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Numerical algorithm based on Bernstein polynomials for solving nonlinear fractional diffusion-wave equation

Research Article

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Abstract: In this paper a simple numerical algorithm based on Bernstein polynomials is presented and analyzed for obtaining numerical solution of non-linear time fractional diffusion and wave-diffusion problems. The method is essentially based on reducing the differential equations with their initial and boundary conditions to a system of algebraic equations in the expansion coefficients of the sought for solutions. The equations are solved by collocation method. Two numerical examples are considered to ascertain the validity, wide applicability and efficiency of the proposed method. The obtained numerical results are compared with those obtained from known analytical solutions and are found to be very accurate and better than those obtained by others employing different techniques.

MSC: 35L05 • 35R11

Keywords: Bernstein polynomials • Fractional calculus • Fractional diffusion-wave equation

1. Introduction

The first appearance of the derivatives of non-integer order was introduced in a letter by the famous mathematician Leibnitz in 1695 to Guillaume de l'Hospital where the meaning of the derivative of order $\frac{1}{2}$ was discussed. Leibnitz's note influenced Liouville, Grünwald, Letnikov and Riemann to study about the theory of derivatives and integrals of arbitrary order by the end of 19^{th} century. As far as the existence of such a theory is concerned, the groundwork of the subject were laid by Liouville in a paper from 1832. For three centuries the theory of fractional derivatives developed mainly as a pure theoretical field of mathematics and physics.

In the past few decades, the study of numerical solutions for approximating fractional derivatives has been extensively studied by many authors. Fractional calculus has been the focus of many researches due to its frequent appearance in numerous seemingly diverse and widespread fields of mathematical physics and engineering applications, for example, in the fields of viscoelasticity, diffusing procedures, electro-chemistry, fluid flow in porous materials, anomalous diffusion transport, wave propagation, signal processing, financial theory, electric conductance of biological system, rheology, dynamical processes in self-similar and porous structures, probability and statistics, control theory of dynamical systems, chemical physics, optics, thermoelasticity and so on (cf. Podlubny[1], Kilbas et al.[2], Das[3], Parthiban and Balachandran[4], Belarbi et al.[5], Kumar et al.[6], Singh et al.[7], Ebadian et al.[8], Misra et al.[9, 10], Jassim[11]). Recently, different numerical methods have been proposed in the literature to solve fractional differential equations (FDEs)(cf. Sweilam et al.[12], Sweilam and Khader[13], Al-Bar[14]) and others etc. The purpose of this paper is to develop a numerical algorithm to solve nonlinear fractional diffusion-wave equation problems.

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Various forms of fractional differential equations have been presented in mathematical models, and there has been several methods to solve them. These methods include Laplace transforms (cf. Podlubny[1]), adomian decomposition method (ADM) (cf. Momani[15]), variational iteration method(VIM) (cf. Sweilam et al.[12], Das[16]), fractional differential transform method (FDTM) (cf. Arikoglu and Ozkol[17]), fractional difference method (FDM)(cf. Meerschaert and Tadjeran[18]), fractional sub-equation method(cf. Kadkhoda and Jafari[19]), theta method(cf. Aslefallah et al.[20]), homotopy perturbation method(cf.Gupta et al.[21], Jassim[11]) etc.

The time fractional diffusion-wave equation is obtained from the traditional diffusion or wave equation after replacing the first- or second-order time derivative by a fractional derivative. It was introduced to describe diffusion in special type of porous media. It is due to the fact that a realistic modeling of a physical phenomenon having dependence on not only at the time instant, but also the preceding time history can be successfully accomplished by using fractional calculus. In recent years a great progress has been made in extending the different models for diffusion incorporating this fractional diffusion. The tools of fractional calculus have proven to be very useful in these developments, for the applicability of fractional constitutive laws.

Here we have considered the following class of time fractional diffusion and wave-diffusion equations

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}+\frac{\partial u(x,t)}{\partial t}=\frac{\partial^2 u(x,t)}{\partial x^2}+v(x,t), \quad (x,t)\in[0,1]\times[0,1], \quad 1<\alpha\le 2 \tag{1}$$

subject to the initial conditions

$$u(x,0)=f_0(x), \qquad \frac{\partial u(x,0)}{\partial t}=f_1(x), \qquad x\in[0,1] \tag{2}$$

and boundary conditions

$$u(0,t)=g_0(t), \qquad u(1,t)=g_1(t), \qquad t\in[0,1] \tag{3}$$

where x and t are the space and time variables, v is a known function in $L^2([0,1]\times[0,1])$ and f_0, f_1, g_0, g_1 are given functions in $L^2[0,1]$ and $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ denotes the α^{th} order Caputo fractional derivative of $u(x,t)$.

