Int. J. Adv. Appl. Math. and Mech. 5(1) (2017) 1 – 6 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

IJAAMM

International Journal of Advances in Applied Mathematics and Mechanics

Stopping oscillations of a simple harmonic oscillator using an impulse force

Research Article

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Received 18 November 2016; accepted (in revised version) 16 June 2017

Abstract: The harmonic oscillator is famously used among many systems to model the motion of a mass-spring system. A neither damped nor driven harmonic oscillator is known as a simple harmonic oscillator. Its solution commonly known as simple harmonic motion comprises of continuous sinusoidal oscillations with a constant amplitude and period. In this paper, we determine the magnitude of an impulse force required to stop the oscillations of a simple harmonic oscillator of a mass spring system. The magnitude of the impulse force is obtained as the product of the mass, frequency and amplitude of the oscillations. Depending on the direction and time instant this impulse force is applied, the resulting amplitude of the oscillations can attain a minimum value of zero or a maximum value of double the initial amplitude. We propose the optimal time instants and direction to apply the impulse force in order to stop the oscillations.

MSC: 34K11 • 40D10

Keywords: Simple harmonic oscillator • Mass-spring system • Impulse force

1. Introduction

The harmonic oscillator is widely used to model systems from various fields of science and engineering such as the mass spring system in mechanics, the RLC series circuits in the study of electrical circuits, and the blood glucose level regulatory system in the human body [1–3]. The harmonic oscillator equation for a mass spring system is given by the second-order, linear, constant coefficients ordinary differential equation

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = F(t), \quad x(0) = x_0,\ \dot{x}(0) = v_0. \tag{1}$$

Here $\ddot{x}(t)$ is the acceleration, $\dot{x}(t)$ the velocity, $x(t)$ the displacement of the mass m from the equilibrium position at time t, $k > 0$ the spring constant, $c \geq 0$ the viscous damping constant, $F(t)$ the driving force applied to the system, x_0 and v_0 the initial displacement and velocity, respectively [1–4]. The mass is taken to start from a point below the equilibrium position if $x_0 > 0$ and vice versa [1]. For better analysis of the solutions, equation (1) is often expressed as

$$\ddot{x}(t) + 2\xi\omega_n\dot{x}(t) + \omega_n^2 x(t) = \tfrac{1}{m}F(t), \quad x(0) = x_0,\ \dot{x}(0) = v_0, \tag{2}$$

after dividing by m and letting $\xi = \dfrac{c}{2\sqrt{km}}$, and $\omega_n = \sqrt{\dfrac{k}{m}}$, be the damping ratio and the natural frequency of the system, respectively [5]. The behaviour of the solution of (2) is solely determined by the value of the damping ratio, ξ [5]. In the absence of both damping ($\xi = 0$), and the driving force ($F(t) = 0$) we have what is known as a simple harmonic

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oscillator or an undamped harmonic oscillator. Its solution referred to as simple harmonic motion is characterised by continuous sinusoidal oscillations with a constant amplitude and period [1–3].

An impulse force is a force that is defined to be zero elsewhere except at the time instant at which it is applied. It is often denoted by

$$F(t) = F_0\,\delta(t - t_0), \tag{3}$$

where F_0 is the magnitude of the force and t_0 the time instant at which it is applied [2, 6]. For a mass-spring system an impulse force can be thought as a sharp blow or a nudge.

In this paper, we determine the magnitude of an impulse force needed to stop the oscillations of a simple harmonic oscillator. The correct time instants and direction of the force are also established.

2. Simple harmonic oscillator driven by an impulse force

In the absence of damping the harmonic oscillator equation (2) driven by an impulse force is given by

$$\ddot{x}(t) + \omega_n^2\, x(t) = \tfrac{1}{m} F_0\,\delta(t - t_0), \quad \text{with} \quad x(0) = x_0,\, \dot{x}(0) = v_0. \tag{4}$$

