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A new modification to homotopy perturbation method for solving Schlömilch's integral equation

Research Article

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Abstract: In this study we introduce a modification to the homotopy perturbation method to solve Schlömilch's integral equations. As a result of this modification, we obtain solutions for various kinds of Schlömilch's integral equations, including the linear, nonlinear, and generalized Schlömilch's integral equations. The solutions are represented by the well-known gamma function. Illustrative examples are provided to show the simplicity and applicability of the proposed algorithm.

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1. Introduction

The linear and nonlinear Schlömilch's integral equations are considered to be important and useful equations in atmospheric and terrestrial physics. The equations and their solutions have been used for some ionospheric problems. For a comprehensive treatment of the equation and its applications in physics, we refer the reader to [1–6].

The standard *linear Schlömilch's integral equation* has the following form:

$$g(x) = \frac{2}{\pi}\int_0^{\pi/2} f(x\sin\theta)\,d\theta, \quad -\pi \le x \le \pi. \tag{1}$$

It's known that this equation has a solution which admits the following form:

$$f(x) = g(0) + x\int_0^{\pi/2} g'(x\sin\theta)\,d\theta,$$

where the derivative is taken with respect to the argument $x\sin\theta$ [1–4].

In addition to standard linear Schlömilch's integral equation, there are also two other forms of Schlömilch's integral equation which will be investigated in this article.

The first one is the *generalized Schlömilch's integral equation* and it admits the following form

$$g(x) = \frac{2}{\pi}\int_0^{\pi/2} f(x\sin^n\theta)\,d\theta, \quad n \ge 1.$$

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The second one is the *nonlinear Schlömilch's integral equation* and it takes the form

$$g(x) = \frac{2}{\pi}\int_0^{\pi/2} F\Big(f(x\sin\theta)\Big)\,d\theta,$$

where $F\big(f(x\sin\theta)\big)$ is a nonlinear function of $f(x\sin\theta)$ and g is a continuous differential function on $-\pi \le x \le \pi$. Unlike the theoretical analysis of ionospheric problems, research on computational aspects of them has started only recently. The Schlömilch's integral equation is no exception. A list of some recent studies on Schlömilch's integral equation is as follows:

In [5], the author introduced the Regularization-Adomian method to solve the linear and the nonlinear Schlömilch's integral equation and their generalized form. [7] used the generalized Chebyshev orthogonal functions collocation method to solve different types of Schlömilch's integral equations including the linear, nonlinear, and generalized Schlömilch's integral equations and the Schlömilch-type equations. [8] obtained a closed-form expression for solutions of the linear and the nonlinear Schlömilch's integral equations in terms of the well-known gamma function when the data function g has a special form.

Since the Schlömilch's integral equations are of the form of Fredholm integral equations of the first kind, the methods that are available for solving Fredholm integral equations of the first kind can successfully be used [9, 10]. An important such technique is the homotopy perturbation method. This method and its variations have been applied to solve many application-based problems emerging from different branches of sciences such as engineering, physics, mathematics, etc. In [11], the authors established an elegant modification to homotopy perturbation method to solve Fredholm integral equations. Based on this, we introduce a new modification to homotopy perturbation method for solving Schlömilch's integral equations.

The rest of the article is organized as follows: The next section reviews the homotopy perturbation method (HPM). In section 3, a new modification to HPM for solving Schlömilch's integral equations of different types is introduced. Section 4 is reserved for comparison and discussion. Final section concludes the article.