When $0<\alpha<1$, Eq. (1) represents the time fractional diffusion-wave equation. When $\alpha=1$, it becomes the traditional diffusion equation and when $\alpha=2$, it illustrates a traditional wave equation.

Recently, Ebadian et al.[8] proposed triangular function(TF) method for solving this type of nonlinear fractional diffusion-wave equation where fractional operational matrix of integration for the TFs was derived.

In this paper, we have developed a simple numerical algorithm based on the method of polynomial approximation in the Bernstein polynomial basis. Here we have introduced a truncated expansion for the unknown function in terms of Bernstein polynomials and used it to reduce the problem into a system of algebraic equations after using collocation points. Bernstein polynomials have been used in the literature to solve several linear and nonlinear differential equations, ordinary and partial, approximately (cf. Bhatta and Bhatti[22], Bhatti and Bracken[23]) and to solve various integral equations(cf. Mandal and Bhattacharya[24], Bhattacharya and Mandal[25]). These polynomials defined on an interval form a complete basis over the interval. The sum of these polynomials is unity, each of them being positive.

The coefficients of the truncated expansion are obtained by solving the reduced system of algebraic equations and the approximate series expansion of $u(x,t)$ are then calculated. This method is applied to two mathematical models and the calculated absolute errors of the proposed computational method at some points are represented in tables. Both the numerical experiments confirm that the Bernstein polynomial approximation method is computationally highly efficient and user-friendly for solving such problems.

2. Preliminaries

2.1. Basic definitions of fractional integrals and derivative operators

Definition 2.1.
A function $f(x)\in\mathbb{R},\ x>0$ is said to be in the $\mathbb{C}_\mu$ space, $\mu\in\mathbb{R}$ if there exists a real number $p>\mu$, such that $f(x)=x^p g(x)$, where $g(x)\in[0,\infty)$ and it is said to be in the space $\mathbb{C}_\mu^m$ iff $f^{(m)}\in\mathbb{C}_\mu,\ m\in\mathbb{N}$.

Definition 2.2.
The Riemann-Liouville fractional integral operator J_a^α of order α, generalized from the repeated n-fold integration by Gamma function for the factorial expression is defined on $L_1[a,b]$ by

$$J_a^\alpha f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x (x-\tau)^{\alpha-1}f(\tau)d\tau,\ \alpha>0,\ a\le x\le b. \tag{4}$$

Definition 2.3.
The expression for Riemann-Liouville fractional derivative operator D_a^α of order α $(n-1<\alpha\leq n)$(left hand definition (LHD)) is given by (cf. Podlubny[1])

$$D_a^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_a^x\frac{f(\tau)}{(x-\tau)^{\alpha+1-n}}d\tau,\ n\in\mathbb{N},\ \alpha>0,\ a\leq x\leq b. \tag{5}$$

Definition 2.4.
The expression for Caputo fractional derivative operator ${}^cD_a^\alpha$ of order α $(n-1<\alpha\leq n)$(right hand definition (RHD)) is given by (cf. Caputo and Mainardi[26])

$${}^cD_a^\alpha f(x)=\frac{1}{\Gamma(n-\alpha)}\int_a^x\frac{f^{(n)}(\tau)}{(x-\tau)^{\alpha+1-n}}d\tau,\ n\in\mathbb{N},\ \alpha>0,\ a\leq x\leq b. \tag{6}$$

Properties:
Caputo fractional derivative operator is a linear operator similar to integer order differentiation so that

$$D^\alpha(\lambda f(x)+\mu g(x))=\lambda D^\alpha f(x)+\mu D^\alpha g(x) \tag{7}$$

where λ and μ are constants.
Caputo derivative satisfies

$$D^\alpha C=0,\ C \text{ being a constant},$$

$$D^\alpha x^\beta=\begin{cases}0 & \text{for } \beta\in\mathbb{N}_0 \text{ and } \beta<\lceil\alpha\rceil\\ \frac{\Gamma(\beta+1)}{\Gamma(\beta+1-\alpha)}x^{\beta-\alpha} & \text{for } \beta\in\mathbb{N}_0 \text{ and } \beta\geq\lceil\alpha\rceil\end{cases} \tag{8}$$

where the ceiling function $\lceil\alpha\rceil$ denotes the smallest integer greater than or equal to α and $\mathbb{N}_0=\{0,1,2,....\}$. For $\alpha\in\mathbb{N}$, Caputo differential operator coincides with the usual differential operator of integer order.

