Int. J. Adv. Appl. Math. and Mech. 5(3) (2018) 1 – 14 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Numerical algorithm based on Bernstein polynomials for solving boundary value problems involving singular, singularly perturbed type differential equations

Research Article

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Received 11 January 2018; accepted (in revised version) 08 February 2018

Abstract: In this article a simple numerical algorithm based on Bernstein polynomials is presented and analyzed for obtaining numerical solution of linear and non-linear boundary value problems. The method is essentially based on reducing the differential equations with their boundary conditions to a system of algebraic equations in the expansion coefficients of the sought for solutions. The equations are solved by collocation method. A number of illustrative examples including singular and singularly perturbed boundary value problems are considered to ascertain the validity, wide applicability and efficiency of the proposed method. The obtained numerical results are compared with those obtained from known analytical solutions and are found to be very accurate and better than those obtained by some others employing different techniques.

MSC: 34B05 • 34B15 • 65L10

Keywords: Bernstein polynomials • Boundary value problems • Singular differential equations • Singularly perturbed differential equations

1. Introduction

Over the years, the study of numerical solutions of boundary value problems has been the focus of many researches due to its frequent appearance in many areas of mathematical physics such as fluid dynamics, chemical reactions, structural mechanics etc. The purpose of this paper is to develop a numerical algorithm to solve singular and singularly perturbed boundary value problems.

Singular boundary value problems have attracted the attention of scientists and engineers as those problems have shown to be adequate models for various physical phenomena in areas like boundary layer theory, biological flow, control theory, astrophysics, optimization problems etc. Here we have considered the following class of singular boundary value problems

$$y^{(2r)}(x) + \frac{k_1}{x} y'(x) + \frac{k_2}{x^2} y(x) = f(x, y(x), y'(x), \dots, y^{(2r-1)}(x)), \quad 0 < x < 1 \tag{1}$$

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subject to the boundary conditions

$$\begin{aligned} y(0) &= y'(0) = 0, \\ y^{j}(0) &= \alpha_j, \quad j = 2,3,\dots,r-1, \\ y^{i}(1) &= \beta_i, \quad i = 0,1,2,\dots,r-1. \end{aligned} \tag{2}$$

where f is a known function and $r,\ k_1,\ k_2,\ \alpha_j(j=2,\dots,r-1),\ \beta_i(i=0,\dots,r-1)$ are given constants.

In recent years, various spline methods(cf. Ali et al.[1], Caglar and Caglar[2], Mittal and Jain[3], Gamel and Shamy[4]), finite difference methods(cf. Jain and Jain[5], Aziz et al.[6]) and Hermite wavelet method(cf. Usman and Mohyud-Din[7]) have been used to obtain the numerical solutions of singular boundary value problems of the form (1). This type of mathematical model arises mostly in the studies of chemical reactors, theory of stellar interiors, isothermal gas spheres, atomic structures, positive radial solutions of elliptic equations etc.

Also, singularly perturbed boundary value problems have been used as mathematical models to describe chemical reactions, some topics of fluid mechanics, neurobiology, elasticity, quantum mechanics, geophysics, aerodynamics, oceanography, diffusion processes, combustion etc. A few notable examples are boundary layer problems, WKB theory, the modeling of steady and unsteady viscous flow problems with large Reynolds number and convective heat transport problems with large Peclet numbers.

Here, we have considered the self-adjoint singularly perturbed problem

$$-\epsilon y''(x) + f(x)y(x) = g(x), \quad f(x) \geq 0, \quad x \in [0,1] \tag{3}$$

with the boundary conditions

$$y(0) = \alpha_0, \quad y(1) = \alpha_1 \tag{4}$$

where α_0 and α_1 are given constants and ϵ is a small parameter such that $0 < \epsilon << 1$ and $f(x),\ g(x)$ are smooth, bounded, real functions, $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$. The approximate solution of boundary value problems characterized by the presence of a small parameter ϵ affecting highest derivative of the equation is derived. The solution of such a problem exhibits a multi-scale character i.e. there are thin transition layers where the solution varies rapidly, while away from the layer it behaves regularly and varies slowly. The occurrence of stiff boundary layers as ϵ, the coefficient of highest derivative, approaches zero creates difficulty for most standard numerical schemes.

