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Stability analysis of a non-linear HIV/AIDS epidemic model with vaccination and antiretroviral therapy

Research Article

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Abstract: In this paper, we like to propose and analyze a non-linear HIV/AIDS epidemic model with vaccination and antiretroviral therapy. For our convenient study we have divided the total populations into five classes such as susceptible class, unaware HIV infected class, aware HIV infected class, pre AIDS class and AIDS class respectively. For our present purpose we have taken only the disease spread through horizontal transmission into consideration. In this paper we have tried to develop a non-linear HIV/AIDS mathematical model to study the transmission dynamics of HIV at four compartments of the populations with vaccination and antiretroviral therapy and to prove the positivity and boundedness of its solutions. In this paper we have added a treatment procedure i.e. antiretroviral therapy and tried to find out its effect. We have also analyzed the stability behaviour of the system. Finally we have found that vaccination and antiretroviral therapy is an effective way to control the disease transmission. The mathematical model solved numerically by using an iterative numerical recipe which supports the theoretical or analytical results.

MSC: 92B05

Keywords: HIV/AIDS Epidemic model • Antiretroviral therapy • Basic reproduction number • Stability • Vaccination • Numerical results

1. Introduction

HIV/AIDS is one of the most enormous diseases humankind has already ever faced all over the World. The number of people living with HIV is around 8 million in 1990 and it increases to 34 million at the end of 2011 (USAID, 2012 [1]. The first AIDS (Acquired Immunodeficiency Syndrome) case was documented by Centers of Disease Control (CDC) of America in 1981 [2]. AIDS is very dangerous disease that suppresses the normal function of the immune system of human body and such types of virus is called Human Immunodeficiency Virus or HIV. Such types of diseases now spread Worldwide [3, 4]. The first known case of HIV in India was diagnosed amongst female sex workers in Chennai in 1986. In the next year 1987 more or less than 135 new cases were reported. The first National AIDS Control Programme was launched in 1992, and National AIDS Control Organisation (NACO) was constituted to implement the programme in India through 35 HIV/AIDS Prevention and Control Societies [5].

Mathematical models play a crucial part to study the disease transmission and shows the path to treat the HIV/AIDS affected patient. Many a Mathematician have already introduced various mathematical model about such disease [6–10]. From the initial models of May and Anderson [11–13] various refinements have been added into modelling

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frameworks and specific issues have been expected by researchers [14–16]. Their preliminary endeavour is how to prevent AIDS from spreading [17]. There is so many difficulty to treating HIV infected patient. Numerous health organization has taken initiative to arrange programme to slow down the spreading of this virus [27]. Mainly, AIDS is spread by sexual transmission and vertical transmission from mother to child at birth and through breast milk [18]. Vaccination and antiretroviral therapy are also reduces the transmission rate of HIV [19].

Treatment (vaccination and antiretroviral therapy) procedure can not eliminate HIV virus completely from the body but it can provide some resistance or slow down the processes of disease transmission. Although the treatment procedure is very slow and currently there is no effective vaccine. Antiretroviral therapy is very expensive and routine access of ART medicine is not available in all countries of the World [20, 21]. If we can avail the treatment processes properly then the transmission rate HIV/AIDS will be diminished. In this paper we have developed a non-linear HIV/AIDS Mathematical model to study the transmission dynamics of HIV disease at four compartments of the populations with vaccination and antiretroviral therapy. The Mathematical model is solved numerically by using an iterative numerical recipe, which support the theoretical results.

2. The mathematical model

Here we consider a non-linear HIV/AIDS epidemic model with Vaccination and antiretroviral therapy. We divided the total populations into five subclasses that is; the susceptible class $S(t)$, the HIV infected un-aware class $U(t)$, the HIV infected aware class $W(t)$, the pre AIDS class $P(t)$, the AIDS class $A(t)$. Our assumption is that the disease spread only by horizontal transmission i.e. sexual contact, blood transfusion. The disease is spread by unprotected sexual contact at optimum level. Here we may assume that aware HIV infected class does not spread the disease by unprotected sexual contact due to their proper awareness about the disease. Our mathematical model has been built up on the basis of this assumptions.

$$\left.\begin{aligned}
\frac{dS}{dt} &= R_s - \frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\sigma)S,\\
\frac{dU}{dt} &= \frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\gamma+\eta)U,\\
\frac{dW}{dt} &= \eta U - (\mu+\gamma_1)W,\\
\frac{dP}{dt} &= \gamma U + \gamma_1 W - (\mu+\varepsilon)P,\\
\frac{dA}{dt} &= \varepsilon P - (\mu+\alpha)A.
\end{aligned}\right\} \tag{1}$$

