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Solve the groundwater model equation using Fourier transforms method

Research Article

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Abstract: The aim of this paper is to solve a mathematical model of groundwater equation. Groundwater is not static, it flows in an aquifer and its flow can be described using partial differential equation associated initial-boundary conditions. The work considered three dimensional steady state groundwater flows case. Then the model of groundwater equation solved using Fourier transforms method, which cannot be solved in other method when used to solve differential equation.

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1. Introduction

The model of groundwater flow is one of the important equations in order to obtain water sources, it is useful for agricultural, and other fields. It may be noted that in many countries, groundwater is the main source of water so as to lack of rivers; therefore it is necessary to resort to groundwater as a second source of water sources. Good the groundwater management requires the ability to predict the system of underground water movement and knowledge of the changes imposed by nature and human activities on this water. The past 20 years or so have also seen some major technological breakthroughs in groundwater hydrology. One technological growth area has been in the development and use of deterministic, distributed-parameter, computer simulation models for analysing flow and solute transport in groundwater systems. These developments have somewhat paralleled the development and widespread availability of faster, larger memory, more capable, yet less expensive computer systems. Another major technological growth area has been in the application of isotopic analyses to groundwater hydrology, wherein isotopic measurements are being used to help interpret and define groundwater flow paths, ages, leakage, and interactions with surface water [1–4]. The Fourier Transform is an important mathematical transform that is used widely in many application areas such as applied mathematics, physics, engineering, and computer science. Here we focus on the use fourier transforms for solving linear partial differential equations (PDE). The Fourier transform is beneficial in differential equations because it can transform them into equations which are easier to solve. In addition, many transformations can be made simply by applying predefined formulas to the problems of interest.

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2. The Mathematical Model

There are many different mathematical models are used to study the natural phenomena, Rogers (1992), used Artificial Neural Networks and the Genetic Algorithm [5], Seyam (2009), use the Artificial Neural Networks to solve model of ground water (see [6]), Willis (2014) used the U.S. Geological Survey's (USGS) to describe mathematical model to analyze the possible impacts associated with sea level rise [7], Uchdadiya and Patel (2014) studied the seepage losses through the canals [8], there are many researcher study and solve the model equation of ground water such [9–11], shallow water is also having a part of studies as is the case with groundwater [12, 13].

This paper, study and solved the standard equations that govern groundwater flow are derived using the principle of continuity and Darcy's law without resourse is as follow [11]:

$$T_x \frac{\partial^2 h}{\partial x^2} + T_x \frac{\partial^2 h}{\partial y^2} + T_z \frac{\partial^2 h}{\partial z^2} = S \frac{\partial h}{\partial t} \tag{1}$$

With initial condition $h(x, y, z, 0) = f(x, y, z)$; $x, y, z \in R$ and $t > 0$. Where;
T_i : transmissivity in the i-direction (L^2/T).
S : storage coefficient (-).
h : hydraulic head or simply "head" or pressure head (also known as piezometric head) (L).

There are many different techniques to solve ground water model equation, such finite element method [14], finite difference method [15]. A complex transformation is used to change the time-dependent linear aquifer model into an equivalent time-independent problem of a complex function for more details see [10], the homotopy decomposition method [16], and collocation method based on Laplace-homotopy perturbation method [11]. In this paper we will use the Fourier Transform method to solve type of ground water model equation, i.e., problem (1).

3. Fourier Analysis

The subject of Fourier analysis (Fourier series and Fourier transform) is an old subject in mathematical analysis and is of great importance to mathematicians, scientist, and engineers alike, a function can be expressed as a series of sine and cosine functions. Such a representation is called a Fourier series (expansion) [17]. It is, transform the signal from time domain to frequency domain, so it can be analyzed and processed. This classical method is used in many applications. However, these methods fail to provide efficient representations for certain types of functions which have discontinuities. Another problem with this sum is that it is infinite. In use, only a finite number of terms can be used. More accuracy requires more terms in the series, but more terms require more time to compute and more space to store.

3.1. Fourier Transforms

The Fourier transform can be thought of as a continuous form of Fourier series. It can be thought as a limiting case of Fourier series. Let $f(x)$ be a function defined on $(-\infty, \infty)$. The aim is to construct a Fourier expansion for $f(x)$ in terms of basic trigonometric functions. One evident approach is to construct its Fourier series on progressively longer and longer intervals, and then take the limit as their lengths go to infinity. This limiting process converts the Fourier sums into integrals, and the resulting representation of a function is renamed the Fourier transform. Since we are dealing with an infinite interval, there are no longer any periodicity requirements on the function $f(x)$. Moreover, the frequencies represented in the Fourier transform are no longer constrained by the length of the interval, and so we are effectively decomposing a quite general, non-periodic function into a continuous superposition of trigonometric functions of all possible frequencies. There are two types of Fourier transforms, which are the continuous Fourier transform, and the discrete Fourier transform. We shall give the definition of Fourier transforms in one and two - dimensional dimensional spaces of continuous function.