Recently, a number of investigators have considered various non-classical methods including B-splines and cubic splines(cf. Aziz and Khan[8], Kadalbajoo and Patidar[9], Khan et al.[10]), uniform mesh methods(cf. Kadalbajoo and Bawa[11]), reproducing kernel method(RKM)(cf. Geng[12]), finite element method(cf. Lenferink[13]), finite difference methods(cf. Lubuma and Patidar[14]), operational Haar wavelet method(cf. Shah and Abass[15]) to solve singularly perturbed boundary value problems.

In this paper, we have developed a simple numerical algorithm based on the method of polynomial approximation in the Bernstein polynomial basis. Here we have introduced a truncated expansion for the unknown function in terms of Bernstein polynomials and used it to reduce the singular and singularly perturbed boundary value problems into a system of algebraic equations after using collocation points. Bernstein polynomials have been used in the literature to solve several linear and non-linear differential equations, ordinary and partial, approximately (cf. Bhatta and Bhatti[16], Bhatti and Bracken[17]) and to solve various integral equations (cf. Mandal and Bhattacharya[18]). These polynomials defined on an interval form a complete basis over the interval. The sum of these polynomials is unity, each of them being positive.

The coefficients of the truncated expansion are obtained by solving the reduced system of algebraic equations and the approximate series expansion of $y(x)$ are then calculated. Some illustrative examples of singular boundary value problems of higher order with comparison between exact solutions and approximate solutions obtained by B-spline method(cf. Gamel and Shamy[4]) and the present method are displayed in tables. Also there are numerical examples of self-adjoint singularly perturbed second order boundary value problems to demonstrate the usefulness of the present method. Comparison between the maximum absolute errors for different values of ϵ obtained by operational Haar wavelet method(cf. Shah and Abass[15]) and our method are presented in tables. Also, a comparison between the exact solution and the approximate solution obtained by the proposed method are depicted graphically in figures for some particular values of ϵ. All the numerical experiments confirm that the Bernstein polynomial approximation method is computationally highly efficient and user-friendly for both singular and singularly perturbed boundary value problems.

2. Bernstein polynomials and their properties

The Bernstein polynomials of degree n are defined on the interval $[a,b]$ as

$$B_{i,n}(x) = \binom{n}{i}\frac{(x-a)^i(b-x)^{n-i}}{(b-a)^n}, \quad i = 0,1,\dots,n \tag{5}$$

where $\binom{n}{i}$ is a binomial coefficient.

The Bernstein basis polynomials of degree n form a basis for the vector space $\prod_n$ of polynomials of degree at most n. These polynomials defined on an interval form a complete basis over the interval $[a,b]$. The sum of these polynomials is unity, each of them being positive.

3. General method of solution

In this section, we first establish a numerical algorithm using Bernstein polynomials as basis functions for solving boundary value problems. To apply the Bernstein polynomials in the interval $[0,1]$ we note that $B_{i,n}(x)$ is defined as

$$B_{i,n}(x)=\binom{n}{i}x^i(1-x)^{n-i},\ i=0,1,2,...,n \tag{6}$$

Since $\{B_{i,n}(x)\}_{i=0}^{n}$ in Hilbert space $L_2[0,1]$ is a complete basis, we can write any polynomial $y_m(x)$ of degree m in terms of linear combination of $\{B_{i,m}(x)\}_{i=0}^{m}$ as

$$y_m(x)=\sum_{k=0}^{m}c_kB_{k,m}(x). \tag{7}$$

Another form of Bernstein polynomials of degree n in the interval $[0,1]$ is given by

$$B_{i,n}(x)=\sum_{p=0}^{n-i}\binom{n}{i}\binom{n-i}{p}(-1)^px^{i+p}. \tag{8}$$

Since differentiation is a linear operation we have from (7)

$$\frac{d^r}{dx^r}(y_m(x))=\sum_{k=0}^{m}c_k\frac{d^r}{dx^r}(B_{k,m}(x)),\ r\in\mathbb{N}. \tag{9}$$

Using (8) we get

$$\frac{d^r}{dx^r}(B_{k,m}(x))=\sum_{p=0}^{m-k}\binom{m}{k}\binom{m-k}{p}(-1)^p\frac{d^r}{dx^r}(x^{k+p}).$$