Where $S(0)>0$, $U(0)>0$, $W(0)>0$, $P(0)>0$ and $A(0)>0$, $\forall\ t\geq 0$.
Since the state variable A of the system (1) does not appear in the first four equations. Therefore A class do not transmit infection and we consider the following subsystem:

$$\left.\begin{aligned}
\frac{dS}{dt} &= R_s - \frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\sigma)S,\\
\frac{dU}{dt} &= \frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\gamma+\eta)U,\\
\frac{dW}{dt} &= \eta U - (\mu+\gamma_1)W,\\
\frac{dP}{dt} &= \gamma U + \gamma_1 W - (\mu+\varepsilon)P,
\end{aligned}\right\} \tag{2}$$

Where, R_s is the constant recruitment rate of susceptible class, β is the per capita contact rate between susceptible class and un-aware HIV infected class, α_1, α_2 are two positive constants, γ is the conversion rate of un-aware HIV infected class to AIDS class, γ_1 is the conversion rate of aware HIV infected class to pre AIDS class, σ is the successful vaccination rate of the susceptible individuals, ε is the rate by which pre AIDS infective individuals develop AIDS, μ is the natural mortality rate of adult class, α is the AIDS related death rate. η is the antiretroviral therapy rate of an un-aware HIV infected class. The system (2) has to be analysed with the following initial conditions, $S(0)>0$, $U(0)>0$, $W(0)>0$ and $P(0)>0$.

3. Basic properties of the model

3.1. Positivity of the Solutions

Since our proposed mathematical model indicates human population we must show that all the state variables remain non-negative $\forall$ times.

Lemma 3.1.
All the solutions S, U, W and P for the system (2) is non-negative $\forall\ t \geq 0$ *with the initials condition* $S(0) > 0$, $U(0) > 0$,$W(0) > 0$ *and* $P(0) > 0$ *in the region* Γ.

Proof. Taking the first equation of the system (2) in consideration, we get

$$\frac{dS}{dt} = R_s - \frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\sigma)S = R_s - \left\{\frac{\beta U}{1+\alpha_1 U+\alpha_2 U^2} + (\mu+\sigma)\right\} S.$$

$$\frac{dS}{dt} > -\left\{\frac{\beta U}{1+\alpha_1 U+\alpha_2 U^2} + (\mu+\sigma)\right\} S.$$

$$S(t) > S(0)\, e^{-\int\left\{\frac{\beta_1 U(\xi)}{1+\alpha_1 U(\xi)+\alpha_2 U^2(\xi)}+(\mu+\delta)\right\}d\xi} \geqslant 0$$

Considering the second equation of the system (2), we get

$$\frac{dU}{dt} = \frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\gamma+\eta)U > -(\mu+\gamma+\eta)U.$$

$$U(t) > U(0)e^{-(\mu+\gamma+\eta)t} \geqslant 0.$$

Similarly we can prove that $W(t) \geq W(0)\ e^{-(\mu+\gamma_1)t} \geq 0$ and $P(t) \geq P(0)\ e^{-(\mu+\varepsilon)t} \geq 0$
Therefore all solutions of the system (2) are non-negative $\forall\ t \geq 0$. □

3.2. Invariant Region and Boundedness

Lemma 3.2.
All the non-negative solutions of system (2) are contained in the feasible region

$$\Gamma = \left\{(S,\ U,\ W,\ P) \in R_+^4 : 0 \leq S+U+W+P \leq \frac{R_s}{(\mu+\sigma)}\right\}$$

is bounded.

Proof. We have from the system (2)

$$\frac{dS}{dt} + \frac{dU}{dt} + \frac{dW}{dt} + \frac{dP}{dt} = R_s - \mu(S+U+W+P) - \sigma S - \varepsilon P \leqslant R_s - \mu(S+U+W+P).$$

Hence $\lim_{t\to\infty} \sup\ (S+U+W+P) \leq \frac{R_s}{(\mu+\sigma)}$.
Therefore all the non-negative solutions of the system (2) ultimately lies in the region

$$\Gamma = \left\{(S,\ U,\ W,\ P) \in R_+^4 : 0 \leq S+U+W+P \leq \frac{R_s}{(\mu+\sigma)}\right\}.$$

Hence the vector field points into the interior of $\Gamma\ \forall\ t \geq 0$ is positively invariant. □