2.2. Bernstein polynomials and their properties

The Bernstein polynomials of degree n are defined on the interval $[a,b]$ as

$$B_{i,n}(x)=\binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n},\quad i=0,1,......,n \tag{9}$$

where $\binom{n}{i}$ is a binomial coefficient.
The Bernstein basis polynomials of degree n form a basis for the vector space $\prod_n$ of polynomials of degree at most n. These polynomials defined on an interval form a complete basis over the interval $[a,b]$. The sum of these polynomials is unity, each of them being positive.

3. General method of solution

In this section, we introduce a numerical algorithm using Bernstein polynomials as basis functions for solving non-linear fractional diffusion wave equation of the form (1). In order to apply the Bernstein polynomials in the interval $[0,1]$, $B_{i,n}(x)$ is now defined as

$$B_{i,n}(x)=\binom{n}{i}x^i(1-x)^{n-i},\ i=0,1,2,......,n \tag{10}$$

Since the set $\{B_{i,n}(x)\}_{i=0}^n$ in Hilbert space $L_2[0,1]$ is a complete basis, we can write any polynomial $u_{m,q}(x,t)$ of degree $m+q$ in terms of linear combination of $\{B_{i,m}(x)\}_{i=0}^m$ and $\{B_{l,q}(x)\}_{l=0}^q$

$$u_{m,q}(x,t)=\sum_{k=0}^m\sum_{l=0}^q c_{k,l}B_{k,m}(x)B_{l,q}(t). \tag{11}$$

Another form of Bernstein polynomials of degree n in the interval $[0,1]$ is given by

$$B_{i,n}(x)=\sum_{p=0}^{n-i}\binom{n}{i}\binom{n-i}{p}(-1)^px^{i+p}. \tag{12}$$

Since Caputo fractional differentiation is a linear operation we have from (11)

$$D^{\alpha}(u_{m,q}(x,t)) = \sum_{k=0}^{m}\sum_{l=0}^{q} c_{k,l}B_{k,m}(x)D^{\alpha}(B_{l,q}(t)) \tag{13}$$

where $D^{\alpha} = \frac{\partial^{\alpha}}{\partial t^{\alpha}}$.
Using (12) and (6) we calculate $D^{\alpha}(B_{l,q}(t))$ as

$$\begin{aligned} D^{\alpha}(B_{l,q}(t)) &= \sum_{r=0}^{q-l}\binom{q}{l}\binom{q-l}{r}(-1)^{r}D^{\alpha}(t^{l+r}) \\ &= \sum_{r=0}^{q-l}\binom{q}{l}\binom{q-l}{r}(-1)^{r}\frac{\Gamma(l+r+1)}{\Gamma(l+r+1-\alpha)}t^{l+r-\alpha} \ \text{ for } (l+r)\geq\lceil\alpha\rceil. \end{aligned} \tag{14}$$

Since $B_{l,q}(t)$ is a polynomial of degree q, we have

$$D^{\alpha}(B_{l,q}(x)) = 0 \text{ for all } q = 0,1,2,.....,\lceil\alpha\rceil-1,\ \alpha>0. \tag{15}$$

Merging of (13), (14), (15) produces the following form

$$D^{\alpha}(u_{m,q}(x,t)) = \sum_{k=0}^{m}\sum_{l=0}^{q}\sum_{r=\lceil\alpha\rceil-l\geq 0}^{q-l} A^{(\alpha)}_{l,q,r}c_{k,l}B_{k,m}(x)t^{l+r-\alpha} \tag{16}$$

where $A^{(\alpha)}_{k,m,p}$ is given by

$$A^{(\alpha)}_{l,q,r} = \binom{q}{l}\binom{q-l}{r}(-1)^{r}\frac{\Gamma(l+r+1)}{\Gamma(l+r+1-\alpha)}. \tag{17}$$

