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How much time is sufficient for benefiting of awareness programs in epidemics prevention? A free final time optimal control approach

Research Article

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Abstract: In this paper, we focus on Susceptible-Infected-Removed (SIR), SIRS, S-Exposed-I-R (SEIR) and SEIRS epidemic compartmental models. We introduce into these models, a control function which represents the effectiveness rate of awareness programs that could be followed by some heath institutions and organizations for sensitizing people about danger of an epidemic and providing them sufficient information about precautions they could take for avoiding infection. The models are in the form of differential systems with an awareness control variable, and are supposed to describe epidemics that do not require treatment and/or vaccination, or epidemics that do not have remedies at all. In our opinion, the awareness could be effective partially or totally for most kinds of epidemics. Based on epidemic models where there is presence of the compartment associated to removed people, we try to show the impact of awareness campaigns in maximizing the number of removed population while minimizing the number of infected people. In parallel, we seek the optimal final time in which we could stop the proposed optimal awareness strategy. For this, the final time is supposed to be free (non-fixed). We apply Pontryagin's maximum principle for the characterization of the optimal awareness control and we state the necessary condition for the characterization of the sought optimal duration. Finally, we discuss obtained numerical results of the free final time optimal control approach between the studied models.

MSC: 93C15 • 34B15

Keywords: Epidemics • SIR model • SIRS model • SEIR model • SEIRS model • Optimal control • Free final time • Awareness

1. Introduction

In most optimal control approaches applied to epidemic models in literature, either in topics related to biomathematics as in [1],[2], or in mathematical epidemiology as in [3],[4],[5],[6],[7],[8],[9],[10],[11] it is supposed that the final time of any followed strategy for preventing an epidemic to spread, is fixed. However, we often face situations where some epidemics could emerge a population in unprecedented durations, maybe due to an important value of the transmission rate, insufficient immune responses in infected populations, a rapid travel of some pathogens, or particular conditions of a region (geographical location, climate, sanitation). This means it becomes sometimes difficult and not obvious to fix periods of epidemics lives and also periods of optimal control strategies which are thought to be adequate for stopping them, without estimations based on mathematical optimization. Additionally, in the war of epidemics, health policy-makers need sometimes to know when it is appropriate to stop a control policy. This information is necessary for managing their resources, preparing sufficient remedies quantities, and also for having preconceived ideas about costs of strategies aiming to follow.

For all these assumptions, we assume here, there is an epidemic which threatens a susceptible population that becomes infected once it meets infected individuals and moves to the compartment of removed population due to

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awareness programs which encourage them to take necessary precautions against the epidemic, while the infected population is supposed to become removed due to natural recoveries only, for exhibiting the impact of awareness strategies when they are followed alone. In fact, as an example of the effectiveness of awareness campaigns when they are not mixed with any other control strategy such as vaccination or treatment, Al Basir et al. revealed, based on their study in [12], that in the case of mosaic disease, the action of mass media have a direct impact in the prevention of this epidemic, either partially by containing it, or totally when, in specific conditions, it could even eradicate the disease. In two other forms of epidemic dynamics, we consider that the removed category could move after a short time, to the susceptible one, due a loss of immunity, supposed here to be related to a reduction of effectiveness of vaccination or treatment after a short time. We also consider two models where we add a compartment which concerns individuals who are just exposed to infection and could move to the infected category after the appearance of some infection symptoms or once they become infectious.

Many mathematical modelers have showed an interest in highlighting the importance of awareness on susceptible population behavior and its impact on epidemic dynamics [2],[13]-[26]. We should also note here, that we are more interested in the study of models which describe the spread of an epidemic, and in which we could prove the potential of an awareness optimal control strategy on reducing the infection, without adding many compartments, and also, without assuming the existence of other possible interactions between the different classes of a population, in an attempt to exhibit the utility of the proposed mathematical optimization approach when it is applied to general and classical models in epidemiology. For this, we consider simplified Susceptible-Infected-Removed (SIR), SIRS, S-Exposed-I-R (SEIR) and SEIRS epidemic compartmental models and in which a control is introduced to represent the effectiveness rate of awareness programs, followed for avoiding the infection. Then, we try to find the optimal awareness control strategy for each model, by applying Pontryagin's maximum principle, and we seek also, the optimal final time in which an awareness optimal strategy is preferably to be stopped without prolongation and loss of costs, based on a unique free final time criterion for all models. Our considerations aim also to compare simulations of the optimal control function, optimal final time and costs, between basic models in epidemiology which consider that a removed population could move to the susceptible compartment as in SIRS and SEIRS systems, and models where there is absence of this hypothesis as in SIR and SEIR systems.

The paper is organized as follows: In section 2., we present the SIR mathematical model with an awareness control variable and we suggest a free final time optimal control approach for this system with presentations of the numerical results. In sections 3., 4. and 5., we propose similar optimal control strategies for SIRS, SEIR and SEIRS models respectively, and finally, in section 6., we compare between all obtained results.

2. SIR model case

2.1. The model with awareness and free final time optimal control approach

Kermack and McKendrick devised in 1927, a model in which they supposed a fixed population with functions of time; $S(t)$, $I(t)$ and removed $R(t)$ associated to susceptible, infected and removed populations respectively, as the only compartments of their model. Here, we consider the arrival of new susceptible individuals into the population assuming that birth and death rates are equal. Second, we introduce a control function u which represents the effectiveness rate of awareness programs that aim to raise awareness of people about the danger of the epidemic and to clarify them precautions they could take for the prevention from infection.

For the sake of readability, hereafter, we will use the following notations through the rest of paper

$$S(t) \to S$$
$$E(t) \to E \textit{(needed in SEIR and EIRS cases)}$$
$$I(t) \to I$$
$$R(t) \to R$$
$$u(t) \to u$$
$$\lambda_k(t) \to \lambda_k, k = 1,2,3,4 \textit{(needed below)}$$

Thus, the mathematical model with control is described based on the following differential system

$$\dot{S} = -\beta \frac{I}{N} S + (N - S)\, d - uS \tag{1}$$
$$\dot{I} = \beta \frac{I}{N} S - \gamma I - dI \tag{2}$$
$$\dot{R} = \gamma I - dR + uS \tag{3}$$

with $S_0 \geqslant 0$, $I_0 \geqslant 0$ and $R_0 \geqslant 0$, and $N = S + I + R$ is the total population size, and where all parameters are described in Table 1, presented in the discussions section.

In order to minimize the number of infected people and maximize the number of removed people while minimizing the cost of the optimal awareness strategy, and seeking the optimal final time, we consider the following objective function

$$J(u,t_f) = g(t_f,S,I,R) + \int_0^{t_f} \left[k_I I(t) - k_R R(t) + \frac{k_u}{2}(u(t))^2 \right] dt \tag{4}$$

with

$$g(t_f,S,I,R) = k_I I(t_f) - k_R R(t_f) + t_f^2 \tag{5}$$

and where k_I, k_R and k_u are constant severity weights associated to I, R and u respectively.

Then, we seek the optimal pair (u^*, t_f^*) such that

$$J(u^*,t_f^*) = \min_{(u,t_f)\in U\times R^+} J(u,t_f) \tag{6}$$

$U = \{u\ measurable / u_{min} \le u(t) \le u_{max},\ u_{max} \leqslant 1\ u_{min} \geqslant 0, t \in [0,t_f], t_f\ free\}$.
For this, let define the Hamiltonian function associated to system (1)-(3) as

$$\begin{aligned}\mathcal{H}(t,S,I,R,u) = {} & k_I I(t) - k_R R(t) + \frac{k_u}{2}(u(t))^2 + \lambda_1\left(-\beta\frac{I}{N}S + (N-S)d - uS\right) + \lambda_2\left(\beta\frac{I}{N}S - \gamma I - dI\right)\\ & + \lambda_3\left(\gamma I - dR + uS\right)\end{aligned}$$

Based on all these considerations, we obtain the following theorem.