2. Review of the HPM

In this section, we review the HPM and discuss its application to Schlömilch's integral equations. The HPM is a combination of perturbation method and homotopy in topology. This powerful combination has been sucessfuly applied to obtain analytical or numerical solutions for many problems arising form different branches of science [12–17]. To explain the basic idea of the HPM, let us consider (1) as

$$L(f) = \frac{2}{\pi}\int_0^{\pi/2} f(x\sin\theta)\,dt - g(x) = 0$$

with solution $f(x)$. Then a convex homotopy with an embedding parameter $p \in [0,1]$ can be defined by

$$H(f,p) = (1-p)F(f) + pL(f), \tag{2}$$

where F(f) is a functional operator with known solution, say u_0. As it can be easily observed that

$$H(f,p) = 0 \tag{3}$$

implies

$$H(f,0) = F(f) \quad \text{and} \quad H(f,1) = L(f).$$

The equations (2) and (3) can be interpreted as follows: As the embedding parameter monotonically increases from 0 to 1, then the trival problem (H(f,0)=F(f)=0) deforms to the original problem (L(f)=0) [18, 19]. The parameter p can also be considered as expanding parameter [20] since it is used to obtain

$$u = u_0 + pu_1 + p^2u_2 + p^3u_3 + \dots. \tag{4}$$

As $p \to 1$, (3) becomes an approximate solution of (1)[11]. That is,

$$u = \lim_{p\to 1} u = u_0 + u_1 + u_2 + u_3 + \dots. \tag{5}$$

For most of the cases, (5) will be a convergent series and the rate of convergence will be based on $L(u)$[21, 22]. For a more thorough treatment of this method, the reader is referred to [18, 19, 23, 24].

3. Modified homotopy perturbation method

In this section we provide a new modification to HPM to solve Schlömilch's integral equations of various kinds with polynomial data function. The main purposes of this and many other modifications existed in the literature are to accelerate the rate of the convergence of HPM and to obtain more simple and efficient algorithms.

3.1. The linear Schlömilch's integral equation

We consider the standard linear Schlömilch's integral equation (1), i.e.,

$$g(x) = \frac{2}{\pi}\int_0^{\pi/2} f(x\sin\theta)\,d\theta, \quad -\pi \le x \le \pi.$$

with g being a polynomial function of any degree.

Theorem 3.1 ([8]).
g is a polynomial function of degree N if and only if the solution of (1) is a polynomial function of the same degree.

Aiming to find out this solution as $g = \sum_{k=0}^{n} a_k x^k$, we define a new homotopy as follows:

$$H(f,p,m) = (1-p)F(f) + pL(f) + p(1-p)\sum_{i=0}^{n} m_i x^i,$$

where $m_i, i = 0,1,\ldots,n$ are constants to be determined.
Setting $H(f,p,m) = 0$, i.e.,

$$H(f,p,m) = (1-p)F(f) + pL(f) + p(1-p)\sum_{i=0}^{n} m_i x^i = 0,$$

where $F(f) = f$ and $L(f) = \frac{2}{\pi}\int_0^{\pi/2} f(x\sin\theta)\,dt - g$.
Equivalently,

$$H(f,p,m) = (1-p)f + p\Big(\frac{2}{\pi}\int_0^{\pi/2} f(x\sin\theta)\,dt - g\Big) + p\sum_{i=0}^{n} m_i x^i - p^2\sum_{i=0}^{n} m_i x^i = 0,$$

Expanding $f = f_0 + pf_1 + p^2 f_2 + \ldots$ as (4) and equating the like terms, we have

$$\begin{aligned}
p^0 &: f_0 = 0,\\
p^1 &: f_1 = \sum_{i=0}^{n}(a_i - m_i)x^i,\\
p^2 &: f_2 = \sum_{i=0}^{n}\left(a_i - \frac{\Gamma(\frac{i+1}{2})(a_i - m_i)}{\sqrt{\pi}\,\Gamma(\frac{i}{2}+1)}\right)x^i\\
&\ \vdots\\
p^{k+1} &: f_{k+1} = f_k - \frac{2}{\pi}\int_0^{\pi/2} f_k(x\sin t)\,dt, k \ge 2.
\end{aligned} \tag{6}$$