4. Equilibrium points and basic reproduction Number

The system (2) possesses the following equilibria namely,

(i) The disease free equilibrium $E_0 = (\frac{R_s}{(\mu+\sigma)}, 0, 0, 0)$,

(ii) The endemic equilibrium $E^* = (S^*, U^*, W^*, P^*)$ with

$$S^* = \frac{R_s(1+\alpha_1 U^*+\alpha_2 U^{*2})}{(\mu+\sigma)(1+\alpha_1 U^*+\alpha_2 U^{*2})+\beta U^*},$$

$$W^* = \frac{\eta U^*}{(\gamma_1+\mu)},$$

$$P^* = \frac{(\mu\gamma+\gamma\gamma_1+\eta\gamma_1)U^*}{(\mu+\varepsilon)(\gamma_1+\mu)}$$

and U^* is the root of the quadratic equation $a'U*^2 + b'U^* + c' = 0$,
where

$$a' = \alpha_2(\mu+\sigma)(\mu+\gamma+\eta) > 0,$$

$$b' = \alpha_1(\mu+\sigma)(\mu+\gamma+\eta) + \beta(\mu+\gamma+\eta) > 0,$$

$$c' = (\mu+\sigma)(\mu+\gamma+\eta)(1-R_0).$$

If $R_0 > 1$, then $c' < 0$. Therefore the above quadratic equation has a positive root if $R_0 > 1$, where R_0 is a basic reproduction number given as follows

$$R_0 = \frac{R_s\beta}{(\mu+\sigma)(\mu+\gamma+\eta)}.$$

Hence

$$U^* = \frac{-b' + \sqrt{\triangle}}{2a'}$$

where,

$$\triangle = \alpha_1(\mu+\sigma)(\mu+\gamma+\eta) + \beta(\mu+\gamma+\eta)^2 - 4\alpha_2(\mu+\sigma)(\mu+\gamma+\eta)(\mu+\sigma)(\mu+\gamma+\eta)(1-R_0).$$

If $R_0 < 1$, then there is no positive equilibrium. The local stability of E_0 and E^* is governed by the basic reproduction number R_0 which we may find by using the next generation matrix. The non-negative matrix, f' of the infection terms and the non singular matrix v' of the transition terms are

$$f' = \begin{pmatrix} \frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} \\ 0 \\ 0 \end{pmatrix}$$

and

$$v' = \begin{pmatrix} (\mu+\gamma+\eta)U \\ -\eta U + (\mu+\gamma_1)W \\ -\gamma U - \gamma_1 W + (\mu+\varepsilon)P \end{pmatrix}$$

We get, f = Jacobian of f' at disease free equilibrium = $\begin{pmatrix} \frac{\beta R_s}{(\mu+\sigma)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and

v = Jacobian of v' at disease free equilibrium = $\begin{pmatrix} (\mu+\gamma+\eta) & 0 & 0 \\ -\eta & (\mu+\gamma_1) & 0 \\ -\gamma & -\gamma_1 & (\mu+\varepsilon) \end{pmatrix}$.

Therefore $R_0 = \rho f v^{-1}$.
Spectral of the matrix $R_0 = \frac{\beta R_s}{(\mu+\sigma)(\mu+\gamma+\eta)}$.

5. Stability Analysis

5.1. Local Stability

In this current section we have found the local stability of the disease free and the endemic equilibrium point. We have presented the local stability of the disease free equilibrium point $E_0 = \left\{\frac{R_s}{(\mu+\sigma)}, 0, 0, 0\right\}$, in the following theorem.

Theorem 5.1.
The disease free equilibrium $E_0 = (\frac{R_s}{(\mu+\sigma)}, 0, 0, 0)$, *is locally asymptotically stable if,* $R_0 < 1$, *and unstable while* $R_0 > 1$.

Proof. The variational matrix around the disease free equilibrium point E_0 is given by

$$J_0 = \begin{pmatrix} -(\mu+\sigma) & -\frac{\beta R_s}{(\mu+\sigma)} & 0 & 0 \\ 0 & \frac{\beta R_s}{(\mu+\sigma)} - (\mu+\gamma+\eta) & 0 & 0 \\ 0 & \eta & -(\mu+\gamma_1) & 0 \\ 0 & \gamma & \gamma_1 & -(\mu+\varepsilon) \end{pmatrix}$$