3.2. Definitions and Properties of the Continuous Fourier Transform

Let $f(x)$ be a continuous function the n- dimensional Fourier transform defined as:

$$\hat{f}(k) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{-ix.k} f(x) dx \tag{2}$$

The inverse Fourier transform is defined as:

$$f(x) = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{R^n} e^{ix.k} \hat{f}(k) dk \tag{3}$$

Where $k, x \in R^n$

Some its important properties for $k, x \in R$, are;

1. The Fourier transform of Delta function $\delta(x)$ is $\hat{\delta}(k) = 1$.
2. The Fourier transform of Derivative $\frac{\partial^n f}{\partial x^n}$ is $\widehat{\frac{\partial^n f}{\partial x^n}}(k) = (ik)^n \hat{f}(k)$.
3. The Fourier transform of Gaussian e^{-ax^2} is $\frac{1}{(2a)^{1/2}} e^{-k^2/4a}$.
4. The Fourier transform of Translation $f(x-a)$ is $e^{-iak}\hat{f}(k)$.
5. The Fourier transform of Dilation $f(ax)$ is $\hat{f}(k/a)/a$.
6. The Fourier transform of of Convolution $f(x)*g(x)$ is $\hat{f}(k)\hat{g}(k)$.

However, the continuous Fourier transform in two-dimensional is given by:

$$\hat{f}(k_1,k_2) = \frac{1}{2\pi}\int\int_{R^2} e^{-i(k_1x+k_2y)} f(x,y)dxdy \tag{4}$$

In addition, its inverse is given by:

$$f(x,y) = \frac{1}{2\pi}\int\int_{R^2} e^{i(k_1x+k_2y)} \hat{f}(k_1,k_2)dk_1dk_2 \tag{5}$$

4. Suggested Solution of the Problem

Now we will suggest the Fourier transformations on the space, i.e., on the variable x, y and z, as follow:

$$\hat{h}(k_1,k_2,k_3,t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y+k_3z)} h(x,y,z,t)dxdydz \tag{6}$$

In addition, the inverse Fourier transform as

$$h(x,y,z,t) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)} \hat{h}(k_1,k_2,k_3,t)dk_1dk_2dk_3 \tag{7}$$

From this definition we have the following:

$$\widehat{\frac{\partial^2 h}{\partial x^2}}(k_1,k_2,k_3,t) = -k_1^2\hat{h}(k_1,k_2,k_3,t) \tag{8}$$

$$\widehat{\frac{\partial^2 h}{\partial y^2}}(k_1,k_2,k_3,t) = -k_2^2\hat{h}(k_1,k_2,k_3,t) \tag{9}$$

$$\widehat{\frac{\partial^2 h}{\partial z^2}}(k_1,k_2,k_3,t) = -k_3^2\hat{h}(k_1,k_2,k_3,t) \tag{10}$$

And

$$\begin{aligned}\widehat{\frac{\partial h}{\partial t}}(k_1,k_2,k_3,t) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y+k_3z)}\frac{\partial h}{\partial t}(x,y,z,t)dxdydz \\ &= \frac{\partial}{\partial t}\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_1x+k_2y+k_3z)} h(x,y,z,t)dxdydz\right]\end{aligned}$$

Then for the t variable we have

$$\widehat{\frac{\partial h}{\partial t}}(k_1,k_2,k_3,t) = \frac{\partial \hat{h}}{\partial t}(k_1,k_2,k_3,t) \tag{11}$$

Now apply Fourier transform on Eq. (1) and substitute (8), (9), (10) and (11) we get:

$$-T_xk_1^2\hat{h} - T_yk_2^2\hat{h} - T_zk_3^2\hat{h} = S\frac{\partial \hat{h}}{\partial t} \Rightarrow -(T_xk_1^2 + T_yk_2^2 + T_zk_3^2)\hat{h} = S\frac{\partial \hat{h}}{\partial t}$$

$$\Rightarrow \frac{-(T_xk_1^2 + T_yk_2^2 + T_zk_3^2)}{S} = \frac{1}{\hat{h}}\frac{\partial \hat{h}}{\partial t} \tag{12}$$

And IC is $\hat{h}(k_1,k_2,k_3,0) = f(k_1,k_2,k_3)$

Then Eq. (12) is an ODE and its solution is:

$$\hat{h}(k_1,k_2,k_3,t) = Ae^{\frac{-(T_x k_1^2+T_y k_2^2+T_z k_3^2)}{S}t} \tag{13}$$