Since $B_{k,m}(x)$ is a polynomial of degree m, we have

$$\frac{d^r}{dx^r}(B_{k,m}(x))=0 \text{ for all } m=0,1,2,.....,r-1. \tag{10}$$

Hence,

$$\frac{d^r}{dx^r}(B_{k,m}(x))=\sum_{p=r-k}^{m-k}\binom{m}{k}\binom{m-k}{p}(-1)^p\frac{\Gamma(k+p+1)}{\Gamma(k+p+1-r)}x^{k+p-r} \text{ for } k+p\geq r \tag{11}$$

where Γ denotes usual gamma function. Combining (9), (10) and (11) we obtain

$$\frac{d^r}{dx^r}(y_m(x))=\sum_{k=0}^{m}\sum_{p=r-k}^{m-k}\binom{m}{k}\binom{m-k}{p}(-1)^p\frac{\Gamma(k+p+1)}{\Gamma(k+p+1-r)}c_kx^{k+p-r} \tag{12}$$

i.e.

$$\frac{d^r}{dx^r}(y_m(x))=\sum_{k=0}^{m}\sum_{p=r-k}^{m-k}A_{k,p}^{m,r}c_kx^{k+p-r} \tag{13}$$

where $A_{k,p}^{m,r}$ is given by

$$A_{k,p}^{m,r}=\binom{m}{k}\binom{m-k}{p}(-1)^p\frac{\Gamma(k+p+1)}{\Gamma(k+p+1-r)}. \tag{14}$$

Next we will derive a discretization formula of (1) and (3) using simple collocation method. We have already approximated $y(x)$ as

$$y(x)\approx y_m(x)=\sum_{k=0}^{m}c_kB_{k,m}(x). \tag{15}$$

3.1. Singular linear or non-linear higher order differential equation

Using (13) and (15) in (1) we obtain

$$y_m^{(2r)}(x)+\frac{k_1}{x}y_m'(x)+\frac{k_2}{x^2}y_m(x)=f(x,y_m(x),y_m'(x),...,y_m^{(2r-1)}(x)),\ x\in(0,1)$$

i.e.

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{p=2r-k}^{m-k}c_kA_{k,p}^{m,2r}x^{k+p-2r}+\frac{k_1}{x}\sum_{k=0}^{m}\sum_{p=1-k}^{m-k}c_kA_{k,p}^{m,1}x^{k+p-1}+\frac{k_2}{x^2}\sum_{k=0}^{m}c_kB_{k,m}(x)=\\&f\left(x,\sum_{k=0}^{m}c_kB_{k,m}(x),\sum_{k=0}^{m}\sum_{p=1-k}^{m-k}c_kA_{k,p}^{m,1}x^{k+p-1},...,\sum_{k=0}^{m}\sum_{p=(2r-1)-k}^{m-k}c_kA_{k,p}^{m,2r-1}x^{k+p-(2r-1)}\right).\end{aligned}\tag{16}$$

After collocating at points $x_i,\ i=0,1,...,m-2r$ we get

$$\begin{aligned}&\sum_{k=0}^{m}\sum_{p=2r-k}^{m-k}c_kA_{k,p}^{m,2r}x_i^{k+p-2r}+\frac{k_1}{x_i}\sum_{k=0}^{m}\sum_{p=1-k}^{m-k}c_kA_{k,p}^{m,1}x_i^{k+p-1}+\frac{k_2}{x_i^2}\sum_{k=0}^{m}c_kB_{k,m}(x_i)=\\&f\left(x_i,\sum_{k=0}^{m}c_kB_{k,m}(x_i),\sum_{k=0}^{m}\sum_{p=1-k}^{m-k}c_kA_{k,p}^{m,1}x_i^{k+p-1},...,\sum_{k=0}^{m}\sum_{p=(2r-1)-k}^{m-k}c_kA_{k,p}^{m,2r-1}x_i^{k+p-(2r-1)}\right).\end{aligned}\tag{17}$$

Let us now substitute (13) in the boundary conditions (2). Then we get further $2r$ equations given by

$$\begin{aligned}&\sum_{k=0}^{m}c_k\delta_{k,0}=0,\\&\sum_{k=0}^{m}A_{k,1-k}^{m,1}c_k=0,\\&\sum_{k=0}^{m}A_{k,j-k}^{m,j}c_k=\alpha_j,\qquad j=2,3,...,r-1,\\&\sum_{k=0}^{m}\sum_{p=i-k}^{m-k}A_{k,p}^{m,i}c_k=\beta_i,\quad i=0,1,...,r-1.\end{aligned}\tag{18}$$

Hence Eq. (17),together with $2r$ equations of (18) gives $(m+1)$ equations which can be solved for the $(m+1)$ unknowns $c_k,\ k=0(1)m$ using an appropriate numerical method. Finally, the function $y(x)$ approximated by (7) can be obtained.