Theorem 2.1.
If there exist optimal control u^* and t_f^* which minimize J along with the optimal solutions S^*, I^* and R^* and associated to the differential system (1)-(3), there exist adjoint variables $\lambda_k,\ k = 1,2,3$, satisfying the following equations

$$\begin{aligned}\dot{\lambda}_1 &= -\left[\beta\frac{I}{N}(\lambda_2-\lambda_1) + u(\lambda_3-\lambda_1) - d\lambda_1\right]\\ \dot{\lambda}_2 &= -\left[k_I + \beta\frac{S}{N}(\lambda_2-\lambda_1) + \gamma(\lambda_3-\lambda_2) - d\lambda_2\right]\\ \dot{\lambda}_3 &= -[-k_R - d\lambda_3]\end{aligned}$$

with the transversality conditions $\lambda_1(t_f^*) = 0, \lambda_2(t_f^*) = k_I$ and $\lambda_3(t_f^*) = -k_R$

Furthermore,

$$u^* = min\left(max\left(u_{min}, \frac{(\lambda_1-\lambda_3)S^*}{k_u}\right), u_{max}\right) \tag{7}$$

and

$$t_f^* = -\frac{\mathcal{H}}{2} \tag{8}$$

Proof. * **Optimal awareness** Let $\mathcal{H}$, $a\ brief\ notation\ of\ the\ hamiltonian\ function\ \mathcal{H}(t,S,I,R,u))$.
Using the Pontryagin's maximum principle as in [2],[1], we have

$$\begin{aligned}\dot{\lambda}_1 &= -\frac{\partial\mathcal{H}}{\partial S}\\ &= -\left[\beta\frac{I}{N}(\lambda_2-\lambda_1) + u(\lambda_3-\lambda_1) - d\lambda_1\right]\\ \dot{\lambda}_2 &= -\frac{\partial\mathcal{H}}{\partial I}\\ &= -\left[k_I + \beta\frac{S}{N}(\lambda_2-\lambda_1) + \gamma(\lambda_3-\lambda_2) - d\lambda_2\right]\\ \dot{\lambda}_3 &= -\frac{\partial\mathcal{H}}{\partial R}\\ &= -[-k_R - d\lambda_3]\end{aligned}$$

and the transversality conditions are found as follows

$$\lambda_1(t_f^*) = -\frac{\partial g}{\partial S}(t_f^*) = 0,\ \lambda_2(t_f^*) = -\frac{\partial g}{\partial I}(t_f^*) = k_I \text{ and } \lambda_3(t_f^*) = -\frac{\partial g}{\partial R}(t_f^*) = -k_R$$

The optimality condition implies that at $u = u^*$

$$\frac{\partial \mathcal{H}}{\partial u} = 0$$
$$\Rightarrow k_u u - \lambda_1 S + \lambda_3 S = 0$$
$$\Rightarrow u = \frac{(\lambda_1 - \lambda_3) S}{k_u}$$

Taking into account the bounds of the control we get

$$u^* = min\left(max\left(u_{min}, \frac{(\lambda_1 - \lambda_3) S^*}{k_u}\right), u_{max}\right)$$

* **Optimal final time**

As $J(u, t_f)$ reaches its maximum at u^* and t_f^*, we have

$$\lim_{\delta \mapsto 0} \frac{J\left(u^*, t_f^* + \delta\right) - J\left(u^*, t_f^*\right)}{\delta} = 0$$

or equivalently, by setting $u = u^*$ and $t_f = t_f^*$

$$\lim_{\delta \mapsto 0} \frac{1}{\delta}\left[g(t_f + \delta, S(t_f + \delta), I(t_f + \delta), R(t_f + \delta)) + \int_0^{t_f + \delta} k_I I(t) - k_R R(t) + \frac{k_u}{2}(u(t))^2\, dt\right.$$
$$\left. - g(t_f, S(t), I(t), R(t)) - \int_0^{t_f} k_I I(t) - k_R R(t) + \frac{k_u}{2}(u(t))^2\, dt\right] = 0$$
$$\lim_{\delta \mapsto 0}\left[\frac{g(t_f + \delta, S(t_f + \delta), I(t_f + \delta), R(t_f + \delta)) - g(t_f, S(t), I(t), R(t))}{\delta} + \frac{1}{\delta}\int_{t_f}^{t_f + \delta} k_I I(t) - k_R R(t) + \frac{k_u}{2}(u(t))^2\, dt\right] = 0$$
$$\frac{\partial g}{\partial t}(t_t) + \frac{\partial g}{\partial S}(t_f)\dot{S}(t_f) + \frac{\partial g}{\partial I}(t_f)\dot{I}(t_f) + \frac{\partial g}{\partial R}(t_f)\dot{R}(t_f) + k_I I(t_f) - k_R R(t_f) + \frac{k_u}{2}\left(u(t_f)\right)^2 = 0$$
$$2t_f + \mathcal{H}\left(t_f, S, I, R, u\right) = 0$$
$$t_f + \frac{\mathcal{H}\left(t_f, S, I, R, u\right)}{2} = 0$$

Finally, we have $t_f = -\dfrac{\mathcal{H}}{2}$ □

2.2. Numerical results

For the numerical simulations, we follow the Forward-Backward Sweep Method (FBSM) [31] with the incorporation of fourth order Runge-Kutta progressive-regressive schemes to solve our two-point boundary value problem.

The algoritheorem we utilize, consists four steps of numerical calculus

2.3. Algoritheorem

Step 0:

. Guess an initial estimation to t_f and u

Step 1:

. Use the initial condition of state variables S, E, I and R, and the stocked values by u.
. Find S, E, I and R which iterate forward in time.

Step 2:

. Use the transversality conditions $\lambda_k(t_f), k = 1,2,3$ and the stocked values by u.

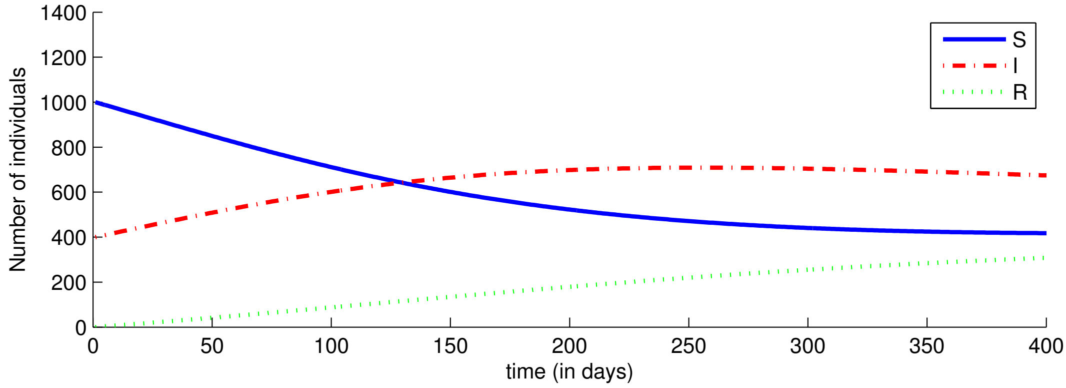

Fig. 1. SIR dynamics without awareness control.

. Find $\lambda_k, k = 1,2,3$ which iterates backward in time.

Step 3:
. Update the control utilizing new values of S, E, I and R, and $\lambda_k, k = 1,2,3$ obtained in the previous step, in the characterization of u^* in (7).
. Set $T(t) = t + \frac{\mathscr{H}(t,S,I,R,u)}{2}$ and seek the optimal final time t_f^* satisfying $T(t_f^*) = 0$.

2.4. Test the convergence

Execute the previous steps several times and test if the values of the sought variables in each iteration are sufficiently close to their values in the previous iteration, check out the recent values as solutions. Else, go back to step 1.

Fig. 1 depicts dynamics of the SIR populations, based on the mathematical model (1) and when the control $u = 0$. We can observe from this figure that the susceptible population decreases from its initial condition towards 400 people due to the increase of the removed people which recover naturally and grows from their initial zero condition towards 275 individuals, while the infection is observed to remain important since the number of infected people increases from 400 people towards values that are close to 675 individuals.