We solve (4) using the Laplace transform method as it is suitable for discontinuous forcing functions such as impulse functions [2, 6]. Taking the Laplace transform both sides of (4) we have

$$\mathscr{L}\left\{\ddot{x}(t) + \omega_n^2\, x(t)\right\} = \mathscr{L}\left\{\tfrac{1}{m} F_0\,\delta(t - t_0)\right\} \tag{5}$$

which simplifies to

$$X(s) = \overbrace{\frac{x_0\, s + v_0}{s^2 + \omega_n^2}}^{free\ Response} + \overbrace{\frac{F_0\, e^{-t_0\, s}}{m(s^2 + \omega_n^2)}}^{impulse\ response} \tag{6}$$

$$= \overbrace{\frac{x_0\, s + v_0}{s^2 + \omega_n^2}}^{free\ Response} + \overbrace{\frac{F_0}{m}\, e^{-t_0\, s} F(s)}^{impulse\ response} \tag{7}$$

where $X(s)$ is the Laplace transform of $x(t)$, and $F(s) = \dfrac{1}{s^2 + \omega_n^2}$ [2, 6, 7]. Taking the inverse Laplace transform both sides of (7) we have

$$\mathscr{L}^{-1}\{X(s)\} = \mathscr{L}^{-1}\left\{ \overbrace{\frac{x_0\, s + v_0}{s^2 + \omega_n^2}}^{free\ Response} + \overbrace{\frac{F_0}{m}\, e^{-t_0\, s} F(s)}^{impulse\ response} \right\} \tag{8}$$

which gives

$$x(t) = \overbrace{x_0 \cos(\omega_n\, t) + \tfrac{v_0}{\omega_n} \sin(\omega_n\, t)}^{free\ response} + \overbrace{\tfrac{F_0}{m\omega_n}\, u(t - t_0) \sin\left[\omega_n (t - t_0)\right]}^{impulse\ response} \tag{9}$$

where $u(t - t_0)$ is a unit step function [2, 3, 6, 7]. More explicitly the general solution is given by

$$x(t) = \begin{cases} x_0 \cos(\omega_n\, t) + \tfrac{v_0}{\omega_n} \sin(\omega_n\, t), & t < t_0, \\ x_0 \cos(\omega_n\, t) + \tfrac{v_0}{\omega_n} \sin(\omega_n\, t) + \dfrac{F_0}{m\,\omega_n} \sin\left[\omega_n (t - t_0)\right], & t \geq t_0. \end{cases} \tag{10}$$

Expanding the term $\sin[\omega_n (t - t_0)]$, the solution (10) simplifies to

$$x(t) = \begin{cases} x_0 \cos(\omega_n\, t) + \dfrac{v_0}{\omega_n} \sin(\omega_n\, t), & t < t_0, \\ \left[x_0 - \dfrac{F_0}{m\omega_n} \sin(\omega_n\, t_0)\right] \cos(\omega_n\, t) + \left[\dfrac{v_0}{\omega_n} + \dfrac{F_0}{m\omega_n} \cos(\omega_n\, t_0)\right] \sin(\omega_n\, t), & t \geq t_0. \end{cases} \tag{11}$$

For both the cases $t < t_0$, and $t \geq t_0$ the solution (11) can be expressed in amplitude-phase form

$$x(t) = A \cos(\omega_n\, t - \phi), \tag{12}$$

where A is the amplitude, ω_n is the frequency and ϕ is the phase angle of the oscillations [1, 2, 4]. The solution having the form (12) is called simple harmonic motion. Its period is given by $T = \frac{2\pi}{\omega_n}$ and is independent of the initial conditions x_0 and v_0. The sine function can also be used instead of cosine [1, 2]. The mass passes through the equilibrium position when the displacement is zero. From (12), $x(t)$ is zero when $\cos(\omega_n t - \phi) = 0$ giving the time values as

$$t = \frac{1}{\omega_n}\left[\frac{\pi}{2} + 2n\pi + \phi\right], \quad t = \frac{1}{\omega_n}\left[\frac{3\pi}{2} + 2n\pi + \phi\right], \quad n \in \mathbb{Z}. \tag{13}$$