We set $f_2 = 0$ to find m_i for $i = 0,1,2,\ldots$. The reason for this setup is that if $f_2 = 0$, then from (6), $f_{k+1} = 0$ for any $k \ge 2$. This will in turn gives us the exact solution, $f = f_1$. Notice that this process reduces a great deal amount of calculations. Instead of a series solution which would have been obtained if the HPM was used, the solution is obtained directly. Setting that $f_2 = 0$, we obtain

$$m_i = \frac{\Gamma(\frac{i+1}{2}) - \sqrt{\pi}\,\Gamma(\frac{i}{2}+1)}{\Gamma(\frac{i+1}{2})}a_i, \quad i = 0,1,2,\ldots n. \tag{7}$$

Example 3.1.
Consider the linear Schlömilch's integral equation

$$g(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x\sin t)\,dt, -\pi \le x \le \pi,$$

where $g(x) = 1 + x + \pi x^2$.

Since this is the first example, we provide the details of the application of the proposed method. For the subsequent examples, we only use (7).

We start by setting the homotopy as follows:

$$H(f,p,m) = (1-p)F(f) + pL(f) + p(1-p)\sum_{i=0}^{2} m_i x^i = 0, \tag{8}$$

where $F(f) = f$ and $L(f) = \dfrac{2}{\pi}\displaystyle\int_0^{\pi/2} f(xsint)\,dt - (1+x+\pi x^2)$.

Expanding f as $f = f_0 + pf_1 + p^2 f_2 + \ldots$ and substitute this into (8), we have

$$H(f,p,m) = (1-p)(f_0 + pf_1 + p^2 f_2 + \ldots) + p\Bigg(\frac{2}{\pi}\int_0^{\pi/2} f_0(xsint) + pf_1(xsint) + p^2 f_2(xsint) + \ldots\,dt - (1+x+\pi x^2),$$
$$+\sum_{i=0}^{2} m_i x^i - p\sum_{i=0}^{2} m_i x^i\Bigg) = 0.$$

Equating the coefficients of like terms, we obtain

$$f_0 = 0,$$
$$f_1 = 1 + x + \pi x^2 - m_0 - m_1 x - m_2 x^2 = 1 - m_0 + (1-m_1)x + (\pi - m_2)x^2,$$
$$f_2 = f_1 - \frac{2}{\pi}\int_0^{\pi/2} f_1(xsint)\,dt + \sum_{i=0}^{2} m_i x^i.$$

To find m_0, m_1, and m_2, we set $f_2 = 0$. Thus, we have $m_0 = 0, m_1 = \frac{2-\pi}{2}$, and $m_2 = -\pi$. Substituting these values into f_1 produces the solution, namely,

$$f = f_1 = 1 - m_0 + (1-m_1)x + (\pi - m_2)x^2,$$
$$= 1 + \frac{\pi}{2}x + 2\pi x^2.$$

Example 3.2.

Solve the linear Schlömilch's integral equation

$$g(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x\sin t)dt, -\pi \le x \le \pi,$$

where $g(x) = x + 3x^2$.

Here, $a_0 = 0, a_1 = 1$, and $a_2 = 3$. Using (7), we have $m_0 = 0, m_1 = 1 - \pi/2$, and $m_2 = -3$.

Thus the solution is

$$f = f_1 = (a_0 - m_0) + (a_1 - m_1)x + (a_2 - m_2)x^2$$
$$= \frac{\pi}{2}x + 6x^2.$$

Example 3.3.

Consider the linear Schlömilch's integral equation [5]

$$g(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x\sin t)dt, -\pi \le x \le \pi,$$

where $g(x) = 1 + \pi x^2$. The exact solution is $f(x) = 1 + 2\pi x^2$.

Here, $a_0 = 1, a_1 = 0$, and $a_2 = 3$. Using (7), we have $m_0 = 0, m_1 = 0$, and $m_2 = -\pi$.

