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On the fractional Brownian motion: Hansdorff dimension and Fourier expansion of fractional Brownian motion

Research Article

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Abstract: In this paper, we determine the Hansdorff dimension by using the cantor Set and we give the Fourier expansion of fractional Brownian motion with $0 < H \leq \frac{1}{2}$.

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Keywords: Hansdorff dimension • Cantor Set • EBrownian motion

1. Introduction

The fractional Brownian motion (fBM) $B^H = \{B_t^H, t \geq 0\}$ is a zero mean Gaussian process with covariance

$$E(B_s^H B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |t-s|^{2H})$$

$H \in (0,1)$ is called the Hurst parameter. If $H = \frac{1}{2}$ is a Brownian motion. For $H \neq \frac{1}{2}$, the fractional Brownian motion has correlated increments the autocorrelation function r(n)is

$$r^n = EB_1^H(B_{n+1}^H - B_H^n) = 2\alpha H \int_0^1 \int_n^{n+1} (u-v)^{2\alpha-1} du dv \sim 2\alpha H n^{2\alpha-1}$$

with $\alpha = H - \frac{1}{2}$ as $n \longrightarrow \infty$
If $H > \frac{1}{2}$, then $r_H^n > 0$ and $\sum r_H(n) = \infty$ we have a long memory. If $H < \frac{1}{2}$, then $r_H(n) < 0$ and $\sum |r_H(n)| < \infty$.

The self-similarity and long memory properties make the fractional Brownian motion a surtable imput noise in a variety of models. Recently, f.B.m has been appliedin connection with financial time series, hydrology and telecommunications.

The stochastic calculus of fBm originated with the pioncering work of B. B Mandelhot worked with factional processes during a long period and his later results concerned and scaling were summarized in the book [12] .

Note also that it was proved that the moving average representation of fbm is unique in the class of the right - continuous, non decreasing concave functions in R_+. The first result where fbm appeared as the limit in the skrohod topology of stationary sums of random variablewas obtained by M. Taqqu [14] another scheme of convergence to fBm in the uniform topology was considered in [11].

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It can be explained by various applications of fbm and other long-memory processes in telegraphic, finances, climate and weather derivatives. The paper [10] was one of the first paper devoted to stochastic analysis for fbm. However,it is closely connected with fractional calculus and can be represented as a "fractional integral". Such a representation, together with the gaussian property of fbm and Holder property of its trajectories permits to create an interesting and specific stochastic calculus of fbm.
In order to develop there applications there is a need for stochastic calculus with respect to the fBm. Nevertheless, fbm is neither a semimartingale nor a markov process, and new tools are required in order to handle the differentials of the fbm and to formulate and solve stochastic differential equation driven by a fbm.
They are essentialy two different methods to defined stochastic integrals with to the fbm.

i) A parth - wise approach that was the holder continuity properties of the sample paths, developed from the work by Ciesielskiand hu and Zahle.

ii) The stochastic calculus of variations (Malliavin calculus) for the fbm introced by Decreusefond and Uut*ü*nel in [8].

2. Preliminaries

Theorem 2.1.
Let $(X_t, t \in [A,B]^d)$ be a random field. If there exist tree positive constants, α, β, C such that, for every

$$t,s \in [A,B]^d \qquad E|X_t - X_s|^\alpha \leq C\|t-s\|^{\alpha+\beta},$$

then, there exists a locally $\delta - h\ddot{o}lder$ continuous modification $\tilde{X}$ of X for every $\delta < \frac{\beta}{\alpha}$. It mean that there exist a random variable $h(\omega)$, and a constant $\delta > 0$ such that

$$P[\omega, \sup_{\|t-s\|\leq h(\omega)} \frac{|\tilde{X}_t(\omega) - \tilde{X}_s(\omega)|}{\|t-s\|^\delta} \leq \delta] = 1$$

