Int. J. Adv. Appl. Math. and Mech. 4(4) (2017) 15 – 19 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

# Heat conduction and the multivariable *I*-function

**Research Article**

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**Received 09 March 2017; accepted (in revised version) 06 April 2017**

**Abstract:** Chaurasia et al. [3] have studied the Fox's $H$-function and the multivariable $H$-function defined by Srivastava et al. [9, 10] and has obtained a solution to a problem of heat conduction. The object of this paper is to employ the $I$-function defined by Rathie [7] and the multivariable $I$-function defined by Prathima et al. [6] in obtaining a solution of the partial differential equation related to a problem of heat conduction. We shall see the particular cases concerning the $I$-function of two variables defined by Rathie et al. [8] and the multivariable $H$-function.

**MSC:** 33C45 • 33C60 • 26D20

**Keywords:** Multivariable $I$-function • $\bar{I}$-function • Heat conduction • Partial differential equation • $I$-function of two variables • Multivariable $H$-function



## 1. Introduction and preliminaries

As an example of the application of the multivariable $I$-function defined by Prathima et al. [6] in applied mathematics, we shall consider the problem of obtaining a solution of a problem of heat conduction. Consider the partial differential equation:

$$\frac{\partial\phi}{\partial t}=\xi\frac{\partial^2\phi}{\partial t^2}-\xi\phi x^2 \tag{1}$$

where $\phi(x,t)\to 0$ for large of $t$ and when $x\to\infty$ ; this equation is related to the problem of heat conduction given by Churchill [4].

$$\frac{\partial\phi}{\partial t}=\xi\frac{\partial^2\phi}{\partial t^2}-\eta\left(\phi-\phi_0\right) \tag{2}$$

provided that $\phi_0=0$ and $\eta=\xi x^2$.

In this paper we shall assume that,

$$f(x)=x^{2\sigma}e^{-x^2}\left(yx^{2h}\ \middle|\ \begin{matrix}(a_j,\alpha_i;A'_j)_{n,n+1},(a_j,\alpha_i;A'_j)_p\\(b_j,\beta_j;B'_j)_{m,m+1},(b_j,\beta_j;B'_j)_q\end{matrix}\right)I\begin{pmatrix}z_1x^{2k_1}\\ \vdots\\ z_rx^{2k_r}\end{pmatrix} \tag{3}$$

The $\bar{I}$-function, introduced by Rathie [7], however the notation and complete definition is presented here in the following manner in terms of the Mellin-Barnes type integral:

$$\bar{I}(z)=\bar{I}^{m,n}_{p,q}\left(z\ \middle|\ \begin{matrix}(a_j,\alpha_i;A'_j)_{n,n+1},(a_j,\alpha_i;A'_j)_p\\(b_j,\beta_j;B'_j)_{m,m+1},(b_j,\beta_j;B'_j)_q\end{matrix}\right)=\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p,q}(s)z^s ds \tag{4}$$

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for all $z$ different to 0 and

$$\Omega_{p,q}^{m,n}(s)=\frac{\prod_{j=1}^{m}\Gamma^{B'_i}(b_j-\beta_j s)\prod_{j=1}^{n}\Gamma^{A'_j}(1-a_j+\alpha_j s)}{\prod_{j=n+1}^{p}\Gamma^{A'_i}(a_j-\alpha_j s)\prod_{j=m+1}^{q}\Gamma^{B'_j}(1-b_j+\beta_j s)} \tag{5}$$

The integral (3) converges when $|argz|<\frac{1}{2}\Delta$ ,if $\Delta>0$ where

$$\Delta=\sum_{j=1}^{m}B'_j\beta_j-\sum_{j=m+1}^{q}B'_j\beta_j+\sum_{j=1}^{n}A'_j\alpha_j-\sum_{j=n+1}^{p}A'_j\alpha_j \tag{6}$$

When the poles of $\Gamma(b_j-\beta_j s), j=1,\cdots,m$, are simples the integral (8) can be evaluate with the help of the Residue Theorem. We obtain

$$\bar{I}(z)=\sum_{G=1}^{m}\sum_{g=0}^{\infty}\frac{(-)^g\Omega_{p,q}^{m,n}(s)}{B'_G g!}z^s \tag{7}$$

with $s=\eta_{G,g}=\frac{b_G+g}{B'_G}, p<q, |z|<1$ and $\Omega_{p,q}^{m,n}(s)$ is given in (8). For more detail, see Rathie [7].