In Fig. 2 (a), we can observe that when we introduce an awareness control u, supposed to be different to zero but varies depending on the characterization in (7), that the removed people has increased more than it has been seen in the previous figure. In fact, the awareness program is seen to be effective since the R function increases towards a value which is very close to 1000 individuals and simultaneously the S function decreases towards a value which is close to only 200 individuals. The most important result here, is that the number of infected people decreases from around 675 people at final instant when there is no control, towards 350 people when we consider to follow an awareness program. This means that despite of an increase of the infected people, we can minimize the I function just until the proposed optimal final time (8) which is approximately equal to 40 days as observed in Fig. 2 (b). Thus, the value of the optimal final time obtained here, belongs very well to the time interval when the infection has been seen to decrease, which means for raising awareness in a population threatened by an epidemic, it is sufficient to follow the suggested strategy here, during 1 month and half only, instead of prolonging time of this approach. This result is also important for institutions and organizations that are responsible for these strategies since they will not lose costs when applying such control strategies in a minimal time.

Fig. 3 depicts the shape of the function $T(t)$, exhibiting its exact numerical values and the reduction of these ones in the time interval $[-1000, 1000]$, to show that as more we reduce the range of $T(t)$ values in regard to the Y axis, as more $T(t)$ is close to appear perpendicular to X axis, which is not the case. This is just a clarification for avoiding any misidentification during the observation of the shape of $T(t)$ in Fig. 2(b), Fig. 5(b), Fig. 7(b) and Fig. 9(b).

3. SIRS model case

3.1. The model with awareness and free final time optimal control approach

The SIRS model is an extension of the previous model but it allows members of the removed compartment to become free of infection after a short time due a loss of immunity related to a reduction of effectiveness of other strategies such as vaccination or treatment. and then, after a short time, rejoin the susceptible population. For this,

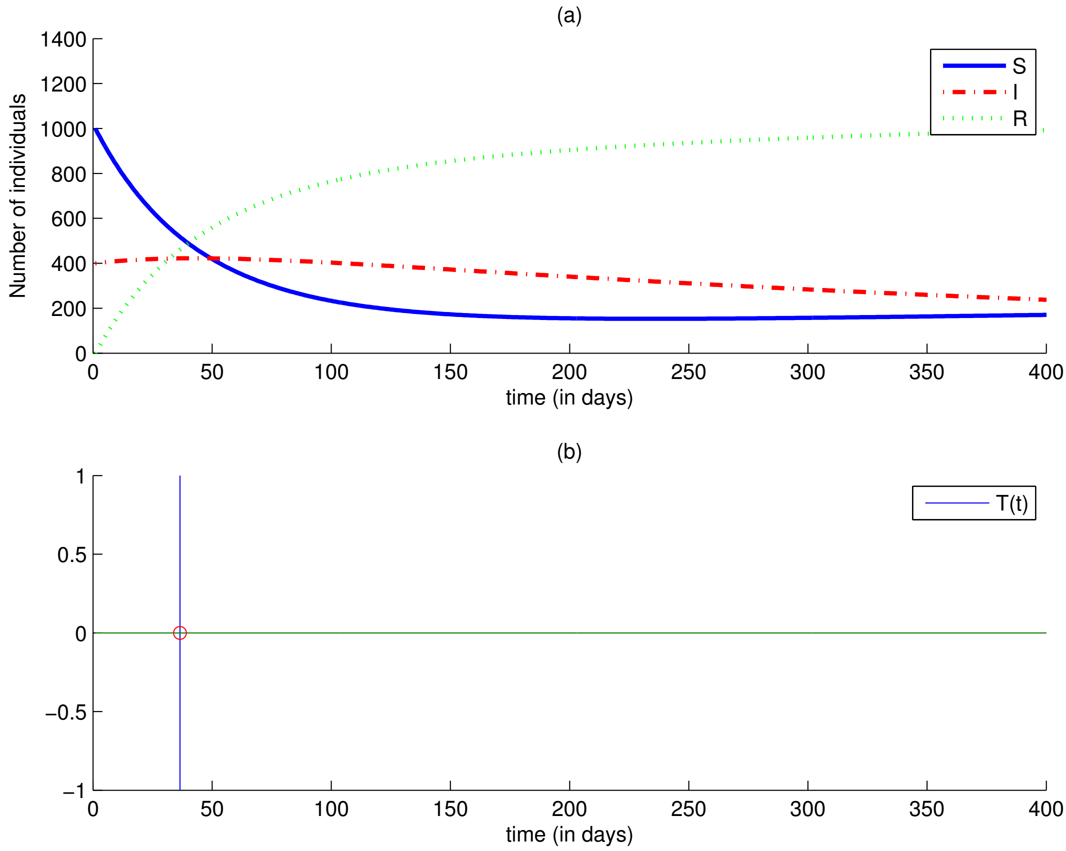

Fig. 2. (a) SIR dynamics with optimal awareness control (7). (b) The numerical location of the optimal final time t_f^* (8) in which the function $T(t)$ is vanished.

we suppose that the SIRS dynamics are described based on the following differential system

$$\dot{S} = -\beta\frac{SI}{N} + \theta R - uS + d(N-S) \tag{9}$$

$$\dot{I} = \beta\frac{SI}{N} - (d+\gamma)I \tag{10}$$

$$\dot{R} = \gamma I - dR - \theta R + uS \tag{11}$$

with $S_0 \geqslant 0$, $I_0 \geqslant 0$ and $R_0 \geqslant 0$, and $N = S + I + R$ is the total population size, and where all parameters are described in 1.

Since our goal always aim to minimize the number of infected people and cost of the awareness optimal control strategy while maximizing the number of removed people and also seeking the optimal final time, we consider same formulations of functions J and g as in (4)-(5). Consequently, we obtain same formulations of the optimal control u^* and optimal final time t_f^*
i.e. we have

$$u^* = min\left(max\left(u_{min}, \frac{(\lambda_1 - \lambda_3)\,S^*}{k_u}\right), u_{max}\right) \tag{12}$$

and

$$t_f^* = -\frac{\mathscr{H}}{2} \tag{13}$$

but with the new Hamiltonian function $\mathscr{H}$ defined as

$$\mathscr{H} = k_I I - k_R R + \frac{k_u u^2}{2} + \left(\theta R - uS + d(N-S) - \beta\frac{SI}{N}\right)\lambda_1 - ((d+\gamma)I - \frac{\beta SI}{N})\lambda_2 + (uS + \gamma I - dR - \theta R)\lambda_3$$