Next we will derive a discretization formula of (1) using simple collocation method. We have already approximated $u(x,t)$ as

$$u(x,t)\approx u_{m,q}(x,t) = \sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{k,m}(x)B_{l,q}(t). \tag{18}$$

Using (17) and (18) in (1) we obtain

$$\frac{\partial^{\alpha}u_{m,q}(x,t)}{\partial t^{\alpha}}+\frac{\partial u_{m,q}(x,t)}{\partial t} = \frac{\partial^{2}u_{m,q}(x,t)}{\partial x^{2}}+v_{m,q}(x,t),\ (x,t)\in[0,1]\times[0,1], \qquad 1<\alpha\leq 2$$

i.e.

$$\begin{aligned} \sum_{k=0}^{m}\sum_{l=0}^{q}\sum_{r=\lceil\alpha\rceil-l\geq 0}^{q-l} A^{(\alpha)}_{l,q,r}c_{k,l}B_{k,m}(x)t^{l+r-\alpha} + \sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{k,m}(x)\frac{\partial}{\partial t}(B_{l,q}(t)) \\ = \sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{l,q}(t)\frac{\partial^{2}}{\partial x^{2}}(B_{k,m}(x))+v_{m,q}(x,t) \end{aligned} \tag{19}$$

After collocating at points $x_i,\ i=0,1,...,m-\lceil\alpha\rceil,\ t_j,\ j=0,1,...,q-\lceil\alpha\rceil$ we get

$$\begin{aligned} \sum_{k=0}^{m}\sum_{l=0}^{q}\sum_{r=\lceil\alpha\rceil-l\geq 0}^{q-l} A^{(\alpha)}_{l,q,r}c_{k,l}B_{k,m}(x_i)t_j^{l+r-\alpha} + \sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{k,m}(x_i)\frac{\partial}{\partial t}(B_{l,q}(t_j)) \\ = \sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{l,q}(t_j)\frac{\partial^{2}}{\partial x^{2}}(B_{k,m}(x_i))+v_{m,q}(x_i,t_j) \end{aligned} \tag{20}$$

Let us now substitute (16) and (18) in the initial conditions (2) and boundary conditions (3). Then we get further $\lceil\alpha\rceil(2-\lceil\alpha\rceil+m+q)$ equations given by

$$\begin{aligned} &\sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{k,m}(x_i)\delta_{l,0} = f_0(x_i), \qquad \sum_{k=0}^{m}\sum_{l=0}^{q}A^{\lceil\alpha\rceil}_{l,q,\lceil\alpha\rceil-l}c_{k,l}B_{k,m}(x_i) = f_1(x_i), \\ &\sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{l,q}(t_j)\delta_{k,0} = g_0(t_j), \qquad \sum_{k=0}^{m}\sum_{l=0}^{q}c_{k,l}B_{l,q}(t_j)\delta_{k,m} = g_1(t_j), \\ &\qquad\qquad i=0,1,...,m+1-\lceil\alpha\rceil,\ j=0,1,...,q+1-\lceil\alpha\rceil. \end{aligned} \tag{21}$$

Hence Eq. (20), together with $\lceil\alpha\rceil(2-\lceil\alpha\rceil+m+q)$ equations of (21) gives $(m+1)(q+1)$ equations which can be solved for the $(m+1)(q+1)$ unknowns $c_{k,l},\ k=0(1)m,\ l=0(1)q$ using an appropriate numerical method. Finally, the function $u(x,t)$ approximated by (11) can be obtained.

4. Numerical examples

In this section, we present two examples of fractional diffusion wave equation problems, to illustrate the method discussed here.

Example 4.1.
Let us consider the following fractional diffusion wave equation studied by Ebadian et al.[8]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{6t^{3-\alpha}}{\Gamma(4-\alpha)} e^x + 3t^2 e^x - t^3 e^x, \qquad (x,t) \in [0,1]\times[0,1], \quad 1<\alpha\le 2$$

subject to the initial conditions

$$u(x,0) = 0, \qquad \frac{\partial u(x,0)}{\partial t} = 0, \qquad x \in [0,1]$$

and boundary conditions

$$u(0,t) = t^3, \qquad u(1,t) = et^3, \qquad t \in [0,1]$$

whose exact solution is

$$u_{\text{exact}}(x,t) = t^3 e^x.$$

We implement our method with $m = 15,\ q = 15$ for $\alpha = 1.2,\ 1.4,\ 1.6,\ 1.8$. The obtained absolute errors by the present method at some points within $(x,t) \in [0,1]\times[0,1]$ given by $|u_{\text{exact}}(x,t) - u_{\text{BPM}}(x,t)|$ are presented in Table 1 which show that our method is highly accurate.