The characteristics equation associated with the Jacobian matrix J_0 is given by

$$\{-(\mu+\varepsilon)-\lambda\}\{-(\mu+\gamma_1)-\lambda\}\{-(\mu+\sigma)-\lambda\}\left\{\frac{\beta R_s}{(\mu+\sigma)} - (\mu+\gamma+\eta) - \lambda\right\} = 0$$

The eigenvalues associated to J_0 are

$$\lambda_1 = -(\mu+\sigma) < 0,$$
$$\lambda_2 = -(\mu+\gamma_1) < 0,$$
$$\lambda_3 = -(\mu+\varepsilon) < 0, \quad \text{and}$$
$$\lambda_4 = \frac{\beta R_s}{(\mu+\sigma)} - (\mu+\gamma+\eta) = (R_0-1)(\mu+\gamma+\eta) < 0.$$

It is clear the all the eigenvalues of the Jacobian matrix J_0 is negative when $R_0 < 1$. So the disease free equilibrium is locally asymptotically stable. Further when $R_0 > 1$, then there exists one positive and three negative eigenvalues, so an saddle unstable equilibrium exists. □

5.2. Global stability of disease free equilibrium

Lemma 5.1.
If $R_0 < 1$, then the disease free equilibrium point E_0 of the model (2) is globally asymptotically stable in Γ and unstable if $R_0 > 1$.

Proof. Define Lyapunov function:

$$L = \eta U + (\mu+\gamma+\eta)W$$
$$L' = \eta U' + (\mu+\gamma+\eta)W'$$
$$L' = \eta\left\{\frac{\beta SU}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\gamma+\eta)U\right\} + (\mu+\gamma+\eta)\left\{\eta U - (\mu+\gamma_1)W\right\},$$
$$L' \leqslant \eta(\mu+\gamma+\eta)\left\{\frac{R_0}{1+\alpha_1 U+\alpha_2 U^2} - 1\right\}U \quad \text{Since } S = \frac{R_s}{(\mu+\sigma)}$$

If $U = 0$, $L' = 0$ but if $U \neq 0$ and $R_0 < 1$, $L' < 0$ therefore, the disease free equilibrium is globally asymptotically stable [23]. □

5.3. Local stability of an endemic equilibrium

Lemma 5.2.
The endemic equilibrium point E^* is locally asymptotically stable, when $R_0 > 1$.

Proof. The Jacobian matrix J^* associate with the endemic equilibrium point E^* is given by,

$$J^* = \begin{pmatrix} -\frac{\beta U^*}{1+\alpha_1 U^*+\alpha_2 U^{*2}} - (\mu+\sigma) & -\frac{\beta S^* - \beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} & 0 & 0 \\ \frac{\beta U^*}{1+\alpha_1 U^*+\alpha_2 U^{*2}} & \frac{\beta S^* - \beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} - (\mu+\gamma+\eta) & 0 & 0 \\ 0 & \eta & -(\mu+\gamma_1) & 0 \\ 0 & \gamma & \gamma_1 & -(\mu+\varepsilon) \end{pmatrix}$$

The characteristics equation related to the Jacobian matrix J^* ,

$$\left\{-(\mu+\varepsilon)-\lambda^*\right\}\left\{-\lambda^* - (\mu+\gamma_1)\right\}\left[\left\{-\frac{\beta U^*}{1+\alpha_1 U^*+\alpha_2 U^{*2}} - (\mu+\sigma) - \lambda^*\right\}\right.$$
$$\left.\left\{\frac{\beta S^* - \beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} - (\mu+\gamma+\eta) - \lambda^*)\right\} + \frac{\beta U^*}{1+\alpha_1 U^*+\alpha_2 U^{*2}}\left\{\frac{\beta S^* - \beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2}\right\}\right] = 0.$$