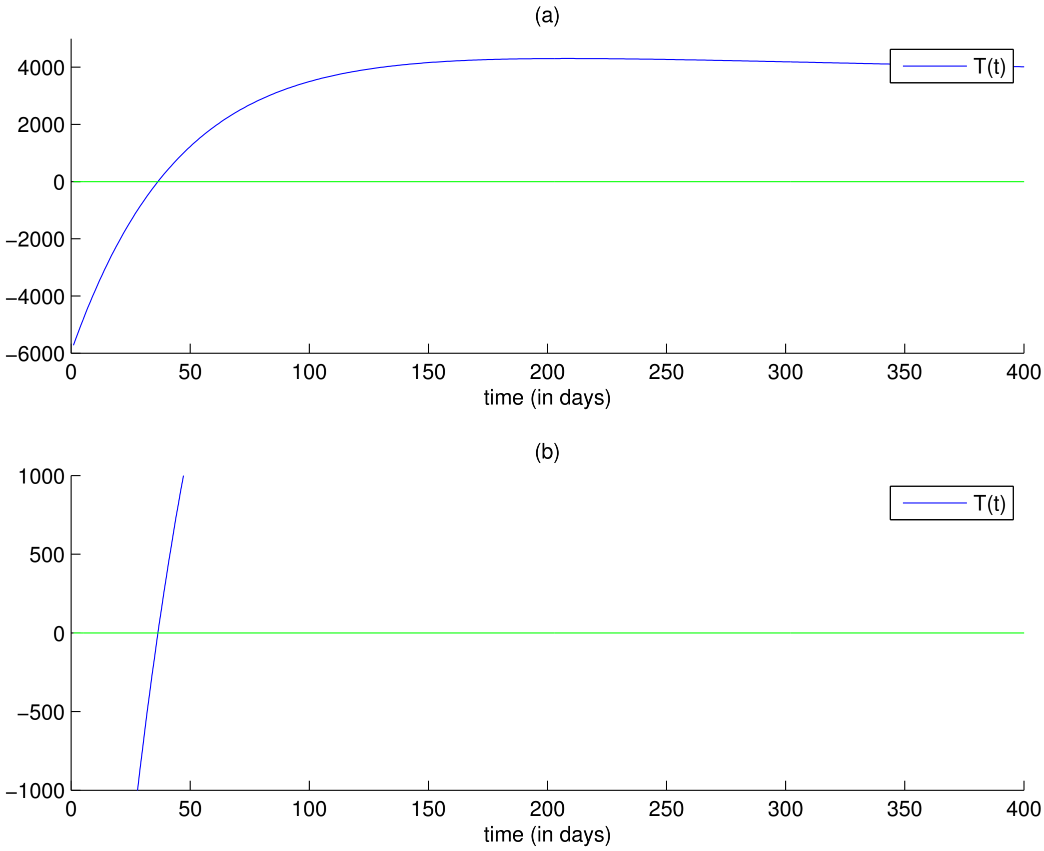

Fig. 3. (a) Shape of the function $T(t)$ with its exact values regarding Y axis (b) Shape of the function $T(t)$ with its exact values regarding just a part of Y axis (i.e. in the interval $[-1000, 1000]$)

and the adjoint differential system which changes to

$$\dot{\lambda}_1 = -\left[u\lambda_3 - \left(d + u + \frac{\beta I}{N}\lambda_1 + \frac{\beta I}{N}\lambda_2\right)\right]$$
$$\dot{\lambda_2} = -\left[k_I + \gamma\lambda_3 - \left(d + \gamma - \frac{\beta S}{N}\lambda_2\right) - \frac{\beta S}{N}\lambda_1\right]$$
$$\dot{\lambda_3} = -\left[\theta\lambda_1 - (d+\theta)\lambda_3 - K_R\right]$$

with the transversality conditions $\lambda_1(t_f^*) = 0, \lambda_2(t_f^*) = k_I$ and $\lambda_3(t_f^*) = -k_R$

3.2. Numerical results

Based on numerical values in Table 1, and using the same algoritheorem presented before, we comment on numerical results associated to this case as follows:

In Fig. 4, we consider that there is yet no awareness strategy, and that the removed population moves to the susceptible compartment after a short time of being healed from infection, maybe due to the cause mentioned above. Compared to the case of Fig. 1 where it has not been supposed that a removed population could become susceptible again, we can observe that the I function increases more while the R function increases less and the S function decreases less. In Fig. 5(a), we can see that once the optimal values of the control u are introduced into the differential system (9)-(11), the I function decreases towards 675 individuals, which is a value bigger than the case of Fig. 2(a). Additionally, the removed people increases to only 225 which is a value, smaller than the case of the model (1)-(3), and then, the susceptible function decreases simultaneously towards a value which is close to 200. As regards to the optimal final time t_f^*, we can observe in Fig. 5(b) that it takes a numerical value equaligng to 88 days and then, which is a more important value than the one obtained in **??**(b). In fact, we can see again that when we consider that a removed population could move to the susceptible compartment, the optimal final time obtained, becomes important since it means that we need more time for sensitizing people.

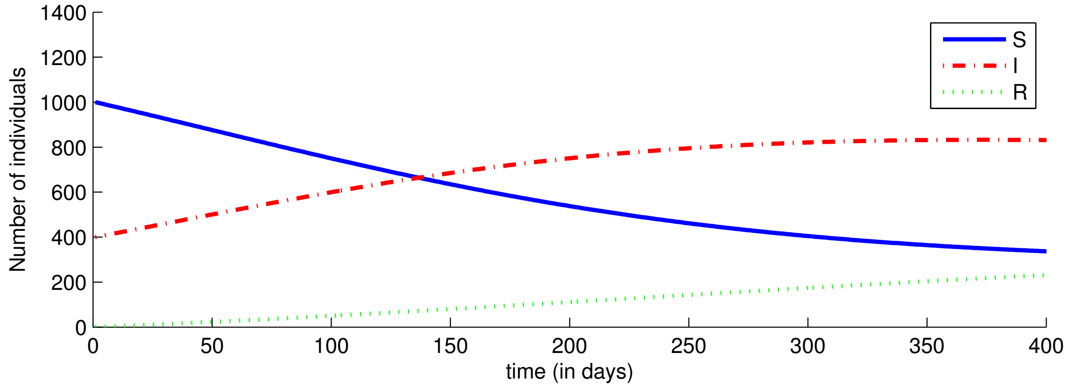

Fig. 4. SIRS dynamics without awareness control.

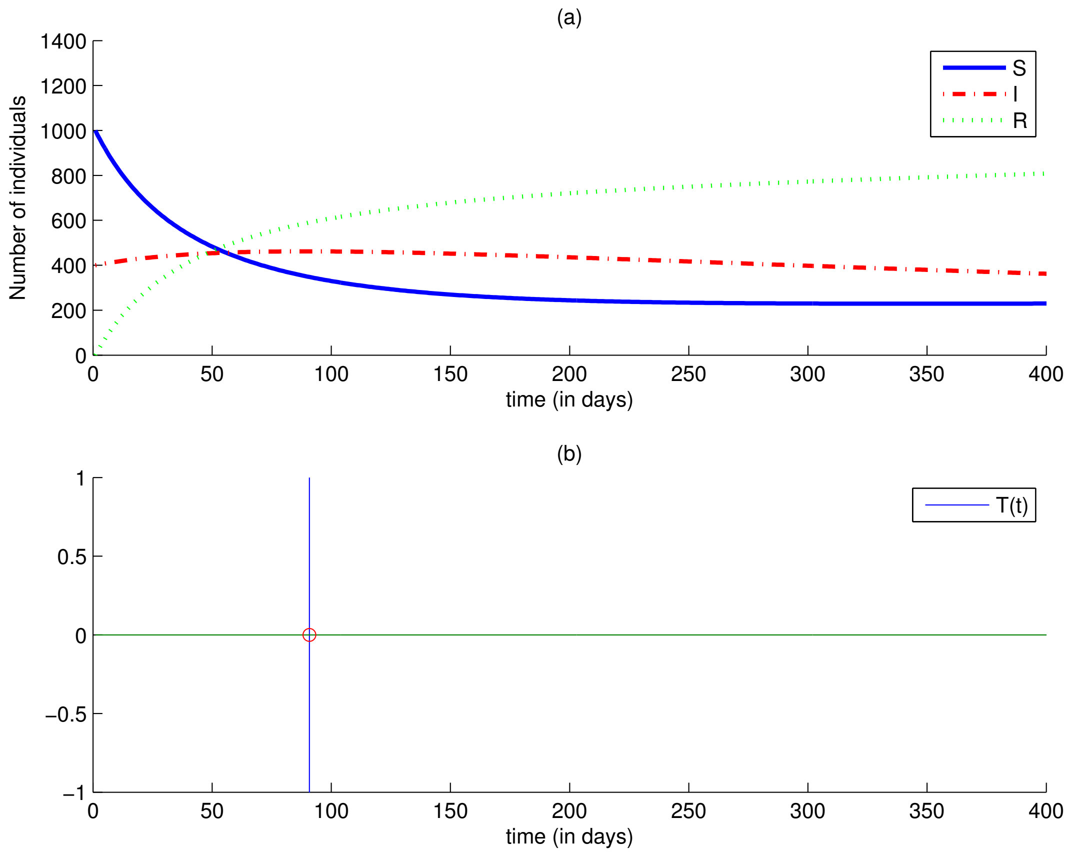

Fig. 5. (a) SIRS dynamics with optimal awareness control (12). (b) The numerical location of the optimal final time t_f^* (13) in which the function $T(t)$ is vanished.