The multivariable $I$-function is defined in term of multiple Mellin-Barnes type integral:

$$I(z_1,\cdots,z_r)=I_{\mathbf{p},\mathbf{q}:p_1,q_1;\cdots;p_r,q_r}^{0,\mathbf{n}:m_1,n_1;\cdots;m_r,n_r}\left(\begin{matrix}z_1\\ \vdots\\ z_r\end{matrix}\left|\begin{matrix}(a_j;\alpha_j^{(1)},\cdots,\alpha_j^{(r)};A_j)_{1,p}:(c_j^{(1)},\gamma_j^{(1)};C_j^{(1)})_{1,p_1};\cdots;(c_j^{(r)},\gamma_j^{(r)};C_j^{(r)})_{1,p_r}\\ (b_j;\beta_j^{(1)},\cdots,\beta_j^{(r)};B_j)_{1,q}:(d_j^{(1)},\delta_j^{(1)};D_j^{(1)})_{1,q_1};\cdots;(d_j^{(r)},\delta_j^{(r)};D_j^{(r)})_{1,q_r}\end{matrix}\right.\right) \tag{8}$$

$$=\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi(s_1,\cdots,s_r)\prod_{i=1}^{r}\theta_i(s_i)z_i^{s_i}ds_1\cdots ds_r \tag{9}$$

where $f(s_1,L,s_r),\theta_i(s_i), i=1,...,r$ are given by,

$$\phi(s_1,\cdots,s_r)=\frac{\prod_{j=1}^{n}\Gamma^{A_j}\left(1-aj+\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)}{\prod_{j=n+1}^{p}\Gamma^{A_j}\left(a_j-\sum_{i=1}^{r}\alpha_j^{(i)}s_j\right)\prod_{j=1}^{q}\Gamma^{B_j}\left(1-bj+\sum_{i=1}^{r}\beta_j^{(i)}s_j\right)} \tag{10}$$

$$\theta_i(s_i)=\frac{\prod_{j=1}^{n_i}\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}+\gamma_j^{(i)}s_i\right)\prod_{j=1}^{m_i}\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right)}{\prod_{j=n_i+1}^{p_i}\Gamma^{C_j^{(i)}}\left(c_j^{(i)}-\gamma_j^{(i)}s_i\right)\prod_{j=m_i+1}^{q_i}\Gamma^{D_j^{(i)}}\left(1-d_j^{(i)}+\delta_j^{(i)}s_i\right)} \tag{11}$$

For more details, see Prathima et al. [6].

Following the result of Braaksma [2] the $I$-function of $r$ variables is analytic if,

$$U_i=\sum_{j=1}^{p}A_j\alpha_j^{(i)}-\sum_{j=1}^{q}B_j\beta_j^{(i)}+\sum_{j=1}^{p_i}C_j^{(i)}\gamma_j^{(i)}-\sum_{j=1}^{q_i}D_j^{(i)}\delta_j^{(i)}\le 0,\quad i=1,\cdots,r \tag{12}$$

The integral (18) converges absolutely if

$$\left|\arg(z_k)\right|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,r$$

where

$$\Delta_k=-\sum_{j=n+1}^{p}A_j\alpha_j^{(k)}-\sum_{j=1}^{q}B_j\beta_j^{(k)}+\sum_{j=1}^{m_k}D_j^{(k)}\delta_j^{(k)}-\sum_{j=m_k+1}^{q_k}D_j^{(k)}\delta_j^{(k)}+\sum_{j=1}^{n_k}C_j^{(k)}\gamma_j^{(k)}-\sum_{j=n_k+1}^{p_k}C_j^{(k)}\gamma_j^{(k)}>0 \tag{13}$$

The parameters $m_j,n_j,p_j,q_j (j=1,\cdots,r), n,p,q$ are non negative integers (for more details, see Prathima et al. [7]) $\alpha_j^{(i)}(j=1,\cdots,p;i=1,\cdots,r), \beta_j^{(i)}(j=1,\cdots,q;i=1,\cdots,r), \gamma_j^{(i)}(j=1,\cdots,p_i;i=1,\cdots,r)$ and $\delta_j^{(i)}, (j=1,\cdots,q_i;i=1,\cdots,r)$

are assumed to be positive quantities for standardization purpose. $a_j(j=1,\cdots,p), b_j(j=1,\cdots,q), c_j^{(i)}(j=1,\cdots,p_i, i=1,\cdots,r), d_j^{(i)}(j=1,\cdots,q_i, i=1,\cdots,r)$ are complex numbers.