The two eigenvalues associated to the J^* are

$$\lambda_1^* = -(\mu+\varepsilon),$$
$$\lambda_2^* = -(\mu+\gamma_1)$$

and the other two eigenvalues of the quadratic equation

$$p_0\lambda^{*2} + q_0\lambda^* + r_0' = 0$$

where

$$
\begin{aligned}
p_0 &= 1 > 0,\\
q_0 &= \frac{\beta U^*}{1+\alpha_1 U^*+\alpha_2 U^{*2}} + (\mu++\sigma) + (\mu+\gamma+\eta) - \frac{\beta S^*}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} + \frac{\beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2}\\
&= \frac{\beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} + \frac{\beta U^*}{1+\alpha_1 U^*+\alpha_2 U^{*2}} + (\mu++\sigma) + \frac{(\mu+\gamma+\eta)(\alpha_1 U^*+\alpha_2 U^{*2})}{1+\alpha_1 U^*+\alpha_2 U^{*2}} > 0\\
r_0' &= \frac{\beta U^*(\mu+\gamma+\eta)}{1+\alpha_1 U^*+\alpha_2 U^{*2}} + \frac{(\mu+\sigma)\beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} - \frac{\beta S^*(\mu+\sigma)}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} + (\mu++\sigma)(\mu+\gamma+\eta)\\
&= \frac{\beta U^*(\mu+\gamma+\eta)}{1+\alpha_1 U^*+\alpha_2 U^{*2}} + \frac{(\mu+\sigma)\beta S^* U^{*2}\alpha_2}{(1+\alpha_1 U^*+\alpha_2 U^{*2})^2} + \frac{(\mu+\gamma+\eta)(\mu+\sigma)(\alpha_1 U^*+\alpha_2 U^{*2})}{1+\alpha_1 U^*+\alpha_2 U^{*2}} > 0
\end{aligned}
$$

It follows by Routh-Hurwitz Criteria that the roots of the above quadratic equation all have negative real parts. Therefore the endemic equilibrium point E^* is locally asymptotically stable when $R_0 > 1$. □

5.4. Global stability of the endemic equilibrium

Lemma 5.3.
If $R_0 > 1$, then the endemic equilibrium E^* of the model (2) is globally asymptotically stable in Γ.

Proof. We define Lyapunov function V as follows [24, 25].

$$
\begin{aligned}
V &= \left(S - S^* - S^* log\frac{S}{S^*}\right) + \left(U - U^* - U^* log\frac{U}{U^*}\right) + \frac{(\mu+\gamma+\eta)}{\eta}\left(W - W^* - W^* log\frac{W}{W^*}\right)\\
&\quad + \frac{(\mu+\varepsilon)\eta}{(\mu\gamma+\gamma\gamma_1+\eta\gamma_1)}\left(P - P^* - P^* log\frac{P}{P^*}\right)\\
\frac{dV}{dt} &= \left(1-\frac{S^*}{S}\right)\frac{dS}{dt} + \left(1-\frac{U^*}{U}\right)\frac{dU}{dt} + \frac{(\mu+\gamma+\eta)}{\eta}\left(1-\frac{W^*}{W}\right)\frac{dW}{dt} + \frac{(\mu+\varepsilon)\eta}{(\mu\gamma+\gamma\gamma_1+\eta\gamma_1)}\left(1-\frac{P^*}{P}\right)\frac{dP}{dt}\\
&= \left(1-\frac{S^*}{S}\right)\left\{-(\mu+\sigma)(S-S^*) + \frac{\beta S^* U^*}{(1+\alpha_1 U^*+\alpha_2 U^{*2})} - \frac{\beta S U}{(1+\alpha_1 U+\alpha_2 U^2)}\right\}\\
&\quad + \left(1-\frac{U^*}{U}\right)\left\{\frac{\beta S U}{1+\alpha_1 U+\alpha_2 U^2} - (\mu+\gamma+\eta)U\right\} + \frac{(\mu+\gamma+\eta)}{\eta}\left(1-\frac{W^*}{W}\right)\{\eta U - (\mu+\gamma_1)W\}\\
&\quad + \frac{(\mu+\varepsilon)\eta}{(\mu\gamma+\gamma\gamma_1+\eta\gamma_1)}\left(1-\frac{P^*}{P}\right)\left\{\frac{(\mu\gamma+\gamma\gamma_1+\eta\gamma_1)W}{\eta} - (\mu+\varepsilon)P\right\}\\
&= -\left(1-\frac{S^*}{S}\right)(\mu+\sigma)(S-S^*) + \frac{\beta S^* U^*}{(1+\alpha_1 U^*+\alpha_2 U^{*2})} - \frac{\beta S^* U^* S^*}{(1+\alpha_1 U^*+\alpha_2 U^{*2})} + \frac{\beta S^* U}{(1+\alpha_1 U+\alpha_2 U^2)}\\
&\quad - \frac{\beta S U^*}{(1+\alpha_1 U+\alpha_2 U^2)} + (\mu+\gamma+\eta)U^* - \frac{(\mu+\gamma_1)(\mu+\gamma+\eta)W}{\eta} - (\mu+\gamma+\eta)U\frac{W^*}{W} + \frac{(\mu+\gamma_1)(\mu+\gamma+\eta)W^*}{\eta}\\
&\quad + (\mu+\gamma)W - \frac{\eta(\mu+\varepsilon)^2 P}{\mu\gamma+\gamma\gamma_1+\eta\gamma_1} - (\mu+\varepsilon)W\frac{P^*}{P} + \frac{\eta(\mu+\varepsilon)^2 P^*}{\mu\gamma+\gamma\gamma_1+\eta\gamma_1}\\
&= -\left(1-\frac{S^*}{S}\right)(\mu+\sigma)(S-S^*) + (\mu+\gamma+\eta)U^* - \frac{\beta S^* U^* S^*}{(1+\alpha_1 U^*+\alpha_2 U^{*2})} + \frac{\beta S^* U}{(1+\alpha_1 U+\alpha_2 U^2)}\\
&\quad - (\mu+\gamma+\eta)U^* + (\mu+\gamma+\eta)U^* - (\mu+\gamma+\eta)U + (\mu+\gamma+\eta)U^* - (\mu+\gamma+\eta)U\frac{W^*}{W}\\
&\quad - (\mu+\varepsilon)W + (\mu+\varepsilon)W - (\mu+\varepsilon)W\frac{P^*}{P} + (\mu+\varepsilon)W^*\\
&= -\left(1-\frac{S^*}{S}\right)(\mu+\sigma)(S-S^*) + (\mu+\gamma+\eta)U^* - (\mu+\gamma+\eta)U^*\frac{S^*}{S} + (\mu+\gamma+\eta)U\frac{S^*}{S}\\
&\quad - (\mu+\gamma+\eta)U^* + (\mu+\gamma+\eta)U^* - (\mu+\gamma+\eta)U - (\mu+\gamma+\eta)U\frac{W^*}{W} + (\mu+\gamma+\eta)U^*\\
&\quad + (\mu+\varepsilon)W^* - (\mu+\varepsilon)W\frac{P^*}{P}\\
&= -\left(1-\frac{S^*}{S}\right)(\mu+\sigma)(S-S^*) - (\mu+\gamma+\eta)U^*\left(\frac{S^*}{S} + \frac{UW^*}{U^*W} - 2\right) - (\mu+\gamma+\eta)U\left(1-\frac{S^*}{S}\right) - (\mu+\varepsilon)W^*\left(\frac{WP^*}{W^*P} - 1\right)\\
&\leqslant 0
\end{aligned}
$$