4. SEIR model case

4.1. The model with awareness and free final time optimal control approach

The two SIR and SIRS models studied above, could describe the evolution of an epidemic which converts a susceptible individual to the infectious class immediately upon their infection. However, in some particular cases of infection, it exists a class of individuals which consider a susceptible population could live in a latent or exposed phase or incubation period during which these individuals are infected but without appearance of symptoms, or without becoming infectious. For this case of infection dynamics, we consider the following differential system

$$\dot{S} = (N - S)d - Su - \frac{\beta SI}{N} \tag{14}$$

$$\dot{E} = \frac{\beta SI}{N} - (d + \delta)E \tag{15}$$

$$\dot{I} = \delta E - dI - \gamma I \tag{16}$$

$$\dot{R} = \gamma I - dR + uS \tag{17}$$

with $S_0 \geqslant 0$, $E_0 \geqslant 0$, $I_0 \geqslant 0$ and $R_0 \geqslant 0$, and $N = S + E + I + R$ is the total population size, and where all parameters are described in 1.

Our objective does not change here, and still aims to minimize the number of infected people in the I compartment and cost of the awareness optimal control strategy while maximizing the number of removed people and also seeking the optimal final time, we consider again, same formulations of functions J and g as in the two previous cases. Consequently, we obtain same formulation of optimal final time t_f^*
i.e.

$$t_f^* = -\frac{\mathscr{H}}{2} \tag{18}$$

but with a new characterization of the optimal control u^* which changes to

$$u^* = min\left(max\left(u_{min}, \frac{(\lambda_1 - \lambda_4)\, S^*}{k_u}\right), u_{max}\right) \tag{19}$$

due to the consideration of the new hamiltonian function $\mathscr{H}$ defined as

$$\mathscr{H} = k_I I - k_R R + \frac{k_u u^2}{2} + \left(d(N-S) - uS - \beta\frac{SI}{N}\right)\lambda_1 - (\frac{\beta SI}{N} - (d+\gamma)I)\lambda_2 - (\delta E - dI - \gamma I)\lambda_3 + \left(\gamma I - dR + uS\right)\lambda_4$$

and which leads to the following adjoint differential system

$$\dot{\lambda}_1 = -\left[u\lambda_4 - \left(d + u + \frac{\beta I}{N}\right)\lambda_1 + \frac{\beta I}{N}\lambda_2\right]$$

$$\dot{\lambda}_2 = -\left[\delta\lambda_3 - (d+\delta)\lambda_2\right]$$

$$\dot{\lambda}_3 = -\left[k_I + \gamma\lambda_4 - (d+\gamma)\lambda_3 - \frac{\beta S}{N}\lambda_1 + \frac{\beta S}{N}\lambda_2\right]$$

$$\dot{\lambda}_4 = k_R + d\lambda_4$$

with the transversality conditions $\lambda_1(t_f^*) = 0, \lambda_2(t_f^*) = 0, \lambda_3(t_f^*) = k_I$ and $\lambda_4(t_f^*) = -k_R$

4.2. Numerical results

Based on numerical values presented in Table 1 and the algoritheorem of the FBSM, we obtain the numerical simulations of Fig. 6 and Fig. 7.

In Fig. 6, we present simulations of the SEIR dynamics when there is yet no awareness strategy, and we can observe that the removed population increases towards 180 individuals while the function E tends towards the numerical value 190, and simultaneously, the number of infected people 825 individuals. As regards to the susceptible people, they decrease towards 380 people. As regards to Fig. 7(a), we can observe that once the optimal values of the control u, are introduced into the differential system (14)-(17), the function R increases towards 950 people while the function I increases to only 200 individuals, and the number of susceptible people decreases more, towards 175 individuals. The optimal final time t_f^* in Figure 7. (b), equals in this case, to only 25 days, which belongs well to the time interval when the exposed population begins to decrease, and the infected populations begins to tend towards a stable number, and also the removed population begins to tend towards important values, while the susceptible people tend towards smaller values, compared to the case when there is no awareness control.

5. SEIRS model case

5.1. The model with awareness and free final time optimal control approach

In the following, we consider the same assumptions as in the previous case, but with the hypothesis that removed people could loss their immunity, and then, become after a short time, susceptible again. Thus, we suppose that the

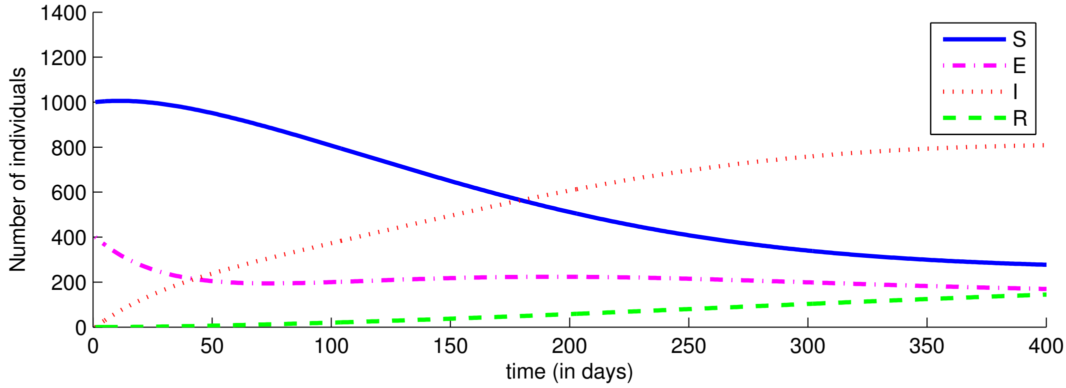

Fig. 6. SEIR dynamics without awareness control.

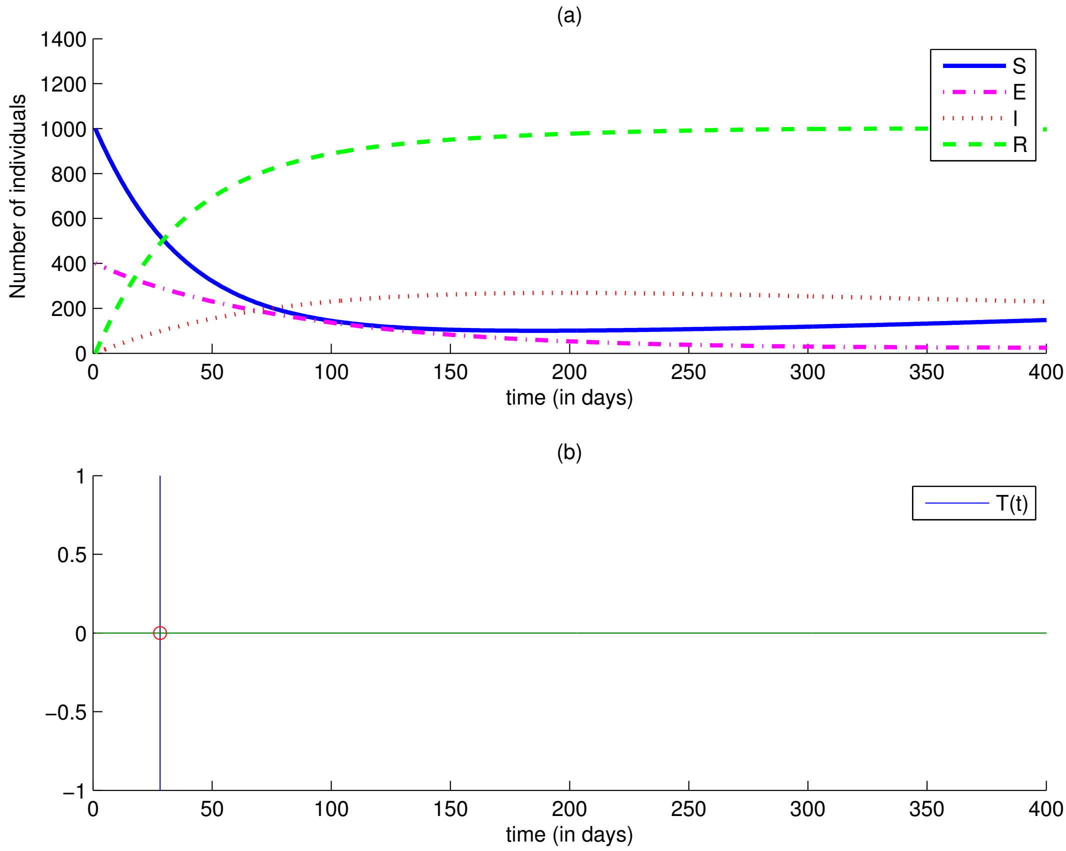

Fig. 7. (a) SIRS dynamics with optimal awareness control (19). (b) The numerical location of the optimal final time t_f^* (18) in which the function $T(t)$ is vanished.