Thus the solution is

$$f = f_1 = (a_0 - m_0) + (a_1 - m_1)x + (a_2 - m_2)x^2$$
$$= 1 + 2\pi x^2.$$

3.2. The generalized Schlömilch's integral equation

We consider

$$g(x) = \frac{2}{\pi}\int_0^{\pi/2} f(x\sin^r\theta)\,d\theta, \quad r \geq 1.$$

We follow similar steps as described above. Let that $g = \sum_{k=0}^{n} a_k x^k$ and define the homotopy as

$$H(f,p,m) = (1-p)F(f) + pL(f) + p(1-p)\sum_{i=0}^{n} m_i x^i,$$

where $m_i, i = 0,1,\ldots,n$ are constants to be determined. Setting $H(f,p,m) = 0$, i.e.,

$$H(f,p,m) = (1-p)F(f) + pL(f) + p(1-p)\sum_{i=0}^{n} m_i x^i = 0,$$

where $F(f) = f$ and $L(f) = \frac{2}{\pi}\int_0^{\pi/2} f(x\sin^r\theta)\,dt - g$. Then

$$H(f,p,m) = (1-p)f + pL(f) + p(1-p)\sum_{i=0}^{n} m_i x^i = 0,$$

or

$$H(f,p,m) = (1-p)f + p\Big(\frac{2}{\pi}\int_0^{\pi/2} f(x\sin^r\theta)\,dt - g\Big) + p\sum_{i=0}^{n} m_i x^i - p^2\sum_{i=0}^{n} m_i x^i = 0,$$

Expanding $f = f_0 + pf_1 + p^2 f_2 + \ldots$ as (4) and equating the like terms, we have

$$\begin{aligned}
&p^0 : f_0 = 0,\\
&p^1 : f_1 = \sum_{i=0}^{n}(a_i - m_i)x^i,\\
&p^2 : f_2 = \sum_{i=0}^{n}\left(a_i - \frac{\Gamma(\frac{ri+1}{2})(a_i - m_i)}{\sqrt{\pi}\Gamma(\frac{ri}{2}+1)}\right)x^i\\
&\quad\vdots\\
&p^{k+1} : f_{k+1} = f_k - \frac{2}{\pi}\int_0^{\pi/2} f_k(x\sin^r\theta)dt, k \geq 2.
\end{aligned}$$

We set $f_2 = 0$ to find m_i for $i = 0,1,2,\ldots$. As in the previous case, the solution is given by

$$f = f_1 = \sum_{i=0}^{n}(a_i - m_i)x^i,$$

where

$$m_i = \frac{\Gamma(\frac{ri+1}{2}) - \sqrt{\pi}\Gamma(\frac{ri}{2}+1)}{\Gamma(\frac{ri+1}{2})}a_i, \quad i = 0,1,2,\ldots,n. \tag{9}$$

Example 3.4.
Solve the generalized Schlömilch's integral equation [5]

$$g(x) = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} f(x\sin^2 t)dt, -\pi \leq x \leq \pi,$$

where $g(x) = x + 3x^2$ and $r = 2$.
Here, $a_0 = 0, a_1 = 1$, and $a_2 = 3$. Using (9), we have $m_0 = 0, m_1 = -1$, and $m_2 = -5$.
Thus the solution is

$$\begin{aligned}
f = f_1 &= (a_0 - m_0) + (a_1 - m_1)x + (a_2 - m_2)x^2\\
&= 2x + 8x^2,
\end{aligned}$$

which is the exact solution.

Example 3.5.
Solve the generalized Schlömilch's integral equation [5]

$$g(x) = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} f(x\sin^3 t)\,dt, -\pi \le x \le \pi,$$

where $g(x) = 4x - \frac{5}{16}x^2$ and $r = 3$.
Here, $a_0 = 0, a_1 = 4$, and $a_2 = -\frac{5}{16}$. Using (9), we have $m_0 = 0, m_1 = 4 - 3\pi$, and $m_2 = \frac{11}{16}$.
Thus the solution is

$$\begin{aligned} f = f_1 &= (a_0 - m_0) + (a_1 - m_1)x + (a_2 - m_2)x^2 \\ &= 3\pi x - x^2, \end{aligned}$$

which is the exact solution.