The exposants $A_j(j=1,\cdots,p), B_j(j=1,\cdots,q), C_j^{(i)}(j=1,\cdots,p_i; i=1,\cdots,r), D_j^{(i)}(j=1,\cdots,q_i; i=1,\cdots,r)$ of various gamma function involved in (10) and (11) may take non integer values.

The contour $L_i$ in the complex $s_i$ -plane is of Mellin Barnes type which runs from $c-i\infty$ to $c+i\infty$ ($c$ real) with indentation, if necessary, in such a manner that all singularities of $\Gamma^{D_j^{(i)}}\left(d_j^{(i)}-\delta_j^{(i)}s_i\right), j=1,\cdots,m_i$ to the right and $\Gamma^{C_j^{(i)}}\left(1-c_j^{(i)}-\gamma_j^{(i)}s_i\right), j=1,\cdots,n_i$ are to the left of $L_i$.

We shall note,

$$X = m_1, n_1; \cdots; m_r, n_r; \quad Y = p_1, q_1; \cdots; p_r, q_r \tag{14}$$

$$A = (a_j; \alpha_j^{(1)}, \cdots, \alpha_j^{(r)}; A_j)_{1,p}; \quad (c_j^{(1)}, \gamma_j^{(1)}; C_j^{(1)})_{1,p_1}; \cdots; (c_j^{(r)}, \gamma_j^{(r)}; C_j^{(r)})_{1,p_r} \tag{15}$$

$$B = (b_j; \beta_j^{(1)}, \cdots, \beta_j^{(r)}; B_j)_{1,q} : (d_j^{(1)}, \delta_j^{(1)}; D_j^{(1)})_{1,q_1}; \cdots; (d_j^{(r)}, \delta_j^{(r)}; D_j^{(r)})_{1,q_r} \tag{16}$$

The multivariable $I$-function write:

$$I(z_1, \cdots, z_r) = \left( \begin{array}{c|c} z_1 & A \\ \vdots & \\ x_r & B \end{array} \right) \tag{17}$$

## 2. Main integral

**Lemma 2.1.**

$$\int_{-\infty}^{\infty} x^{2\sigma} e^{-x^2} H_{2v}(x) dx = \frac{\sqrt{\pi} 4^{v-\sigma} \Gamma(2\sigma+1)}{\Gamma(\sigma - v + 1)} \tag{18}$$

We have the following integral.

**Theorem 2.1.**

$$\int_{-\infty}^{\infty} x^{2\sigma} e^{x^2} H_{2v}(x) \bar{I}_{p,q}^{m,n}\left( yx^{2h} \left| \begin{array}{c} (a_j,\alpha_i;A'_j)_{n,n+1}, (a_j,\alpha_i;A'_j)_p \\ (b_j,\beta_j;B'_j)_{m,m+1}, (b_j,\beta_j;B'_j)_q \end{array} \right. \right) \left( \begin{array}{c} z_1 x^{2k_1} \\ \vdots \\ z_r x^{2k_r} \end{array} \right) dx$$

$$= \sqrt{\pi} 4^{v-\sigma} \sum_{G=1}^{m} \sum_{g=0}^{\infty} \frac{4^{h\eta_{G,g}} (-)^g \Omega_{p,q}^{m,n}(s)}{B_G g!} y^{\eta_{G,g}} I_{p+1,q+1;Y}^{0,n+1;X} \left( \begin{array}{c|c} z_1 4^{-k_1} & (-2\sigma - 2h_{G,g}; 2k_1, \cdots, 2k_r; 1), A \\ \vdots & \cdots \\ z_r 4^{-k_r} & (-v-\sigma - h\eta_{G,g}; k_1, \cdots, k_r; 1), B \end{array} \right) \tag{19}$$