SEIRS dynamics are described based on the following differential system

$$\dot{S} = (N - S)d - Su - \frac{\beta SI}{N} + \theta R \tag{20}$$

$$\dot{E} = \frac{\beta SI}{N} - (d + \delta)E \tag{21}$$

$$\dot{I} = \delta E - dI - \gamma I \tag{22}$$

$$\dot{R} = \gamma I - \theta R - dR + uS \tag{23}$$

with $S_0 \geqslant 0$, $E_0 \geqslant 0$, $I_0 \geqslant 0$ and $R_0 \geqslant 0$, and $N = S + E + I + R$ is the total population size, and where all parameters are described in 1.

Since our goal always aim to minimize the number of infected people and cost of the awareness optimal control strategy while maximizing the number of removed people and also seeking the optimal final time, we consider same

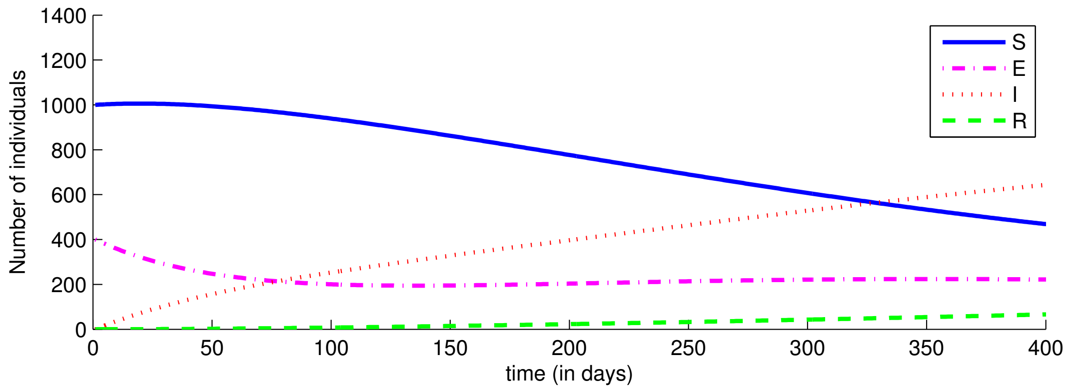

Fig. 8. SEIRS dynamics without awareness control.

formulations of functions J and g as in the previous case. Consequently, we obtain same formulations of the optimal control u^* and optimal final time t_f^* as in the case with there is absence of hypothesis that a removed could become susceptible again,
i.e.

$$u^* = min\left(max\left(u_{min}, \frac{(\lambda_1 - \lambda_3)\,S^*}{k_u}\right), u_{max}\right) \tag{24}$$

and

$$t_f^* = -\frac{\mathscr{H}}{2} \tag{25}$$

but with the new Hamiltonian function $\mathscr{H}$ defined as

$$\mathscr{H} = k_I I - k_R R + \frac{k_u u^2}{2} + \left(d(N-S) - uS + \theta R - \beta\frac{SI}{N}\right)\lambda_1 - \left(\frac{\beta SI}{N} - (d+\gamma)I\right)\lambda_2 - (\delta E - dI - \gamma I)\lambda_3 + \left(\gamma I - \theta R - dR + uS\right)\lambda_4$$

which leads to the following adjoint differential system

$$\begin{aligned}
\dot{\lambda}_1 &= -\left[u\lambda_4 - \left(d + u + \frac{\beta I}{N}\right)\lambda_1 + \frac{\beta I}{N}\lambda_2\right]\\
\dot{\lambda}_2 &= -\left[\delta\lambda_3 - (d+\delta)\lambda_2\right]\\
\dot{\lambda}_3 &= -\left[k_I + \gamma\lambda_4 - (d+\gamma)\lambda_3 - \frac{\beta S}{N}\lambda_1 + \frac{\beta S}{N}\lambda_2\right]\\
\dot{\lambda}_4 &= k_R + (d+\theta)\lambda_4 - \theta\lambda_1
\end{aligned}$$

with the transversality conditions $\lambda_1(t_f^*) = 0, \lambda_2(t_f^*) = 0, \lambda_3(t_f^*) = k_I$ and $\lambda_4(t_f^*) = -k_R$

5.2. Numerical results

Figure 8. depicts dynamics of the SEIRS populations when there is yet no awareness program, and we can observe that the number of exposed people tend towards 200 individuals, which is value, bigger than the case of the previous figure where there was not supposed that removed people could become susceptible after a short time. Simultaneously, and compared to the case in Figure 6., the number of removed people takes a smaller value, equaligng about 50 people, while the number of susceptible people tends a bigger value, equaligng about 500 individuals, and the infected population increases to only 675 people. Once we introduce the optimal control u^* into the differential system (20)-(23), and compared to results of Figure 8., we can observe from Figure 9. (a), that the number of removed people increases more, towards 750 people, and the number of infected people decreases more, towards 380 people, while the exposed population tends towards 50 people and which is bigger than the one obtained in Figure 6. (a). The optimal final time t_f^*, in Figure 9. (b), takes a numerical value which is equal to 30 days, which is bigger than the one observed in Figure 7. (b).

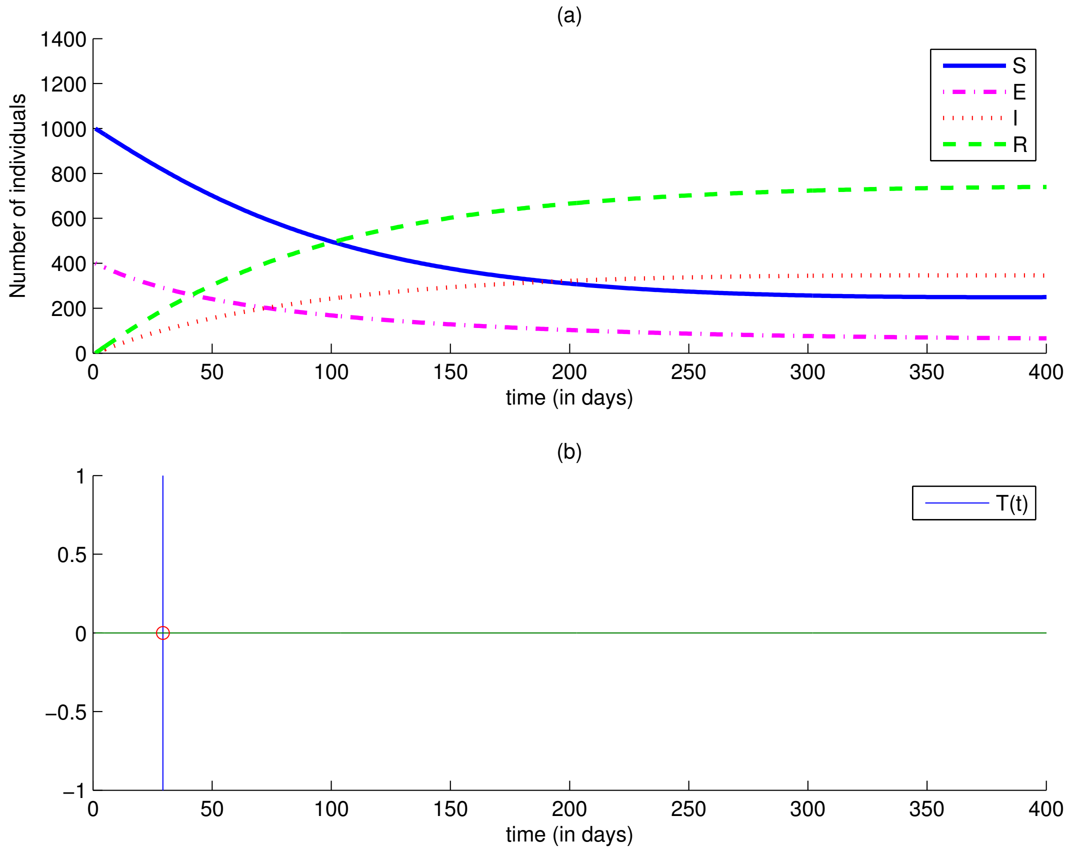

Fig. 9. (a) SIRS dynamics with optimal awareness control (24). (b) The numerical location of the optimal final time t_f^* (25) in which the function $T(t)$ is vanished.