3.3. The nonlinear Schlömilch's integral equation

We consider the nonlinear Schlömilch's integral equation which has the following form:

$$g(x) = \frac{2}{\pi}\int_{0}^{\pi/2} F(f(x\sin\theta))\,d\theta, \quad -\pi \le x \le \pi, \tag{10}$$

where $F(\phi(x\sin\theta))$ is a nonlinear function of $f(x\sin\theta)$.
We assume that F is invertible so that letting that $F(f(x\sin\theta)) = h(x\sin\theta)$ will imply that $f(x\sin\theta) = F^{-1}(h(x\sin\theta))$. Thus, with this transformation, (10) becomes

$$f(x) = \frac{2}{\pi}\int_{0}^{\pi/2} h(x\sin\theta)\,d\theta,$$

which is equivalent to (1). We solve this equation for $h(x)$ and then use the inverse transform F^{-1} to get $f(x)$.

Example 3.6.
Consider the nonlinear Schlömilch's integral equation [5]

$$g(x) = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} f^2(x\sin t)\,dt, -\pi \le x \le \pi, \tag{11}$$

where $g(x) = x^2$.
Let that $h = f^2$. Then (11) becomes

$$x^2 = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} h(x\sin t)\,dt, -\pi \le x \le \pi, \tag{12}$$

Notice that the transformation $h = f^2$ transform the nonlinear equation into a linear equation. So using (7), we can find its solution. That is, since $a_0 = a_1 = 0, a_2 = 1$, we find $m_0 = m_1 = 0$ and $m_2 = -1$.
It follows that the exact solution for h is given by $h = 2x^2$.
Inverting the transformation $h = f^2$, the exact solution is given by $f(x) = \pm\sqrt{2}x$.

4. Comparison and discussion

In this section, we discuss and compare the proposed approach with two methods through an example. As it will be seen clearly in the following example, the proposed method has significant advantageous over both methods when g is a polynomial function.

Example 4.1.
Consider the linear Schlömilch's integral equation

$$g(x) = \frac{2}{\pi}\int_{0}^{\frac{\pi}{2}} f(xsint)dt, -\pi \le x \le \pi, \tag{13}$$

where $g(x) = 1 + x + \pi x^2$.
We solve this problem in three different ways. The methods we use are homotopy perturbation method, Regularization-Adomian method, and the proposed method.

4.1. The HPM method

A homotopy can be readily formed as follows:

$$H(f,p) = (1-p)f(x) + p\left(\frac{2}{\pi}\int_0^{\pi/2} f(xsint)\,dt - g(x)\right) = 0,$$

or

$$H(f,p) = (1-p)(f_0 + pf_1 + p^2 f_2 + \dots) + p\left(\frac{2}{\pi}\int_0^{\pi/2} f_0(xsint) + pf_1(xsint) + p^2 f_2(xsint) + \dots\,dt - g(x)\right) = 0,$$

we have

$$\begin{aligned}
&p^0 : f_0 = 0,\\
&p^1 : f_1 = g(x)\\
&p^2 : f_2 = f_1 - \frac{2}{\pi}\int_0^{\pi/2} f_1(xsint)\,dt,\\
&\quad\vdots\\
&p^{k+1} : f_{k+1} = f_k - \frac{2}{\pi}\int_0^{\pi/2} f_k(x\sin t)dt, k \ge 2.
\end{aligned}$$

So for this particular case, we have

$$\begin{aligned}
&p^0 : f_0 = 0,\\
&p^1 : f_1 = g(x) = 1 + x + \pi x^2,\\
&p^2 : f_2 = f_1 - \frac{2}{\pi}\int_0^{\pi/2} f_1(xsint)\,dt = (1-\frac{2}{\pi})x + \frac{1}{2}\pi x^2\\
&p^3 : f_3 = f_2 - \frac{2}{\pi}\int_0^{\pi/2} f_2(x\sin t)dt = (1-\frac{2}{\pi})^2 x + \frac{1}{4}\pi x^2\\
&\quad\vdots\\
&p^{k+1} : f_{k+1} = f_k - \frac{2}{\pi}\int_0^{\pi/2} f_k(x\sin t)dt = (1-\frac{2}{\pi})^k x + \frac{1}{2^k}\pi x^2.
\end{aligned}$$

