{2n-1} + 4n + 2\right], & \text{if } n \text{ is odd} \\ \left[\frac{2n(4Q_{2n+1}-Q_{2n-1})-4Q_{2n}+Q_{2n-2}+20n+2}{4}\right] + i\left[2nQ_{2n} - Q_{2n-1} - 4n - 2\right], & \text{otherwise} \end{cases}$$

Proof. Let $B_n = \sum_{k=1}^{n} \mathscr{P}_k^2 = \left(\frac{4P_{n+1}P_n - P_nP_{n-1}}{2}\right) + i\left(P_{2n+1} - 2P_{n+1}P_n - 1\right)$,then

$$\begin{aligned}
\sum_{k=1}^{n} k\mathscr{P}_k^2 &= \mathscr{P}_1^2 + 2\mathscr{P}_2^2 + 3\mathscr{P}_3^2 + \ldots + n\mathscr{P}_n^2 \\
&= \sum_{k=1}^{n} \mathscr{P}_k^2 + \sum_{k=2}^{n} \mathscr{P}_k^2 + \sum_{k=3}^{n} \mathscr{P}_k^2 + \ldots + \sum_{k=n}^{n} \mathscr{P}_k^2 \\
&= B_n + (B_n - B_1) + (B_n - B_2) + \ldots + (B_n - B_{n-1}) \\
&= nB_n - \sum_{j=1}^{n-1} B_j = n\left[\left(2P_{n+1}P_n - \frac{1}{2}P_nP_{n-1}\right) + i\left(P_{2n+1} - 2P_{n+1}P_n - 1\right)\right] \\
&\qquad - \sum_{j=1}^{n-1}\left[\left(2P_{j+1}P_j - \frac{1}{2}P_jP_{j-1}\right) + i\left(P_{2j+1} - 2P_{j+1}P_j - 1\right)\right] \\
&= \left[\frac{4n(4P_{n+1}P_n - P_nP_{n-1}) - 4P_{2n+1} + 8P_{n+1}P_n + P_{2n-1} - 2P_nP_{n-1} + 3}{8}\right] \\
&\qquad + i\left[\frac{2n(P_{2n+1} - 2P_{n+1}P_n) + P_{2n+1} - P_{2n} - 2P_{n+1}P_n - 1}{2}\right]
\end{aligned}$$

So, the proof is completed. Accordingly, the value of $\sum_{k=1}^{n} k\mathscr{Q}_k^2$ is computed. □

Corollary 3.1.
$\mathscr{P}_n$ and $\mathscr{Q}_n$ are nth complex Pell and complex Pell-Lucas numbers, we have formulas for $\sum_{k=1}^{n} k\mathscr{P}_k^2$ and $\sum_{k=1}^{n} k\mathscr{Q}_k^2$.

We can derive a formula for $\sum_{k=1}^{n}(n+1-k)\mathscr{P}_k^2$ *and* $\sum_{k=1}^{n}(n+1-k)\mathscr{Q}_k^2$.

$$\begin{aligned}
\sum_{k=1}^{n}(n+1-k)\mathscr{P}_k^2 &= (n+1)\sum_{k=1}^{n}\mathscr{P}_k^2-\sum_{k=1}^{n}k\mathscr{P}_k^2\\
&= (n+1)\left[\left(\frac{4P_{n+1}P_n-P_nP_{n-1}}{2}\right)+i\left(P_{2n+1}-2P_{n+1}P_n-1\right)\right]\\
&\qquad -\left[\frac{4n\left(4P_{n+1}P_n-P_nP_{n-1}\right)-4P_{2n+1}+8P_{n+1}P_n+P_{2n-1}-2P_nP_{n-1}+3}{8}\right]\\
&\qquad -i\left[\frac{2n\left(P_{2n+1}-2P_{n+1}P_n\right)+P_{2n+1}-P_{2n}-2P_{n+1}P_n-1}{2}\right]\\
&= \left(\frac{8P_{n+1}P_n-2P_nP_{n-1}+4P_{2n+1}-P_{2n-1}-3}{8}\right)+i\left(\frac{P_{2n+1}-2P_{n+1}P_n+P_{2n}-2n-1}{2}\right)
\end{aligned}$$

Using the same technique, we can be show that

$$\sum_{k=1}^{n}(n+1-k)\mathscr{Q}_k^2=\begin{cases}\left[\frac{8Q_{2n+1}+4Q_{2n}-2Q_{2n-1}-Q_{2n-2}-40n-42}{4}\right]+i\left[2Q_{2n}+Q_{2n-1}+2\right], & \text{if } n \text{ is odd}\\ \left[\frac{8Q_{2n+1}+4Q_{2n}-2Q_{2n-1}-Q_{2n-2}-40n-22}{4}\right]+i\left[2Q_{2n}+Q_{2n-1}-2\right], & \text{otherwise}\end{cases}$$

References

[1] R. Melham, Sums Involving Fibonacci and Pell Numbers, Portugalie Math. 56(3) (1999) 309 - 317.
[2] A. F. Horadam, Pell Identities, The Fibonacci Quart. 9(3) (1971) 245 - 252.
[3] S. Vajda, Fibonacci and Lucas Numbers and the Golden Section: Theory and Applications, Ellis Horwood Ltd., England, 1989.
[4] T. Koshy, Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, New York, 2001.
[5] T. Koshy, Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.