Here we have used the properties i.e. the arithmetic mean is always greater than or equal to the geometric mean, therefore $(\frac{S^*}{S} + \frac{UW^*}{U^*W} - 2) \geq 0$, $(1 - \frac{S^*}{S}) \geq 0$ and $(\frac{WP^*}{W^*P} - 1) \geq 0$. Then $\frac{dV}{dt} = 0$ only when $S = S^*$, $U = U^*$, $W = W^*$ and $P = P^*$. Hence the maximum invariant set of the system (2) on the set $\left\{(S, U, W, P : \frac{dV}{dt} = 0\right\}$ is the singleton and by using LaSalle invariant principle [22], the endemic equilibrium is globally asymptotically stable if $R_0 > 1$. □

6. Numerical Simulations

In this part of our paper, we have presented some numerical simulations of the system (2) to explain our analytical results.. The results of numerical simulation are exhibited in Fig. 1 to Fig. 10. Fig. 1 shows that the disease free equilibrium point is locally asymptotically stable when $R_0 = 0.9375 < 1$ for the set of parameters value which is taken from Table 1. While $R_0 = 1.0333 > 1$, then Fig. 2 shows that the disease free equilibrium point is unstable for the set of parameters value $R_s = 0.52$, $\eta = 0.05$, $\mu = 0.25$, $\gamma = 0.3$, $\sigma = 0.01$, $\beta = 0.31$ and the other parameters value are taken from Table 1. Also we say that when $R_0 = 4.4183168 > 1$, then Fig. 3 shows that the endemic equilibrium is locally asymptotically stable for the set of parameters value $R_s = 1.7$, $\eta = 0.06$, $\mu = 0.05$, $\gamma = 0.9$, $\sigma = 0.03$, $\beta = 0.21$, $\varepsilon = 0.5$, $\alpha_1 = 0.01$, $\gamma_1 = 0.3$ and $\alpha_2 = 0.02$. Now we choose the set of parameters value $\eta = 0.9$, $\mu = 0.001$, $\gamma = 0.01$, $\sigma = 0.001$, $\beta = 0.09$, $\epsilon = 0.5$, $\gamma_1 = 0.3$ and $\alpha_2 = 0.002$ and the other parameters value are taken from Table 1 with the initials value $S(0) = 4$, $U(0) = 3$, $W(0) = 2$, its phase portrait is given in Fig. 4, which shows that the system (2) has a stable limit cycle. Now for the parameters value $R_s = 1.7$, $\eta = 0.06$, $\mu = 0.05$, $\gamma = 0.9$, $\sigma = 0.01, 0.05, 0.09$, $\beta = 0.21$, $\varepsilon = 0.5$, $\alpha_1 = 0.01$, $\gamma_1 = 0.01$ and $\alpha_2 = 0.02$ then Fig. 5, Fig. 6 and Fig. 7 shows that if we increase the vaccination rate then U, W and P classes are also decreases. We have Fig. 8, Fig. 10 which illustrate that if we increase the antiretroviral therapy then U, and P class are decrease but W class is increase which is given in Fig. 9 for the set of parameters value $R_s = 1.7$, $\eta = 0.02, 0.10, 0.18$, $\mu = 0.05$, $\gamma = 0.9$, $\sigma = 0.01$, $\beta = 0.21$, $\varepsilon = 0.5$, $\alpha_1 = 0.01$, $\gamma_1 = 0.01$ and $\alpha_2 = 0.02$.