2.1. Amplitude and Phase of the general solution

2.1.1. Before applying an impulse force

Expanding (12) and equating like terms to solution (11) for the case before an impulse force is applied ($t < t_0$), we obtain

$$A\cos(\phi) = x_0 \tag{14}$$

and

$$A\sin(\phi) = \frac{v_0}{\omega_n}. \tag{15}$$

From (14) and (15), the initial amplitude of the solution $A = A_{init}$ is obtained as

$$A_{init} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2}. \tag{16}$$

Similarly from (14) and (15), the initial phase angle of the solution $\phi = \phi_{init}$ is obtained as

$$\phi_{init} = \tan^{-1}\left(\frac{v_0}{x_0\,\omega_n}\right). \tag{17}$$

2.1.2. Impulse force applied

Similarly from (11) for the case when an impulse force is applied ($t \geq t_0$), the resulting amplitude of the solution A_{new} is given by

$$A_{new} = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_n}\right)^2 + \left(\frac{F_0}{m\,\omega_n}\right)^2 + \frac{2F_0}{m\,\omega_n}\left[\frac{v_0}{\omega_n}\cos(\omega_n\,t_0) - x_0\sin(\omega_n\,t_0)\right]} \tag{18}$$

which simplifies to

$$A_{new} = \sqrt{A_{init}^2 + \frac{F_0^2}{m^2\,\omega_n^2} - \frac{2F_0\,A_{init}}{m\,\omega_n}\sin(\omega_n\,t_0 - \phi_{init})} \tag{19}$$

where A_{init} and ϕ_{init} are given by (16) and (17), respectively. The resulting phase angle ϕ_{new} is given by

$$\phi_{new} = \tan^{-1}\left(\frac{m\,v_0 + F_0\cos(\omega_n\,t_0)}{m\,\omega_n\,x_0 - F_0\sin(\omega_n\,t_0)}\right). \tag{20}$$

The aim of this paper is to determine the value of F_0 needed to stop the oscillations. That is, the value of F_0 for which A_{new} is zero. So equating (19) to zero and simplifying gives

$$F_0^2 - 2m\,\omega_n\,A_{init}\sin(\omega_n\,t_0 - \phi_{init})F_0 + m^2\,\omega_n^2\,A_{init}^2 = 0. \tag{21}$$

Solving (21) for F_0 gives

$$F_0 = m\,\omega_n\,A_{init}\left[\sin(\omega_n\,t_0 - \phi_{init}) \pm \sqrt{\sin^2(\omega_n\,t_0 - \phi_{init}) - 1}\right]. \tag{22}$$

Equation (22) has real roots when the discriminant $D = \sin^2(\omega_n\,t_0 - \phi_{init}) - 1$ is zero. That is, when

$$\sin(\omega_n\,t_0 - \phi_{init}) = \pm 1, \tag{23}$$

giving the time instants as

$$t_0 = \frac{1}{\omega_n}\left[\frac{\pi}{2} + 2n\pi + \phi_{init}\right], \quad t_0 = \frac{1}{\omega_n}\left[\frac{3\pi}{2} + 2n\pi + \phi_{init}\right], \; n \in \mathbb{Z} \tag{24}$$

and the magnitude of the impulse force as

$$F_0 = \pm\,m\,\omega_n\,A_{init}. \tag{25}$$

The *plus* and *minus* indicates that the force acts in the positive and negative direction, respectively.

2.1.3. Minimum and maximum values of the resulting amplitude, A_{new}

Next we determine the value of F_0 for which A_{new} attains its minimum and maximum values. Mathematically, if A_{new} attains a minimum at some value of F_0, then the derivative $A'_{new}(F_0)$ is zero [8, 9]. Differentiating (19) with respect to F_0 gives

$$A'_{new}(F_0) = \frac{1}{\sqrt{A_{init}^2 + \frac{F_0^2}{m^2\omega_n^2} - \frac{2F_0}{m\omega_n} A_{init}\sin(\omega_n t_0 - \phi_{init})}} \times \left[\frac{F_0}{m^2\omega_n^2} - \frac{A_{init}}{m\omega_n}\sin(\omega_n t_0 - \phi_{init})\right]. \tag{26}$$

Equating (26) to zero we obtain

$$F_0 = m\omega_n A_{init}\sin(\omega_n t_0 - \phi_{init}). \tag{27}$$

Similarly, if A_{new} has a maximum at some value of F_0, then $A'_{new}(F_0) = 0$ [8, 9]. Thus the impulse force giving minimum and maximum values of the resulting amplitude has a maximum value of $F_0 = m\omega_n A_{init}$ which is equivalent to the value given by (25).