Thus the solution admits the following form:

$$\begin{aligned}
f(x) &= f_0 + f_1 + f_2 + \dots,\\
&= 1 + x + \pi x^2 + (1-\frac{2}{\pi})x + \frac{1}{2}\pi x^2 + (1-\frac{2}{\pi})^2 x + \frac{1}{4}\pi x^2 + (1-\frac{2}{\pi})^k x + \frac{1}{2^k}\pi x^2 + \dots,\\
&= 1 + x + \pi x^2 + (1-\frac{2}{\pi})x\left(1 + (1-\frac{2}{\pi}) + (1-\frac{2}{\pi})^2 + \dots\right) + \pi x^2\left(1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots\right),\\
&= 1 + \frac{\pi}{2}x + 2\pi x^2.
\end{aligned}$$

4.2. The Regularization-Adomian method [5]

Using the method of regularization, we transform (13) into

$$f_\alpha(x) = \frac{1+x+\pi x^2}{\alpha} - \frac{2}{\alpha\pi}\int_0^{\frac{\pi}{2}} f_\alpha(x sin t)dt.$$

The recurrence relation for Adomian decomposition method is as follows:

$$f_{\alpha_0}(x) = \frac{1}{\alpha}(1+x+\pi x^2),$$
$$f_{\alpha_{k+1}}(x) = -\frac{2}{\alpha\pi}\int_0^{\pi/2} f_{\alpha_k}(x\sin t)\,dt, k \geq 0.$$

A few components of the regularized solution have the following form:

$$\begin{aligned}
f_{\alpha_0}(x) &= \frac{1}{\alpha}(1+x+\pi x^2),\\
f_{\alpha_1}(x) &= -\frac{1}{\alpha^2} - \frac{2x}{\alpha^2\pi} - \frac{\pi x^2}{2\alpha^2},\\
f_{\alpha_2}(x) &= \frac{1}{\alpha^3} + \frac{4x}{\alpha^3\pi^2} + \frac{\pi x^2}{4\alpha^3},\\
&\vdots
\end{aligned}$$

Adding these components, we have

$$\begin{aligned}
f_\alpha(x) &= \frac{1}{\alpha}\left(\frac{1}{\alpha} - \frac{1}{\alpha^2} + \frac{1}{\alpha^3} - \dots\right) + \frac{1}{\alpha}x\left(1 - \frac{2}{\alpha\pi} + \frac{4}{\alpha^2\pi^2} - \dots\right) + \frac{\pi x^2}{\alpha}\left(1 - \frac{1}{2\alpha} + \frac{1}{4\alpha^2} + \dots\right)\\
&= \frac{1}{\alpha+1} + \frac{\pi x}{\alpha\pi+2} + \frac{2\pi x^2}{2\alpha+1}.
\end{aligned} \tag{14}$$

The limit of (14) gives the exact solution, namely,

$$f(x) = \lim_{\alpha\to 0} f_\alpha(x) = 1 + \frac{\pi}{2}x + 2\pi x^2.$$

4.3. The proposed method

Here, $a_0 = 1, a_1 = 1, a_2 = \pi$. Using the equation (7), we get $m_0 = 0, m_1 = 1 - \frac{\pi}{2}, m_2 = -\pi$. Thus the exact solution is given by

$$\begin{aligned}
f = f_1 &= (a_0 - m_0) + (a_1 - m_1)x + (a_2 - m_2)x^2\\
&= 1 + \frac{\pi}{2}x + 2\pi x^2.
\end{aligned}$$

5. Conclusion

In this article we employed a modification to the homotopy perturbation method for solving Schlömilch's integral equations of various kinds. The proposed method reduced a great deal amount of calculations presented in other methods. Illustrative examples were given to demonstrate the simplicity and applicability of the proposed method.

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