Table 1. A hypothetical set of parameter values.

Parameter	Definition	Default value
R_s	Is the constant recruitment rate of susceptible class.	0.5
α_1	A positive constants.	0.001
α_2	A Positive constants.	0.02
μ	Is the natural mortality rate of adult class.	0.1
σ	Is the successful vaccination rate of the susceptible individuals.	0.1
γ	Is the conversion rate of un-aware HIV infected class to AIDS class.	0.2
γ_1	Is the conversion rate of aware HIV infected class to pre AIDS class.	0.01
β	Is the per capita contact rate between susceptible class and un-aware HIV infected class.	0.3
η	Is the antiretroviral therapy rate of an un-aware HIV infected class.	0.5
ε	Is the rate by which pre AIDS infective individuals develop AIDS.	0.02

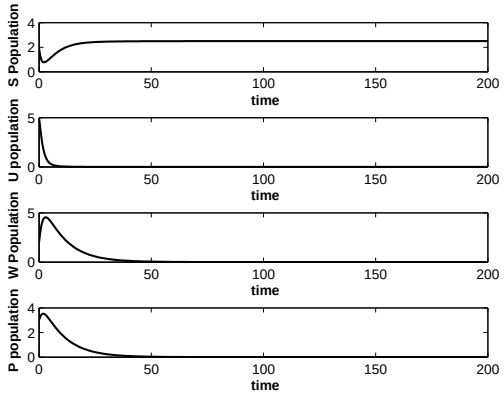

Fig. 1. The equilibrium point E_0 is asymptotically stable while $R_0 < 1$.

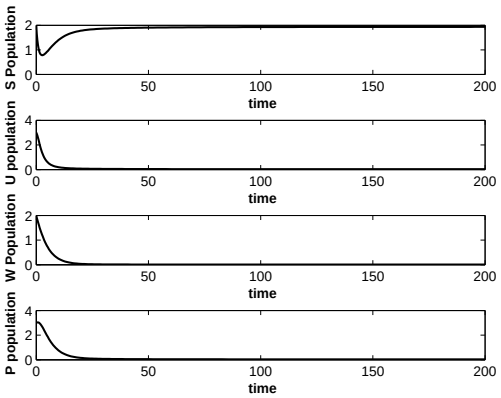

Fig. 2. The equilibrium point E_0 is unstable while $R_0 > 1$.

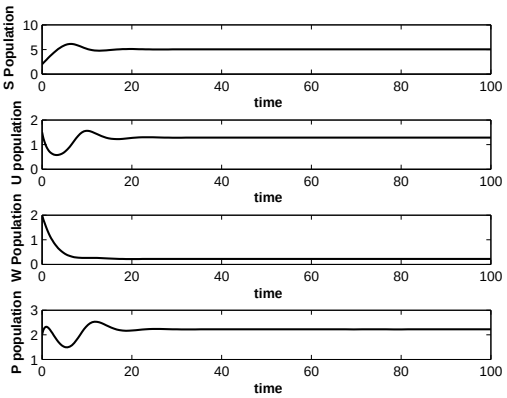

Fig. 3. The equilibrium point E^* is locally asymptotically stable while $R_0 > 1$.