Notice that an impulse force of magnitude $F_0 = m\omega_n A_{init}$ needed to stopping the oscillation can also result in oscillations of various amplitudes. Next we determine the maximum value of the resulting amplitude that can be produced by this impulse force. Substituting for $F_0 = m\omega_n A_{init}$ into (19) and simplifying gives

$$A_{new} = A_{init}\sqrt{2\left[1 - \sin(\omega_n t_0 - \phi_{init})\right]} \tag{28}$$

giving the maximum value of the resulting amplitude as $A_{new}[max] = 2A_{init}$. It occurs when $\sin(\omega_n t_0 - \phi_{init}) = -1$ giving the time instants as

$$t_0 = \frac{1}{\omega_n}\left[\frac{3\pi}{2} + 2n\pi + \phi_{init}\right],\ n \in \mathbb{Z}. \tag{29}$$

Also from (28), the resulting amplitude is zero when $\sin(\omega_n t_0 - \phi_{init}) = 1$ giving the time instants

$$t_0 = \tfrac{1}{\omega_n}\left[\tfrac{\pi}{2} + 2n\pi + \phi_{init}\right],\ n \in \mathbb{Z}. \tag{30}$$

These time instants coincide with those obtained in (24) for a positive impulse force of magnitude $F_0 = m\omega_n A_{init}$. In conclusion, an impulse force needed to stop the oscillations of the simple harmonic motion has magnitude $F_0 = m\omega_n A_{init}$. However, this impulse force can either unchange, reduce or increase the amplitude by a factor less than one when applied at time instants different from those given in (24). For example, using (28) no change in the amplitude occurs ($A_{new} = A_{init}$) when the force is applied at the time instants

$$t_0 = \frac{1}{\omega_n}\left[\frac{\pi}{6} + 2n\pi + \phi_{init}\right],\ t_0 = \frac{1}{\omega_n}\left[\frac{5\pi}{6} + 2n\pi + \phi_{init}\right], n \in \mathbb{Z}. \tag{31}$$

The amplitude is halved ($A_{new} = \frac{1}{2}A_{init}$) when the force is applied at the time instants

$$t_0 = \frac{1}{\omega_n}\left[\sin^{-1}\left(\tfrac{7}{8}\right) + 2n\pi + \phi_{init}\right],\ t_0 = \frac{1}{\omega_n}\left[\pi - \sin^{-1}\left(\tfrac{7}{8}\right) + 2n\pi + \phi_{init}\right], n \in \mathbb{Z}. \tag{32}$$

The amplitude is increased by half ($A_{new} = \frac{3}{2}A_{init}$) when the force is applied at the time instants

$$t_0 = \frac{1}{\omega_n}\left[\pi + \sin^{-1}\left(\tfrac{1}{8}\right) + 2n\pi + \phi_{init}\right],\ t_0 = \frac{1}{\omega_n}\left[2\pi - \sin^{-1}\left(\tfrac{1}{8}\right) + 2n\pi + \phi_{init}\right], n \in \mathbb{Z}. \tag{33}$$