The definition of continuity is not a quantitative one, because it does not say how rapidly the values of $u(y)$ of a function approach its value $u(x)$ as $y \mapsto x$. The modulus of continuity $\omega : [0;\infty] \longrightarrow [0;\infty]$ of a general continuous u, satisfying $|u(x) - u(y)| \leq \omega|x-y|$ may decrease arbitrarity slowly. As a result despite their simple and natural appearance, space of continuous functions are often not suitable for the analysis of PDES which is almost always based on quantitative estimates.
A straight forward and useful way to strenghten the definition of continuity is proportional to a power $|x-y|^\alpha$ for some exponent $0 < \alpha \leq 1$, such functions are said to be $h\ddot{o}lder$ continuous function with exponent α as functions with bounded fractional derivatives of the order α.
Suppose that Ω is a open set in R^n and $0 < \alpha \leq 1$.
A function $u : \Omega \mapsto R$ is uniformly $H\ddot{o}lder$ continuous with exponent α in Ω if the quantity A function $u : \Omega \mapsto R$ is uniformly $H\ddot{o}lder$ continuous with exponent α in Ω if the quantity

$$[[u]_{\alpha,\Omega} = \sup_{x,y\in\Omega_{x\neq y}} \frac{|u(x) - u(y)|}{|x-y|^\alpha} \quad \text{is} \quad \text{finite.}$$

Hansdorff measure and dimension:
Hansdorff dimension has been introduced to avoid the drawbacks of box-counting dimension. As opposed to box-counting dimension, we first define the hansdorff measure and then the associated dimension.
if $\{u_t\}$ is a countable (or finite) collection of sets of diameter at most δ that cover F, i.e $F \subset \cup^\alpha u_i$ with $0 < u_i \leq \delta$ for each i, we say that $\{u_i\}$ is a δ-cover of F. Suppose F is a subset of R^d and s is a non-negative number.
For any $\delta > 0$ we define

$$H_s^\delta(F) = inf\{\sum_i^\infty |u_i|^s, \{u_i\} \quad is \quad a \quad \delta - cover \quad of \quad F\}$$

As δ decreases, the class of permissible covers of F is reduced, therefore $H_\delta^s(F)$ increase as $\delta \mapsto 0$. The following limit exists

$$H^s(F) = \lim_{\delta\to 0} H_\delta^s(F)$$

and is called the s-dimensional Hansdorff measure of F. It can be shown that H^s is an outer measure on Borel sets .
Hansdorff measures generalize lebesgue measures : for integer i, H^i is, up to a constant, the usual lebesgue measures. Actually the limit in 3 is for all s, except eventually one value, null or infinite. First $s \mapsto H_\delta^s(F)$ is clearly a non increasing

function, s_0 is $s \mapsto H^s(F)$
Morever if $t > s$ and $\{u_i\}$ is a δ-cover of F we have

$$\sum_i^{\infty} |u_i|^t \le \delta^{t-s} \sum_i^{\infty} |u_i|^s$$

so

$$H_\delta^t(F) \le \delta^{t-s} H_\delta^s(F).$$

Letting $\delta \mapsto 0$, we see that if $H_\delta^s(F) < \infty$ then $H_\delta^t(F) = 0$.
The Hansdorff dimension of F is the only possible s where $s \mapsto H^s(F)$ jumps from $+\infty$ to 0 more precisely

$$dim_H F = \inf\{s \quad s.t \quad H^s(F) = 0\} = \sup_s\{s \quad s.t \quad H^s(F) = \infty\}$$

3. Hansdorff dimension and fourier expansion of fractional Brownian motion

3.1. Mandelbrot-Van Ness Representation of fbm

Let $W = \{W_t, t \in R\}$ be the two-sided Wiener process, i.e the Gaussian process with independent increments satisfying $EW_t = 0$ and $EW_t W_s = s \wedge t\ s, t \in R$
Denote $k_H(t,u) = (t-u)^\alpha + -(-u)_+^\alpha$, where $\alpha = H - \frac{1}{2}$

Theorem 3.1.
The process $\overline{B}^H = \{\overline{B}_t, t \in R\}$ *defined by*

$$\overline{B}_t^H = C_H^{(2)} \int_R k_H(t,u) dW_u, H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$$

where $C_H^{(2)} = (\int_R ((1+s)^\alpha - s^\alpha)^2 ds + \frac{1}{2H})^{\frac{-1}{2}}$

$$= \frac{(2H sin\pi H \Gamma(2H))^{\frac{1}{2}}}{\Gamma(H + \frac{1}{2})}$$

has a continuous modification which is a normalized two-sided fractional Brownian motion

To demonstrate the Theorem 3.1, we use the folliwing lemma.