6. Discussions

After the introduction of the optimal control u^* (7)-(12)-(19)-(24) in all differential systems (1)-(3), (9)-(11), (14)-(17) and (20)-(23), namely SIR, SIRS, SEIR and SEIRS models, we deduce that, by following awareness programs only, we could reduce the number of the infected and exposed people, and also, maximize the number of the removed individuals. The optimal control approach based on awareness alone, does not vanish the infection totally, but it could help to reduce it gradually, and then, we can say it could be more effective, in the case where it is required to follow simultaneously, other control strategies such as vaccination or treatment which are often used in dangerous epidemics.

In Fig. 10 we present histograms which depict the maximal numerical values of the optimal final time (8)-(13)-(18)-(25), obtained in all models considered in this paper, and we can deduce from this figure that as more we add compartments, as more the optimal final time t_f^* becomes small, and also, we understand that when we suppose a removed population loses its immunity after a short time, t_f^* becomes more important, compared to the case when there is absence of this hypothesis. In Fig. 11 (a), we present simulations of the optimal control u^* (7)-(12) associated to the two cases of SIR and SIRS models, and we can see that in the case when it is not considered that a removed population loses its immunity, the optimal final time is small, while the optimal final time becomes important when there is presence of this hypothesis. In fact, even the optimal control u^* takes important values when the removed population moves to the susceptible one, which means a need to increase the rate of awareness in this case.

As regards to Fig. 11 (b), we present simulations of the optimal control u^* (19)-(24) associated to the two cases of SEIR and SEIRS models, and we can see that as more we go forward in time, as more the optimal control u^* tends towards the same value in both cases, which equals about 3.5×10^{-3}, because in the first case, it begins to stabilize due to the increase of the number of people that have been healed from infection, while in the second case which needs more awareness programs since the removed people moves rapidly to the S compartment, u^* increases towards its numerical value in the first case. We also understand that in the case of other initial conditions and parameters which could lead to bigger numerical values of u^*, the impact of the awareness strategy, would be more important and interesting.

As regards to the numerical values of the optimal final time t_f^*, we mention that its numerical values are more important in the case where there is the relation $R \to S$, than the case where is absence of this movement. Additionally,

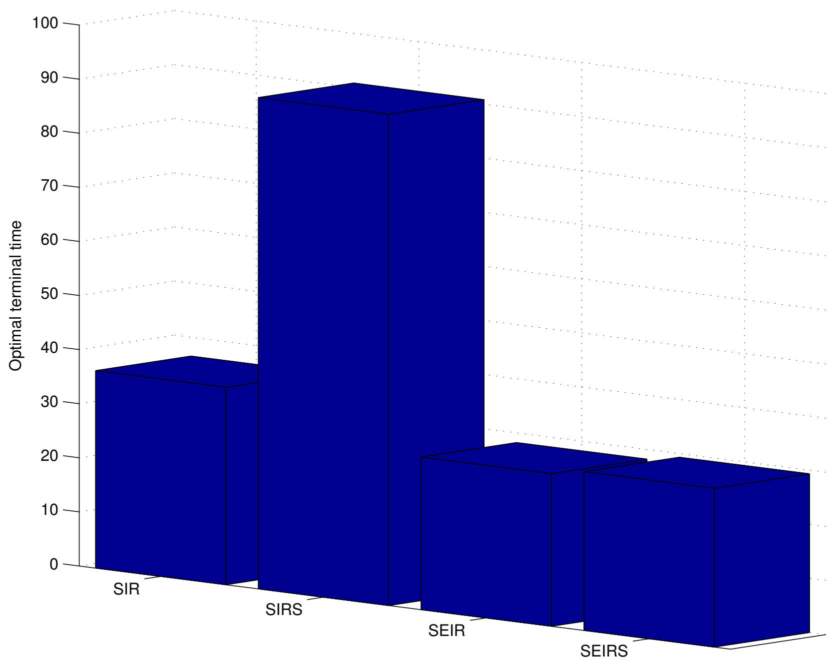

Fig. 10. The values of the cost function J (4) in SIR, SIRS, SEIR and SEIRS models.

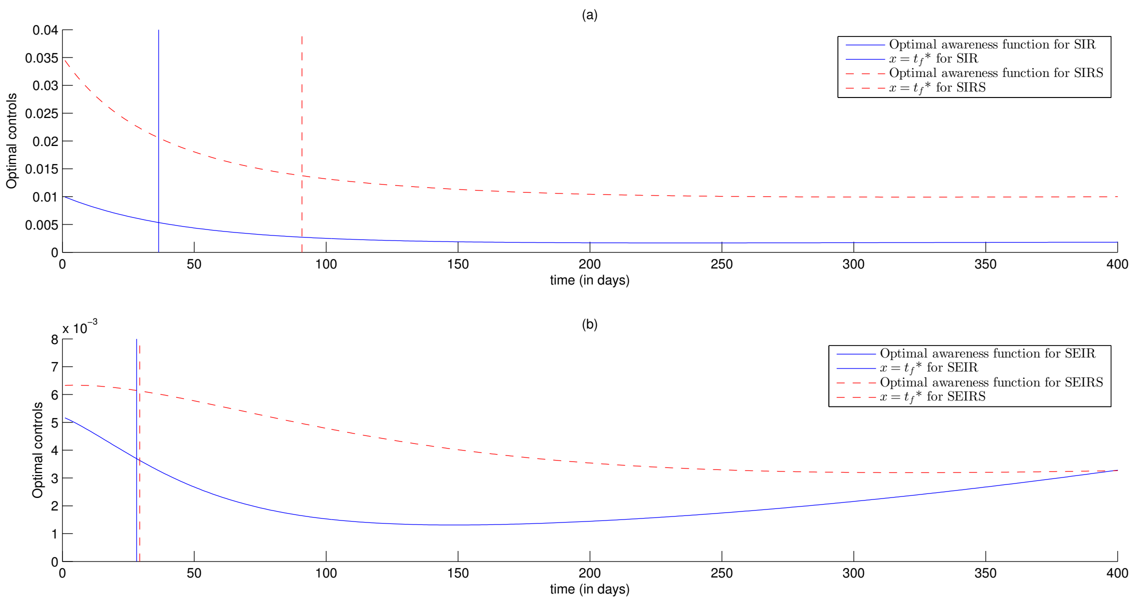

Fig. 11. (a) Optimal control u^* in the case of SIR and SIRS models (b) Optimal control u^* in the case of SEIR and SEIRS models.

as more we add compartment, as more the numerical values of t_f^* are closer. In Table 1, we give numerical values of initial conditions and parameters with their descriptions, utilized in all numerical results of the studied models.

Table 1. Table of numerical values of initial conditions and parameters associated to all differential systems (1)-(3), (9)-(11), (14)-(17), (20)-(23), and which are utilized for all numerical simulations.