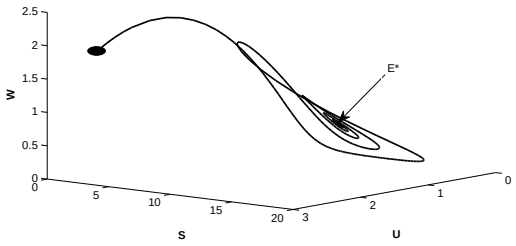

Fig. 4. The interior equilibrium point $E^*(S^*, U^*, W^*)$ is stable while $R_0 > 1$.

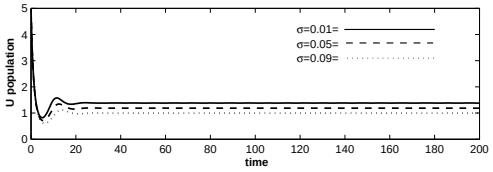

Fig. 5. This figure shows that when σ increases then U class decreases.

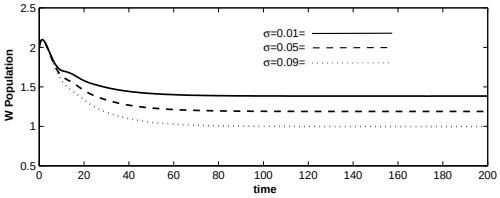

Fig. 6. This figure shows that when σ increases then W class decreases.

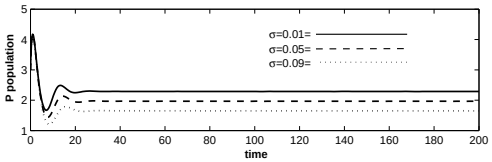

Fig. 7. This figure shows that when σ increases then P class decreases.

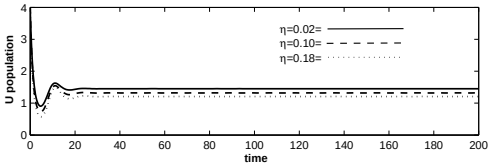

Fig. 8. This figure shows that when η increases then U class decreases.

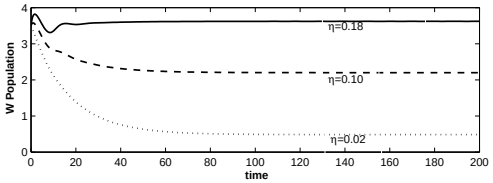

Fig. 9. This figure shows that when η increases then W class increases.

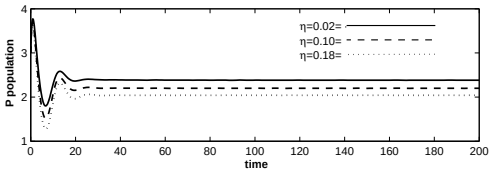

Fig. 10. This figure shows that when η increases then P class decreases.

7. Conclusion

In this paper, we have proposed and analyzed a non-linear HIV/AIDS epidemic model with vaccination and antiretroviral therapy and studied its stability analysis. By analyzing the mathematical model, first we have found a basic reproduction number R_0 because it plays an important role. It is noted that when $R_0 < 1$ then disease dies out and when $R_0 > 1$ then disease become endemic. The model has two non-negative equilibria namely, $E_0 = (\frac{R_s}{(\mu+\sigma)}, 0, 0, 0)$, the disease free equilibrium and $E^* = (S^*, U^*, W^*, P^*)$, the endemic equilibrium. It is noted that the equilibrium state E_0 corresponding to disappearance of disease is locally as well as globally asymptotically stable if $R_0 < 1$ and unstable if $R_0 > 1$. The endemic equilibrium E^*, which exists only when $R_0 > 1$ is locally asymptotically stable by using Jacobian matrix and globally asymptotically stable which proof by using LaSalle invariant principle. If there is no vaccination then R_0 reduces to $R_0' = \frac{\beta R_s}{\mu(\mu+\gamma+\eta)}$ and if there is no antiretroviral therapy then R_0' reduces to $R_0'' = \frac{\beta R_s}{\mu(\mu+\gamma)}$. Therefore we say that $R_0 < R_0' < R_0''$ which ensure that the disease spread among the population rapidly in a very short time. Therefore vaccination and antiretroviral therapy must decrease the disease transmission. We also say that if we increase the vaccination rate then U, W and P class are decreases. How ever if we increase the antiretroviral therapy rate then U, P class are decrease and W class is also increase. Also if we restrict the migration rate into susceptible community, then the spread of the disease can be kept under control.

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