Lemma 3.1.
Note that $(I_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} f(t)(t-x)^{\alpha-1} dt$

$$1_{(a,b)}(t) = \begin{cases} 1, & si \ \ a \le t < b \\ \text{-1}, & si \ \ b \le t < a \\ O, & si \ \ otherwise \end{cases}$$

Let $H \in (0, \frac{1}{2}) \cup (\frac{1}{2}, 1)$ *and* $\alpha = H - \frac{1}{2}$ *then for all* $t \in R$*, we have the equality*

$$(I_-^\alpha 1_{(0,t)})(x) = \frac{1}{\Gamma(1+\alpha)}((t-x)_+^\alpha - (-x)_+^\alpha)$$

Proof. Let $H \in (\frac{1}{2}, 1)$ and for example $x < 0 < t$ the other cases can be considered similarly, then

$$\left(I_-^\alpha 1_{(0,t)}\right)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{+\infty} 1_{(0,t)}(u)(u-x)^{\alpha-1} du = \frac{1}{\Gamma(\alpha)} \int_0^t (u-x)^{\alpha-1} du = \frac{1}{\Gamma(\alpha+1)}((t-x)^\alpha - (-x)^\alpha)$$

Let $H \in (0, \frac{1}{2})$. According to the definition of the fractional derivative and the properties $I_{+-}^\alpha I_{+-}^{-\alpha} f = f$ we must prove that

$$\int_x^{+\infty} ((t-u)_+^\alpha - (-u)_+^\alpha)(u-x)^{-\alpha-1} du = \Gamma(-\alpha)\Gamma(\alpha+1) 1_{(0,1)}(x)$$

Let, for example, $0 < x < t$ then the left-hand side of (1) equals

$$\int_x^t ((t-u)^\alpha - (u-x)^{-\alpha-1} du 1_{(0,t)}(x) = B(\alpha+1,-\alpha)1_{(0,t)}(x) = \Gamma(-\alpha)\Gamma(\alpha+1)1_{(0,t)}(x)$$

□

Lemma 3.2.
$C_H^{(2)}$ is chosen to normalize the f.B.m and

$$C_H^{(2)} = \frac{(2Hsin\pi H\Gamma(2H))^{\frac{1}{2}}}{\Gamma(H+\frac{1}{2})}$$

Proof.

$$\overline{B}_t^H = C_H^{(2)} \int_R k_H(t,u)dW_u = C_H^{(2)}\Gamma(1+\alpha)\int_R (I_-^\alpha 1_{(0,t)})(x)dW_u$$

In using Lemma 3.1. Therefore, the first equality is evident , since

$$\int_R (k_H(t,u))^2 du = \int_\infty^0 ((t-x)^\alpha - (-x)^\alpha)^2 dx + \int_0^t (t-x)^{2\alpha} dx$$

$$= t^{2H}(\int_0^\infty ((1+s)^\alpha - s^\alpha)^2 ds + \frac{1}{2H})$$

When obtain the second equality if we note that

$$\int_R ((I_-^\alpha 1_{(0,t)})^2(x)dx = \frac{1}{2\pi}\int_R \left(1_{(0,t)}(\lambda)|\lambda|^{-\alpha}\exp\left\{\frac{\alpha\pi i}{2}sign\lambda\right\}\right)^2 d\lambda$$

therefore

$$\begin{aligned}\int_R (I_-^\alpha 1_{(0,t)})^2(x)dx &= \frac{1}{2\pi}\int_R |e^{i\lambda t}-1|^2|\lambda|^{-2\alpha-2}d\lambda \\ &= \frac{1}{2\pi}\int_R (1-\cos\lambda t)^2|\lambda|^{-2\alpha-2}d\lambda + \frac{1}{2\pi}\int_R sin^2 t\lambda|\lambda|^{-2\alpha-2}d\lambda \\ &= \frac{1}{\pi}\int_0^\infty \frac{(1-\cos\lambda t)^2}{\lambda^{2\alpha+2}}d\lambda + \frac{1}{\pi}\int_0^\infty \frac{\sin^2 t\lambda}{|t|^{2\alpha+2}} \\ &= t^{2H}\left(\frac{1}{\pi}\int_0^\infty \frac{(1-\cos\lambda)^2}{\lambda^{2\alpha+2}}d\lambda + \frac{1}{\pi}\int_0^\infty \frac{\sin^2 t\lambda}{\lambda^{2\alpha+2}}d\lambda\right) \\ &= \frac{t^{2H}}{2H\sin\pi H\Gamma(2H)}\end{aligned}$$