3.2. Self adjoint singularly perturbed problem

Using (13) and (15) in (3) we obtain

$$Ly_m(x)\equiv-\epsilon\frac{d^2y_m(x)}{dx^2}+f(x)y_m(x)=g(x),\ f(x)\geq0,\ x\in[0,1]$$

i.e.

$$-\epsilon\sum_{k=0}^{m}\sum_{p=2-k}^{m-k}c_kA_{k,p}^{m,2}x^{k+p-2}+f(x)\sum_{k=0}^{m}c_kB_{k,m}(x)=g(x).\tag{19}$$

Let us now collocate (19) at points $x_i,\ i=0,1,......,m-2r(=m-2)$ to obtain

$$-\epsilon\sum_{k=0}^{m}\sum_{p=2-k}^{m-k}c_kA_{k,p}^{m,2}x_i^{k+p-2}+f(x_i)\sum_{k=0}^{m}c_kB_{k,m}(x_i)=g(x_i),\ i=0,1,...,m-2.\tag{20}$$

Substituting (13) in the boundary conditions (4) we get further $2r(=2)$ equations given by

$$\begin{aligned}&\sum_{k=0}^{m}c_k\delta_{k,0}=\alpha_0,\\&\sum_{k=0}^{m}c_k\delta_{k,m}=\alpha_1\end{aligned}\tag{21}$$

Hence Eq. (20),together with (21) gives $(m+1)$ equations for the $(m+1)$ unknowns $c_k,\ k=0(1)m$ which can be solved easily with any of the usual methods. Then the function $y(x)$ approximated by (7) can be obtained.

4. Numerical examples

In this section, we present the numerical results for some examples of singular linear or non-linear higher order differential equations in section 4.2 and self adjoint singularly perturbed problems in section 4.2, to illustrate the presented schemes that are employed in our study.

The performance of the Bernstein polynomial method is measured by the maximum absolute error E_{BPM} which is defined by

$$E_{BPM} = \max\{|y_{exact} - y_{BPM}| : 0 \le x \le 1\}$$

where y_{exact} and y_{BPM} denote the exact solution and approximate solutions obtained by the proposed method respectively.

4.1. Singular higher order boundary value problems

Example 4.1.
Let us consider the linear boundary value problem studied by Gamel and Shamy[4]

$$y^{(8)} + \frac{1}{x}y' + \frac{1}{x^2}y = e^x(x^3 + 25x^2 + 172x + 336),\ 0 < x \le 1, \tag{22}$$

subject to the boundary conditions

$$\begin{aligned} &y(0) = y'(0) = y''(0) = 0,\ y'''(0) = 6,\\ &y(1) = e,\ y'(1) = 4e,\ y''(1) = 13e,\ y'''(1) = 34e, \end{aligned} \tag{23}$$

whose exact solution is

$$y(x) = x^3 e^x.$$

We implement our method with $m = 15$. The exact solution and approximate solution obtained by the proposed method and those obtained by B-spline method are presented in Table 1 and the exact and approximate solution by present method are compared in Fig. 1 which show that our method is highly accurate.

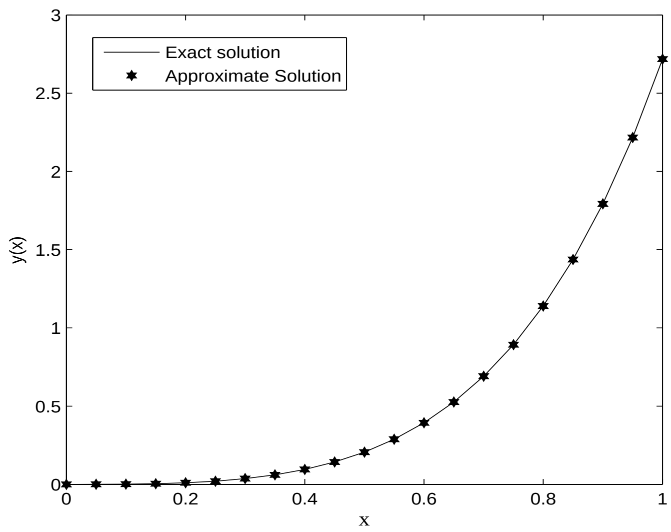

Fig. 1. Comparison of exact and approximate solution for Example 4.1.