From IC we get $A = \hat{f}(k_1,k_2,k_3)$ and then Eq. (13) becomes:

$$\hat{h}(k_1,k_2,k_3,t) = \hat{f}(k_1,k_2,k_3)e^{\frac{-(T_x k_1^2+T_y k_2^2+T_z k_3^2)}{S}t}$$

Now use the inverse of Fourier transform on $\hat{h}$ obtain the solution h as:

$$\begin{aligned} h(x,y,z,t) &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)}\hat{h}(k_1,k_2,k_3,t)dk_1dk_2dk_3 \\ &= \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(k_1x+k_2y+k_3z)}\hat{f}(k_1,k_2,k_3)e^{\frac{-(T_x k_1^2+T_y k_2^2+T_z k_3^2)}{S}t}dk_1dk_2dk_3 \end{aligned}$$

Let

$$K_x = \sqrt{\frac{T_x}{S}t}K_1 - \frac{ix}{2\sqrt{\frac{T_x}{S}t}},\ K_y = \sqrt{\frac{T_y}{S}t}K_2 - \frac{iy}{2\sqrt{\frac{T_y}{S}t}},\ K_z = \sqrt{\frac{T_z}{S}t}K_3 - \frac{iz}{2\sqrt{\frac{T_z}{S}t}}.$$

Then

$$dK_x = \sqrt{\frac{T_x}{S}t}dK_1,\ dK_y = \sqrt{\frac{T_y}{S}t}dK_2,\ dK_z = \sqrt{\frac{T_x}{S}t}dK_3$$

And then

$$dK_1 = \sqrt{\frac{S}{T_x t}}dK_x,\ dK_2 = \sqrt{\frac{S}{T_y t}}dK_y,\ dK_3 = \sqrt{\frac{S}{T_z t}}dK_z$$

Therefore, we have

$h(x,y,z,t) =$

$$\begin{aligned} &\frac{1}{(2\pi)^{\frac{3}{2}}}\sqrt{\frac{S}{T_x t}}\sqrt{\frac{S}{T_y t}}\sqrt{\frac{S}{T_z t}}e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2+k_z^2)}\hat{f}ÌĆ(k_1,k_2,k_3)dk_xdk_ydk_z \\ &= (\frac{S}{2\pi t})^{\frac{3}{2}}\sqrt{\frac{1}{T_xT_yT_z}}\ e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2+k_z^2)}\hat{f}ÌĆ(k_1,k_2,k_3)dk_xdk_ydk_z \end{aligned}$$

Then the solution is:

$$h(x,y,z,t) = B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2+k_z^2)}\hat{f}(k_1,k_2,k_3)dk_xdk_ydk_z \tag{14}$$

Where $B = (\frac{S}{2\pi t})^{\frac{3}{2}}\sqrt{\frac{1}{T_xT_yT_z}}\ e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})}$

Now to simplify the solution we recall

$$\hat{f}(k_x,k_y,k_z) = \frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_xx+k_yy+k_zz)}f(x,y,z)dxdydz$$

Substitute this in Eq. (14) we get:

$$\begin{aligned} h(x,y,z,t) =&B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2+k_z^2)}\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-i(k_xx+k_yy+k_zz)}f(x,y,z)dxdydz\right]dk_xdk_ydk_z \\ =&B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\left[\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2+k_z^2)-i(k_xx+k_yy+k_zz)}dk_xdk_ydk_z\right]dxdydz \end{aligned}$$

It's clear that the magnitude inside the braces is the Fourier transform to the function $e^{-(k_x^2+k_y^2+k_z^2)}$ and by the property (3) above, get the follow:

$$\frac{1}{(2\pi)^{\frac{3}{2}}}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(k_x^2+k_y^2+k_z^2)-i(k_xx+k_yy+k_zz)}dk_xdk_ydk_z = e^{-\frac{x^2+y^2+z^2}{4}}$$

Then the solution of the problem is

$$h(x,y,z,t) = B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)e^{-\frac{x^2+y^2+z^2}{4}}dxdydz \tag{15}$$

Where $B = (\frac{S}{2\pi t})^{\frac{3}{2}}\sqrt{\frac{1}{T_xT_yT_z}}\ e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})}$

5. Applications

Now, three applications are given to illustrate the suggested method.