	SIR	SIRS	SEIR	SEIRS	Description
S_0	1×10^3	1×10^3	1×10^3	1×10^3	Initial susceptible population
E_0	Absent	Absent	4×10^2	4×10^2	Initial exposed population
I_0	4×10^2	4×10^2	0	0	Initial infected population
R_0	0	0	0	0	Initial removed population
β	1.5×10^{-2}	2×10^{-2}	2.3×10^{-2}	2.3×10^{-2}	Infection transmission rate
d	3×10^{-3}	3×10^{-3}	3×10^{-3}	3×10^{-3}	Natural birth and death rate
γ	2×10^{-3}	2×10^{-3}	1×10^{-3}	1×10^{-3}	Natural recovery rate
θ	Absent	1×10^{-5}	Absent	1×10^{-5}	Rate at which removed individuals become susceptible
δ	Absent	Absent	2×10^{-2}	2×10^{-2}	Rate at which exposed individuals become infected
t_f^*	36.4904	90.8045	28.2	29.3297	Optimal final time for stopping the awareness strategy

7. Conclusion

We considered SIR, SIRS, SEIR and SEIRS epidemic models in which we introduced a control variable related to optimal awareness strategies that we showed to have the potential to maximize the number of removed population and to minimize the number of infected people. In addition, we proved that the obtained optimal final times, met very well, the points where we could observe a remarkable change in the shapes of S, E, I and R functions and where we could understand there is no need to prolong the duration of the control policy. We also deduced that the susceptible people needed more time to be sensitized and the cost was more important in models where we supposed that the removed category has lost its immunity, than the case of models where there was absence of this hypothesis.

References

[1] O. Zakary, M. Rachik, I. Elmouki, On Effectiveness of an Optimal Antiviral Bitherapy in HBV-HDV Coinfection Model; International Journal of Computer Applications 127 (12)(Foundation of Computer Science (FCS), NY, US) (2015) 1–10.

[2] O. Zakary, M. Rachik, I. Elmouki, On the impact of awareness programs in HIV/AIDS prevention: an SIR model with optimal control, Int. J. Comput. Appl. 133(9) (2016) 1–6.

[3] M. A. Khan, A. Wahid, S. Islam, i. Khan, S. Shafie, T. Gul, Stability analysis of an SEIR epidemic model with non-linear saturated incidence and temporary immunity, Int. J. Adv. Appl. Math. and Mech. 2(3) (2015) 1–14.

[4] A. A. Mohsen, H. Kasim, The effect of external source of disease on epidemic model, Int. J. Adv. Appl. Math. and Mech. 2(4) (2015) 53–63.

[5] S. C. Mpeshe, N. Nyerere, S. Sanga, Modeling approach to investigate the dynamics of Zika virus fever: A neglected disease in Africa; Int. J. Adv. Appl. Math. and Mech. 4(3) (2017) 14–21.

[6] O. Zakary, A. Larrache, M. Rachik, i. Elmouki, Effect of awareness programs and travel-blocking operations in the control of HIV/AIDS outbreaks: a multi-domains SIR model, Advances in Difference Equations 2016(1) (2016) 1–17.

[7] O. Zakary, M. Rachik, I. Elmouki, On the analysis of a multi-regions discrete SIR epidemic model: an optimal control approach, International Journal of Dynamics and Control (2016) 1–14.

[8] O. Zakary, M. Rachik, I. Elmouki, A new analysis of infection dynamics: multi-regions discrete epidemic model with an extended optimal control approach; International Journal of Dynamics and Control (2016) 1–10.

[9] O. Zakary, M.Rachik, I. Elmouki, A multi-regional epidemic model for controlling the spread of Ebola: awareness, treatment, and travel-blocking optimal control approaches, Mathematical Methods in the Applied Sciences 40(4) (2017) 1265–1279.

[10] I. Abouelkheir, F. El Kihal, M. Rachik, O. Zakary, I. Elmouki, A Multi-Regions SIRS Discrete Epidemic Model With a Travel-Blocking Vicinity Optimal Control Approach on Cells. British Journal of Mathematics and Computer Science 20(4) (2017) 1–16.

[11] F. El Kihal, M. Rachik, O. Zakary, I. Elmouki, A multi-regions SEIRS discrete epidemic model with a travel-blocking vicinity optimal control approach on cells. International Journal of Advances in Applied Mathematics and Mechanics 4(3) (2017) 60–71.

[12] F. A. Basir, E. Venturino, P. K. Roy, Effects of awareness program for controlling mosaic disease in Jatropha curcas plantations, Mathematical Methods in the Applied Sciences 40(7) (2017) 2441–2453.
[13] H. Singh, J. Dhar, H. S. Bhatti, Bifurcation in Disease Dynamics with Latent Period of Infection and Media Awareness, International Journal of Bifurcation and Chaos 26(6) (2016) 1650097:11 pages.
[14] J. Juang, Y. H. Liang, The impact of vaccine success and awareness on epidemic dynamics; Chaos: An Interdisciplinary Journal of Nonlinear Science 26(11) (2016) 113105 12 pages.
[15] G. O. Agaba, Y. N. Kyrychko, K. B. Blyuss, Mathematical model for the impact of awareness on the dynamics of infectious diseases; Mathematical Biosciences 286 (2017) 22–30.
[16] S. Samanta, Effects of awareness program and delay in the epidemic outbreak; Mathematical Methods in the Applied Sciences 40(5) (2016) 1679–1695.
[17] H. Singh, J. Dhar, H. Bhatti, G. P. Sahu, Dynamical behavior of SIRS epidemic model with media awareness as control strategy, International Journal of Infectious Diseases 45 (2016) 286.
[18] P. K. Roy, S. Saha, F. Al Basir, Effect of awareness programs in controlling the disease HIV/AIDS: an optimal control theoretic approach, Advances in Difference Equations 1 (2015) 217.
[19] L. Yan-Ling, J. Guo-Ping, S. Yu-Rong, Epidemic spreading on a scale-free network with awareness, Chinese Physics B (Chicago) 21(10) (2012) 100207.
[20] S. Maharaj, A. Kleczkowski, Controlling epidemic spread by social distancing: Do it well or not at all; BMC Public Health 12(1) (2012) 679.
[21] C. Granell, S. GÃşmez, A. Arenas, Dynamical interplay between awareness and epidemic spreading in multiplex networks, Physical review letters 111(12) (2013) 128701.
[22] C. Granell, S. GÃşmez, A. Arenas, Competing spreading processes on multiplex networks: awareness and epidemics; Physical review E 90(1) (2014) 012808.
[23] S. Samanta, S. Rana, A. Sharma, A. K. Misra, J. Chattopadhyay, Effect of awareness programs by media on the epidemic outbreaks: a mathematical model; Applied Mathematics and Computation 219(12) (2013) 6965–6977.
[24] K. Paarporn, C. Eksin, J. S. Weitz, J. S. Shamma, Epidemic spread over networks with agent awareness and social distancing; In Communication, Control, and Computing (Allerton) 2015 53rd Annual Allerton Conference, IEEE (2015) 51–57.
[25] K. Paarporn, C. Eksin, J. S. Weitz, J. S. Shamma, The effect of awareness on networked SIS epidemics; In Decision and Control (CDC), 2016 IEEE 55th Conference (2016) 973–978.
[26] Q. Guo, Y. Lei, C. Xia, L. Guo, X. Jiang, Z. Zheng, The Role of Node Heterogeneity in the Coupled Spreading of Epidemics and Awareness, PloS one 11(8) (2016).
[27] M. Scatà, A. Di Stefano, P. Liò, A. La Corte, The Impact of Heterogeneity and Awareness in Modeling Epidemic Spreading on Multiplex Networks, Scientific Reports 6 (2016) Article number: 37105.
[28] S. Funk, E. Gilad, C. Watkins, V. A. Jansen, The spread of awareness and its impact on epidemic outbreaks; Proceedings of the National Academy of Sciences 106(16) (2009) 6872–6877.
[29] Q. Guo, Y. Lei, X. Jiang, Y. Ma, G. Huo, Z. Zheng, Epidemic spreading with activity-driven awareness diffusion on multiplex network, Chaos: An Interdisciplinary Journal of Nonlinear Science 26(4) (2016) 043110.
[30] Q. Wu, X. Fu, M. Small, X. J. Xu, The impact of awareness on epidemic spreading in networks; Chaos: an interdisciplinary journal of nonlinear science, 22(1) (2012) 013101.
[31] S. Lenhart, J. Workman, Optimal Control Applied to Biological Models, Boca Raton, Chapman Hall/CRC, 2007.