Table 1. Comparison of results for Example 4.1

(x,t)	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
(0.0,0.0)	2.359E-06	5.424E-05	1.536E-06	5.587E-07
(0.1,0.1)	4.675E-06	1.823E-04	8.961E-06	4.955E-06
(0.2,0.2)	3.313E-06	1.068E-04	4.469E-06	3.483E-06
(0.3,0.3)	2.039E-06	6.178E-05	2.926E-06	2.614E-06
(0.4,0.4)	9.263E-07	2.284E-05	5.679E-07	7.863E-07
(0.5,0.5)	6.099E-10	6.823E-09	2.145E-09	9.924E-10
(0.6,0.6)	8.377E-07	1.737E-05	1.336E-07	2.989E-07
(0.7,0.7)	1.474E-06	8.392E-05	2.852E-06	1.169E-06
(0.8,0.8)	6.173E-06	1.227E-03	2.211E-06	2.339E-06
(0.9,0.9)	3.847E-05	1.376E-02	2.970E-04	6.000E-05

Example 4.2.
Let us consider the following fractional diffusion wave equation with damping studied by Ebadian et al.[8]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + \frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} v(x,t), \qquad (x,t) \in [0,1]\times[0,1], \quad 1<\alpha\le 2$$

where

$$v(x,t) = \frac{2x(1-x)}{\Gamma(3-\alpha)} t^{2-\alpha} + 2xt(1-x) + 2t^2$$

subject to the initial conditions

$$u(x,0) = 0, \qquad \frac{\partial u(x,0)}{\partial t} = 0, \qquad x \in [0,1]$$

and boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = 0, \qquad t \in [0,1]$$

whose exact solution is

$$u_{\text{exact}}(x,t) = t^2 x(1-x).$$

We implement our method with $m = 15,\ q = 15$ for $\alpha = 1.2,\ 1.4,\ 1.6,\ 1.8$. The obtained absolute errors by the present method at some points within $(x,t) \in [0,1]\times[0,1]$ given by $|u_{\text{exact}}(x,t) - u_{\text{BPM}}(x,t)|$ are presented in Table 2 which show that our method is highly effective.

Table 2. Comparison of results for Example 4.2

(x,t)	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
(0.0,0.0)	4.034E-06	2.073E-05	1.024E-05	4.265E-06
(0.1,0.1)	7.994E-06	5.995E-05	4.095E-05	2.645E-05
(0.2,0.2)	5.666E-06	4.075E-05	2.831E-05	1.734E-05
(0.3,0.3)	3.487E-06	2.367E-05	1.854E-05	1.662E-05
(0.4,0.4)	1.584E-06	8.768E-06	4.473E-06	4.511E-06
(0.5,0.5)	1.658E-09	3.878E-09	7.672E-10	2.862E-09
(0.6,0.6)	1.447E-06	5.880E-06	6.492E-07	2.745E-06
(0.7,0.7)	3.393E-06	1.101E-05	2.182E-06	3.774E-06
(0.8,0.8)	1.188E-05	2.446E-05	1.226E-05	4.849E-06
(0.9,0.9)	1.541E-04	2.738E-04	7.291E-05	1.314E-04

5. Conclusion

In this paper, we have presented a simple numerical algorithm to solve fractional diffusion wave equation. Here we approximate the unknown function in terms of a truncated series involving Bernstein polynomials. The properties of Bernstein polynomials were used to reduce the fractional problems to the solution of algebraic equations by avoiding the appearance of ill-conditioned matrices or complicated integrations. By adopting a few terms for the truncated series we observed a great accuracy between the exact and the approximate solutions computed numerically. The accuracy of the obtained solution can be further improved by increasing the number of terms of the Bernstein polynomial.

Thus, the proposed method can be utilized as a powerful solver for the solution of fractional order diffusion wave equation problems. As compared to the other methods, the present method appears more convenient and simple in principle and for computer algorithms.

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