□

Proof. (Proof of Theorem 3.1)
Evidently,$\overline{B}^H$ is a Gaussian process with $\overline{B}_0^H$ and $E\overline{B}_t^H = 0$.
Furthermore, it holds that for $t > 0$

$$E(\bar{B}_t^H)^2 = \left(C_H^{(2)}\right)^2\left(\int_{-\infty}^0 k_H^2(t,u)du + \int_0^t (t-u)^{2\alpha}du\right) == t^{2H}$$

For $t < 0$, we have that

$$E(\bar{B}_t^H)^2 = \left(C_H^{(2)}\right)^2\left(\int_{-\infty}^t k_H^2(t,u)du + \int_t^0 (-u)^{2\alpha}du\right) == (-t)^{2H}$$

Furthermore, for $h > 0$, it holds that

$$\bar{B}_{s+h}^H - \bar{B}_s = C_H^{(2)}\int_{-\infty}^s (k_H(s+h,u) - k(s,u))dW_u + \int_s^{s+h} k_H(s+h,u)dW_u = I_1 + I_2$$

Note that the terms I_1 and I_2 on the right hand side of 2 are independent, and the Wiener process W has stationary increments. Therefore

$$I_1 =^d \int_{-\infty}^{0} (k_H(s,u) - k(0,u))dW_u$$

$$I_2 = \int_0^h k_H(h,u)dW_u$$

and

$$E(\overline{B}^H_{s+h} - \overline{B}^H_s)^2 = E(\overline{B}^H_h) = h^{2H}$$

By combining these results, we obtain that

$$E\bar{B}^H_s\bar{B}^H_t = \frac{1}{2}\left(E\left(\bar{B}^H_s\right) + E\left(\bar{B}^H_s\right) - E\left(\bar{B}^H_t - \bar{B}^H_s\right)^2\right) = \frac{1}{2}\left(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\right)$$

The proof follows immediately from definition □

Theorem 3.2.
Let $F = \{\sum_{n=1}^{+\infty} \frac{x_n}{3^n}$ such $x_n \in \{0,2\}$ } be the middle third Cantor Set. Let X be a fractional Brownian motion of parameter H. The Hansdorff dimension of $\{(x, Xx, x \in F)\} =^{as} 1 + log_3 2 - H$.

Proof.

1) Upper bound We split the interval $[0,1]$ into sub-intervals $[k3^{-n}, (k+1)3^{-n}]$ $k = 0, 3^n - 1$
We need 2^n sub-intervals of cover F. On such a sub-interval, the oscillation of is overestimated by $C3^{-nH'}$ for every $H' < H$. We need $C2^n 3^{n(1-H')}$ squares of sizes 3^{-n} to cover $\{xX(x), x \in F\}$
The Hansdorff dimension of $\{(x, X(x)), x \in F\}$ is overestimated by $1 + log_3 2 - H$

2) Lower bound

$$E((X(t) - X(s))^2 + (t-s)^2)^{-\frac{s}{2}}) \le C|t-s|^{1-s-H}$$

Let μ be a probability measure supported by F and

$$I_s(\mu) = \int\int_{F\times F} ((X(t) - X(s))^2 + (t-s)^2)^{\frac{-s}{2}} d\mu(t)d\mu(s)$$

then

$$EI_s(\mu) = E\int\int_{F\times F} ((X(t) - X(s))^2 + (t-s)^2)^{\frac{-s}{2}} d\mu(t)d\mu(s)$$

$$\le C\int\int_{F\times F} |t-s|^{1-s-H} d\mu(t)d\mu(s)$$

We now apply Frostman's lemma.
Since $dim_H(F) = log_3 2$, for s such that $H + s - 1 > log_3 2$ H(F)=0 then for every $s' < s$ there exists a probability measure μ such that

$$\int\int_{F\times F'} |t-t'|^{1-s'-H} d\mu(t)d\mu(s) < \infty$$

For this measure μ $EI'_s(\mu) < \infty$ and almost surely $I'_s(\mu) < \infty$. This implies $dim\{x, X(x)x \in F\} \ge 1 + log_3 2 - H$, it follow that $dim\{(x, X(x))x \in F\} = 1 + log_3 2 - H$.