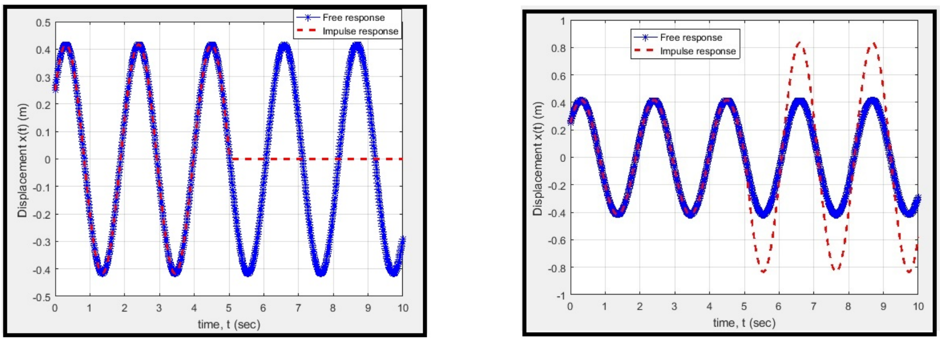
2.2. Direction of impulse force

The direction in which the impulse force is applied is a crucial. Practically, a minimum value of the amplitude can be achieved only if the impulse force acts in the opposite direction of the motion. Similarly, a maximum value of the amplitude can only be obtained when the impulse force acts in the same direction of the motion. Thus if at a certain time instant the resulting amplitude attains a minimum value when the force is applied in one direction, a maximum value is obtained when the same force of the same magnitude is applied in the opposite direction and vice versa.

3. Results and analysis

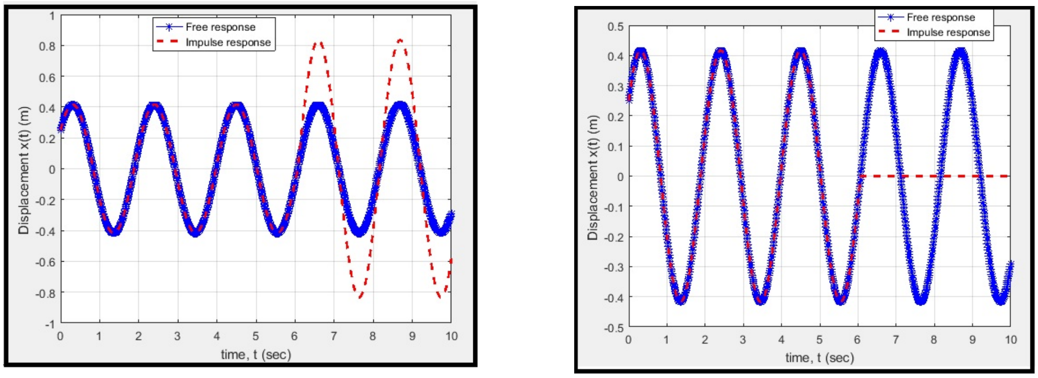
We consider a mass-spring system having a mass of 2 kg with spring constant 18 N/m. The mass is initial displaced a distance of 0.25 m below the equilibrium position and then released with an initial velocity of 1 m/s. The parameter values are: $m = 2$ kg, $k = 18$ N/m, $x_0 = 0.25$ m, $v_0 = 1$ m/s, $\omega_n = 3$ rads/sec, $A_{init} = 0.4167$ m, $F_0 = m\omega_n A_{init} = 2.5000$ N, and $\phi_{init} = 0.9273$ radians. The impulse responses for an impulse force having a magnitude of $F_0 = 2.5000$ N applied in the positive and negative direction for $n = 3$ are given in Fig. 1 and Fig. 2.

From Fig. 1, a positive impulse force at the given time instant terminates the initial amplitude while a negative impulse force doubles the initial amplitude. From Fig. 2, a positive impulse force at the given time instant doubles the initial amplitude while a negative impulse force terminates the amplitude.



(a) $F_0 = +2.5000$ N (b) $F_0 = -2.5000$ N

Fig. 1. Free and impulse responses at $t_0 = \frac{1}{\omega_n}\left[\frac{\pi}{2} + 2n\pi + \phi_{init}\right]$



(a) $F_0 = +2.5000$ N (b) $F_0 = -2.5000$ N

Fig. 2. Free and impulse responses at $t_0 = \frac{1}{\omega_n}\left[\frac{3\pi}{2} + 2n\pi + \phi_{init}\right]$

4. Conclusion

The impulse force needed to stop the oscillations of a mass spring system undergoing simple harmonic motion is equal to the product of the mass, frequency and amplitude of the oscillations. This impulse force must be applied at the time instants corresponding to the equilibrium positions of the displacement and in the opposite direction to the displacement. Oscillations with maximum amplitude of double the initial amplitude are obtained when the force is applied in the same direction as the displacement. Applying this impulse force at time instants other than the equilibrium positions either brings no change, a decrease or increase in the initial amplitude by a factor less than one. A decrease occurs when the force acts in the opposite direction to the displacement and vice versa.

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