$\Omega_{p,q}^{m,n}(s)$ is defined by (4), provided that

$$\min\{h, h, \sigma, k_i\} > 0, i = 1, \cdots, r; Re\left[1 + h\frac{b_j}{\beta_j} + \sum_{i=1}^{r} k_i \min_{1 \le j \le m_i} \frac{d_j^{(i)}}{\delta_j^{(i)}}\right] > 0$$

$\left|arg(z_k)\right| < \frac{1}{2}\Delta_k \pi, k = 1, \cdots, r$ $\Delta_k$ is defined by (12).

$|arg z| < \frac{1}{2}\Delta, \text{if} \Delta > 0$ where $\Delta = \sum_{j=1}^{m} B'_j \beta_j - \sum_{j=m+1}^{q} B'_j \beta_j + \sum_{j=1}^{n} A'_j \alpha_j - \sum_{j=n+1}^{p} A'_j \alpha_j$

*Proof.* To prove (18), first expressing the $I$-function in serie with the help of (7), and we interchange the order of summations and x-integral (which is permissible under the conditions stated). Expressing the $I$-function of r variables defined by Prathima et al. [6] in Mellin-contour integral and interchange the order of integrations which is justifiable due to absolute convergence of the integral involved in the process. Now evaluating the inner x-integral with the help of Lemma. Interpreting the Mellin-Barnes contour integral in multivariable $I$-function, we obtain the desired result (19). □

## 3. Solution

The solution of (1) to be establish is,

$$\phi(x,t)=\sum_{\alpha=0}^{\infty}\sum_{G=1}^{m}\sum_{g=0}^{\infty}\frac{2^{\alpha-2\sigma-2h\eta_{G,g}-\frac{1}{2}}(-)^{g}\Omega_{p,q}^{m,n}(s)}{\alpha!B_G g!}y^{\eta_{G,g}}e^{(1+2\alpha)\xi t-\frac{x^2}{2}}$$

$$I_{p+1,q+1;Y}^{0,n+1;X}\left(\begin{array}{c}z_1 4^{-k_1}\\ \vdots\\ z_r 4^{-k_r}\end{array}\middle|\begin{array}{l}\left(-2\sigma-2h_{G,g};2k_1,\cdots,2k_r;1\right),A\\ \cdots\\ \left(\frac{\alpha}{2}\$-\sigma-h\eta_{G,g};k_1,\cdots,k_r;1\right),B\end{array}\right) \tag{20}$$

Provided that,

$$\min\{h,h,\sigma,k_i\}>0, i=1,\cdots,r;\quad Re\left[1+h\frac{b_j}{\beta_j}+\sum_{i=1}^{r}k_i\min_{1\leq j\leq m_i}\frac{d_j^{(i)}}{\delta_j^{(i)}}\right]>0$$

$$\left|\arg(z_k)\right|<\frac{1}{2}\Delta_k\pi, k=1,\cdots,r,\quad \Delta_k\}\text{ is defined by (12).}$$

$$|\arg z|<\frac{1}{2}\Delta,\text{if}\Delta>0\text{where }\Delta=\sum_{j=1}^{m}B'{}_j\beta_j-\sum_{j=m+1}^{q}B'{}_j\beta_j+\sum_{j=1}^{n}A'{}_j\alpha_j-\sum_{j=n+1}^{p}A'{}_j\alpha_j$$

*Proof.* The solution of (1) can be written as [[1], page 360, Eq. 2.3]

$$\phi(x,t)=\sum_{\alpha=0}^{\infty}A_\alpha e^{(1+2\alpha)\xi t-\frac{x^2}{2}}H_\alpha(x) \tag{21}$$

where $H_\alpha(x)$ is the Hermite polynomial. If $t=0$, then by vertue of (3), we have

$$x^{2\sigma}e^{x^2}H_{2v}(x)\bar{I}_{p,q}^{m,n}\left(yx^{2h}\middle|\begin{array}{l}(a_j,\alpha_i;A'{}_j)_{n,n+1},(a_j,\alpha_i;A'{}_j)_p\\(b_j,\beta_j;B'{}_j)_{m,m+1},(b_j,\beta_j;B'{}_j)_q\end{array}\right)I\left(\begin{array}{c}z_1x^{2k_1}\\ \vdots\\ z_rx^{2k_r}\end{array}\right)=\sum_{\alpha=0}^{\infty}A_\alpha e^{-\frac{x^2}{2}}H_\alpha(x) \tag{22}$$