Example 4.2.
Let us consider the linear boundary value problem(cf. Gamel and Shamy[4])

$$y^{(8)} + \frac{1}{x}y' + \frac{1}{x^2}y = 3 - 4x,\ 0 < x \le 1, \tag{24}$$

with the boundary conditions

$$\begin{aligned} &y(0) = y'(0) = 0,\ y''(0) = 2,\ y'''(0) = -6, \\ &y(1) = 0,\ y'(1) = -1,\ y''(1) = -4,\ y'''(1) = -6, \end{aligned} \tag{25}$$

whose exact solution is

$$y(x) = x^2(1-x).$$

The problem is solved with $m = 15$. The exact solution and approximate solution obtained by the proposed method and those obtained by B-spline method are presented in Table 2. Fig. 2 shows that the present method gives results with an excellent agreement with the exact solution.

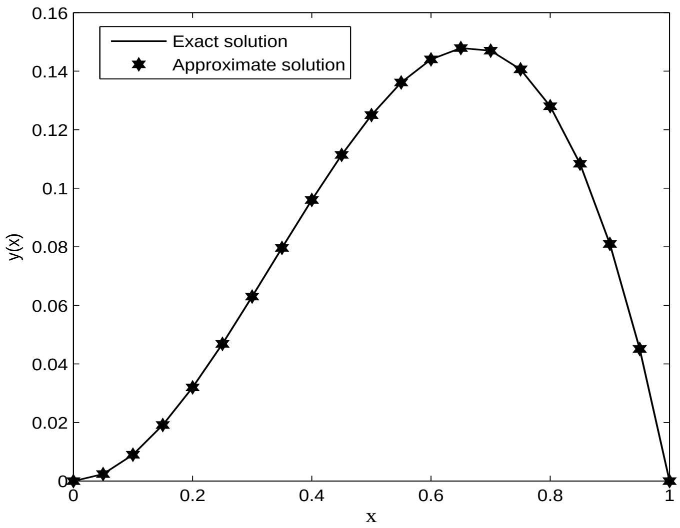

Fig. 2. Comparison of exact and approximate solution for Example 4.2.

Example 4.3.
We consider the non-linear boundary value problem(cf. Gamel and Shamy[4])

$$y^{(6)} + \frac{1}{x}y' + \frac{1}{x^2}y = 3 - e^{-x}y^2 - 4x + x^4e^{-x}(1-x)^2,\ 0 < x \le 1, \tag{26}$$

Table 1. Comparison of results for Example 4.1

x	Exact solution	B-Spline solution($n = 20$)	Bernstein polynomial method
0.1	0.001 105 1709	0.001 105 1713	0.001 105 1709
0.2	0.009 771 2220	0.009 771 2268	0.009 771 2220
0.3	0.036 446 1878	0.036 446 2021	0.036 446 1878
0.4	0.095 476 7806	0.095 476 8054	0.095 476 7806
0.5	0.206 090 1588	0.206 090 1886	0.206 090 1588
0.6	0.393 577 6609	0.393 577 6868	0.393 577 6609
0.7	0.690 717 1787	0.690 717 1943	0.690 717 1787
0.8	1.139 476 9554	1.139 476 9610	1.139 476 9554
0.9	1.793 050 6680	1.793 050 6690	1.793 050 6680
		Maximum Absolute error	5.31797E-12
		CPU time(s)	0.0312

with the boundary conditions

$$\begin{aligned} &y(0) = y'(0) = 0,\ y''(0) = 2, \\ &y(1) = 0,\ y'(1) = -1,\ y''(1) = -4, \end{aligned} \tag{27}$$

whose exact solution is

$$y(x) = x^2(1-x).$$

The problem is solved with $m = 10$. The exact solution and approximate solution obtained by the proposed method and those obtained by B-spline method are presented in Table 3. Fig. 3 shows that the present method works well for non-linear differential equations.