□

Theorem 3.3.
Let $0 < H \le \frac{1}{2}$. We denote $C_n = \int_0^1 x^{2H} cos(n\pi x)dx$ $\forall n \ge 1$

(1) $C_n \le 0$ and $\sum_{n=1}^{+\infty} C_n cos(n\pi x)$ is convergent and there exists C_o such that $|x|^{2H} = C_0 + \sum_{n=0}^{+\infty} C_n cos(n\pi x)$ $\forall x \in [-1,1]$

(2) *Let* (ξ_n, η_n) *a sequence of i.i.d centered Gaussian random vectors in* R^2 *with covariance matrix equal to identity , then the series* $\sum_{n=1}^{+\infty}(\xi_n + i\eta_n)\sqrt{\frac{-C_n}{2}}(e^{in\pi t} - 1)$ *is convergent in* L^2.

(3) *Let us denote by* $X(t)$ *its sum, then* $\Re(X(t))$ *is a* 2π *periodic Gaussian process with stationary increments such that* $E(\Re(X(t)).\Re(X(s)))^2 = |t-s|^{2H}$ *for all t, s such that* $-1 < t - s < 1$
We usually refer to $\Re(X(t))$ *as a fractional Brownian motion indexed by the circle* S^1

Proof.

1) $\int_0^1 x^{2H} cos(2\pi x)dx = \int_0^{\frac{1}{4}} (x^{2H} - (\frac{1}{2} - x)^{2H} + (1-x)^{2H} - (\frac{1}{2} + x)^{2H}) cos 2\pi x dx$

since $x \mapsto x^{2H}$ is concave for $0 < H \leq \frac{1}{2}$, we get $(x^{2H} - (\frac{1}{2} - x)^{2H} + (1-x)^{2H} - (\frac{1}{2} + x)^{2H}) \leq 0$ for $0 \leq x \leq \frac{1}{4}$ and C_2.
One can use similar arguments to show that $C_{2n} \leq 0$ for any integer n . The integral in the definition of C_{2n+1} shall be split into two parts and we use the fact that $x \mapsto x^{2H}$ is increasing to show that $C_{2n+1} \leq 0$

In [15] fractional fields parameterized by Euclidean spheres S^d are studied, and series ewpansion of $\theta \mapsto |\theta|^{2H}$ when $\theta \in S^d$ are used for this study . In this proof X(t) can be considered as a fractional field parametrized by the circle S^1. Because of the 2π periodicity of X, so the results on C_n are a special case of the levy Kinchine formula for θ $x \mapsto \theta^{2H}$ when $\theta \in S^d$ as seen in .

2.) Since $x \mapsto x^{2H}$ is a continuous function C^1 except for $x = 0$, we can apply Dirichlet theorem Fourier series that fields the convergence of the Fourier series to $|x|^{2H}$.

When $x = 0$, we use the Dirichlet criterion, which is in this particular cas $\int_O^1 \frac{x^{2H}}{x} dx < \infty$, to have the convergence of $\sum_{n=1}^{+\infty} C_n$ and we denotes its sum-C_0 which is non positive because of 1.

3.) Let us consider the

$$S(t) = \sum_{n=1}^{+\infty} (\xi_n(\cos(n\pi t) - 1) - \eta_n \sin(n\pi t))\sqrt{\frac{-C_n}{2}}$$

$$\forall t \in [-1, 1] \quad E\left((S(t))^2\right) = \sum_{n=1}^{+\infty} (cos(n\pi t) - 1)C_n = |t|^{2H} - C_0 + C_0$$

Hence the series defining S(t) converge in L^2 and obviously $S(t) = \Re(X(t))$ with the same type of argument one can show the convergence in L^2.
Morever $\forall t, s \in R$

$$X(t) - X(s) = \sum_{n=1}^{+\infty} ([\xi_n + i\eta_n] e^{in\pi s})(e^{in\pi(t-s)} - 1)$$

But

$$(\xi_n + i\eta_n)e^{in\pi s} =^d (\xi_n + i\eta_n)_n,$$

and X has stationary increments. In particular

$$E(\Re(X(t)) - \Re(X(s))^2) = |t-s|^{2H}$$

for all t, s such that $-1 < t - s < 1$.

□

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