Example 5.1.
Consider Eq. (1) with the following initial condition (IC) $h(x,y,z,0) = f(x,y,z) = h_0$, where h_0 is constant. We want to find the solution of Eq. (1) with above IC using Eq. (15), so we get

$$\begin{aligned} h(x,y,z,t) =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)e^{-\frac{x^2+y^2+z^2}{4}}dxdydz \\ =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_0 e^{-\frac{x^2+y^2+z^2}{4}}dxdydz \\ =& Bh_0\left[\int_{-\infty}^{\infty} e^{-\frac{x^2}{4}}dx\right]\left[\int_{-\infty}^{\infty} e^{-\frac{y^2}{4}}dy\right]\left[\int_{-\infty}^{\infty} e^{-\frac{z^2}{4}}dz\right] \\ =& Bh_0\left[2\sqrt{\pi}\right]\left[2\sqrt{\pi}\right]\left[2\sqrt{\pi}\right] \\ \Rightarrow h(x,y,z,t) =& h_0\sqrt{\frac{8S^3}{t^3T_xT_yT_z}}\, e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})} \end{aligned}$$

Example 5.2.
Consider Eq. (1) with the following initial condition (IC) $h(x,y,z,0) = f(x,y,z) = h_0e^{ax^2+by^2+cz^2}$, where $a,b,candh_0$ are constants. We want to find the solution of Eq. (1) with above IC using Eq. (15), so we get

$$\begin{aligned} h(x,y,z,t) =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)e^{-\frac{x^2+y^2+z^2}{4}}dxdydz \\ =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_0e^{ax^2+by^2+cz^2} e^{-\frac{x^2+y^2+z^2}{4}}dxdydz \\ =& Bh_0\left[\int_{-\infty}^{\infty} e^{-(\frac{1}{4}-a)x^2}dx\right]\left[\int_{-\infty}^{\infty} e^{-(\frac{1}{4}-a)y^2}dy\right]\left[\int_{-\infty}^{\infty} e^{-(\frac{1}{4}-a)z^2}dz\right] \\ =& Bh_0\left[\frac{2\sqrt{\pi}}{\sqrt{1-4a}}\right]\left[\frac{2\sqrt{\pi}}{\sqrt{1-4b}}\right]\left[\frac{2\sqrt{\pi}}{\sqrt{1-4c}}\right] \\ \Rightarrow h(x,y,z,t) =& \frac{h_0}{\sqrt{(1-4a)(1-4b)(1-4c)}}\sqrt{\frac{8S^3}{t^3T_xT_yT_z}}\, e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})} \\ & Wherea,b,c \in ïČŐ(-\inftyïĆě,1/4). \end{aligned}$$

Example 5.3.
Consider Eq. (1) with the following initial condition (IC) $h(x,y,z,0) = f(x,y,z) = ax^2 + by^2 + cz^2 + h_0$, where $a,b,candh_0$ are constants. We want to find the solution of Eq. (1) with above IC using Eq. (15), so we get

$$\begin{aligned} h(x,y,z,t) =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)e^{-\frac{x^2+y^2+z^2}{4}}dxdydz \\ =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (ax^2+by^2+cz^2+h_0)e^{-\frac{x^2+y^2+z^2}{4}}dxdydz \\ =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}(ax^2+by^2+cz^2+h_0)e^{-\frac{x^2}{4}}dx\right]e^{-\frac{y^2+z^2}{4}}dydz \\ =& B\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(4a\sqrt{\pi}+2by^2\sqrt{\pi}+2cz^2\sqrt{\pi}+2h_0\sqrt{\pi})e^{-\frac{y^2+z^2}{4}}dydz \\ =& B\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty}(4a\sqrt{\pi}+2by^2\sqrt{\pi}+2cz^2\sqrt{\pi}+2h_0\sqrt{\pi})e^{-\frac{y^2}{4}}dy\right]e^{-\frac{z^2}{4}}dz \\ =& B\int_{-\infty}^{\infty}(8a\pi+8b\pi+4cz^2\pi+4h_0\pi)e^{-\frac{z^2}{4}}dz \\ =& (\frac{S}{2\pi t})^{\frac{3}{2}}\sqrt{\frac{1}{T_xT_yT_z}}\, e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})}(16a\pi+16b\pi+16cz^2\pi+8h_0\pi) \\ \Rightarrow h(x,y,z,t) =& (2a+2b+2c+h_0)\sqrt{\frac{8S^3}{t^3\,T_xT_yT_z}}\, e^{-\frac{S}{4t}(\frac{x^2}{T_x}+\frac{y^2}{T_y}+\frac{z^2}{T_z})} \end{aligned}$$

6. Conclusion

The approximate analytical solution of groundwater flow equation for aquifer was solved by applying Fourier transforms method. We see that the suggested method is efficient, easy implementation compared with other methods. The two important parameters viz., hydraulic conductivity and specific yield S are considered in the present groundwater flow problem. The analytical solution is obtained.

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