Multiplying both sides of (22) by $H_\beta(x)$ and integrating from $-\infty$ to $\infty$ with respect to $x$, and using (19) and the orthogonality property of Hermite polynomials [5], we find.

$$A_\beta=\sum_{G=1}^{m}\sum_{g=0}^{\infty}\frac{2^{\alpha-2\sigma-2h\eta_{G,g}-\frac{1}{2}}(-)^{g}\Omega_{p,q}^{m,n}(s)}{\alpha!B_G g!}y^{\eta_{G,g}}e^{(1+2\alpha)\xi t-\frac{x^2}{2}}I_{p+1,q+1;Y}^{0,n+1;X}\left(\begin{array}{c}z_1 4^{-k_1}\\ \vdots\\ z_r 4^{-k_r}\end{array}\middle|\begin{array}{c}(-2\sigma-2h\eta_{G,g};2k_1,\cdots,2k_r;1),A\\ \cdots\\ \cdots\\ \frac{\alpha}{2}-\sigma-h\eta_{G,g};k_1,\cdots,k_r;1),B\end{array}\right) \tag{23}$$

With the help of (21) and (23), the solution (20) is established. □

## 4. Particular case

a) If $r=0$ the multivariable $I$-function reduces to $I$-function of two variables defined by Rathie et al. [8].

$$\phi(x,t)=\sum_{\alpha=0}^{\infty}\sum_{G=1}^{m}\sum_{g=0}^{\infty}\frac{2^{\alpha-2\sigma-2h\eta_{G,g}-\frac{1}{2}}(-)^{g}\Omega_{p,q}^{m,n}(s)}{\alpha!B_G g!}y^{\eta_{G,g}}e^{(1+2\alpha)\xi t-\frac{x^2}{2}}$$

$$I_{p+1,q+1;Y}^{0,n+1;X}\left(\begin{array}{c}z_1 4^{-k_1}\\ \vdots\\ z_2 4^{-k_2}\end{array}\middle|\begin{array}{c}(-2\sigma-2h\eta_{G,g};2k_1,2k_2;1),A\\ \cdots\\ \cdots\\ \frac{\alpha}{2}-\sigma-h\eta_{G,g};k_1,k_2;1),B\end{array}\right)H_\alpha(x) \tag{24}$$

under the same conditions and notations that (20) with $r=2$.

b) The multivariable $I$-function reduces to multivariable $H$-function defined by Srivastava et al. [9, 10] and we obtain.

$$\phi(x,t)=\sum_{\alpha=0}^{\infty}\sum_{G=1}^{m}\sum_{g=0}^{\infty}\frac{2^{\alpha-2\sigma-2h\eta_{G,g}-\frac{1}{2}}(-)^{g}\Omega_{p,q}^{m,n}(s)}{\alpha!B_G g!}y^{\eta_{G,g}}e^{(1+2\alpha)\xi t-\frac{x^2}{2}}$$

$$H_{p+1,q+1;Y}^{0,n+1;X}\left(\begin{array}{c}z_1 4^{-k_1}\\ \vdots\\ z_r 4^{-k_r}\end{array}\middle|\begin{array}{c}(-2\sigma-2h\eta_{G,g};2k_1,\cdots,2k_r),A\\ \cdots\\ \cdots\\ \frac{\alpha}{2}-\sigma-h\eta_{G,g};k_1,\cdots,k_r),B\end{array}\right)H_\alpha(x) \tag{25}$$

under the same conditions and notations that (20) with $A_j=B_j=C_j^{(i)}=D_j^{(i)}=1$.

## 5. Conclusion

Specializing the parameters of the $I$-function and the multivariable $I$-function, we can obtain a large number of results involving various special functions of one and several variables useful in Mathematics analysis, Applied Mathematics, Physics and Mechanics. The result derived in this paper is of general character and may prove to be useful in several interesting situations appearing in the literature of sciences.

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