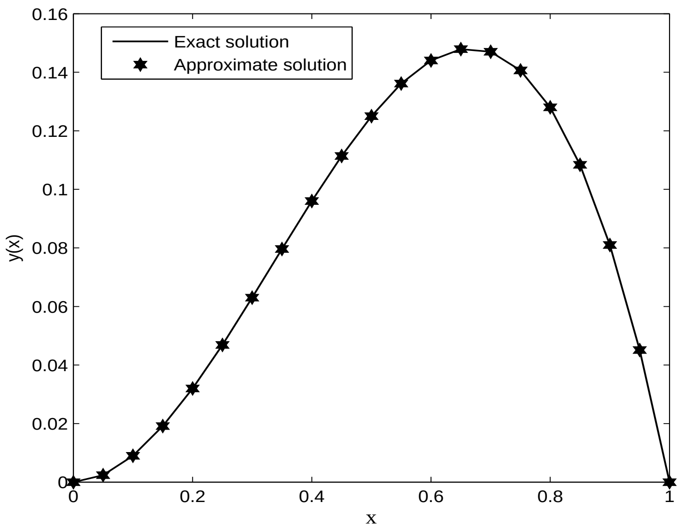

Fig. 3. Comparison of exact and approximate solution for Example 4.3.

Example 4.4.
Let us consider the non-linear boundary value problem(cf. Gamel and Shamy[4])

$$y^{(6)} + \frac{1}{x}y' + \frac{1}{x^2}y = \frac{y}{1+y} + 3 - 4x - \frac{x^2(1-x)}{1+x^2(1-x)},\ 0 < x \le 1, \tag{28}$$

Table 2. Comparison of results for Example 4.2

x	Exact solution	B-Spline solution($n = 20$)	Bernstein polynomial method
0.1	0.009	0.009	0.009
0.2	0.032	0.032	0.032
0.3	0.063	0.063	0.063
0.4	0.096	0.095 999 99	0.096
0.5	0.125	0.124 999 99	0.124 999 9999
0.6	0.144	0.143 999 99	0.143 999 9999
0.7	0.147	0.146 999 99	0.146 999 9999
0.8	0.128	0.127 999 99	0.127 999 9999
0.9	0.081	0.080 999 99	0.080 999 9999
		Maximum Absolute error	4.29168E-12
		CPU time(s)	0.0312

with the boundary conditions

$$\begin{aligned} &y(0) = y'(0) = 0,\ y''(0) = 2, \\ &y(1) = 0,\ y'(1) = -1,\ y''(1) = -4, \end{aligned} \tag{29}$$

whose exact solution is

$$y(x) = x^2(1-x).$$

To solve the problem we implement our method for $m = 10$. The exact solution and approximate solution obtained by the proposed method and those obtained by B-spline method are presented in Table 4. The numerical results are depicted graphically in Fig. 4 which exhibits the effectiveness of the present method.

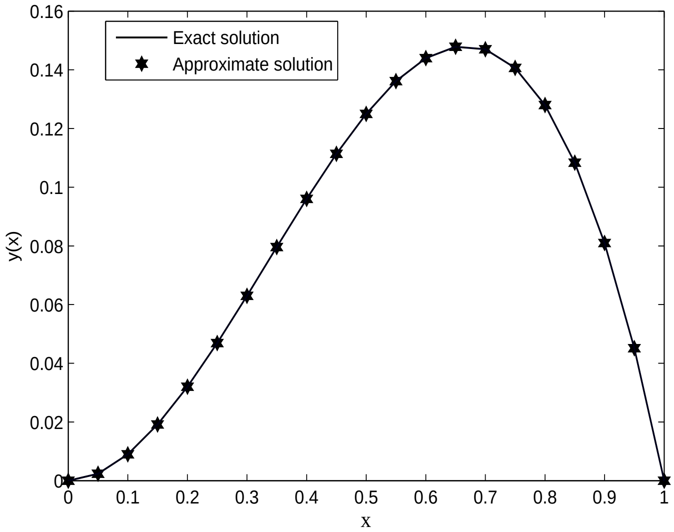

Fig. 4. Comparison of exact and approximate solution for Example 4.4.

4.2. Self adjoint singularly perturbed problem

Next we implement our proposed method to solve singularly perturbed boundary value problems to demonstrate wide applicability and high accuracy of the method.

Table 3. Comparison of results for Example 4.3

x	Exact solution	B-Spline solution($n = 20$)	Bernstein polynomial method
0.1	0.009	0.008 998 64	0.009 000 0014
0.2	0.032	0.031 992 85	0.032 000 0017
0.3	0.063	0.062 984 93	0.063 000 0149
0.4	0.096	0.095 979 16	0.096 000 0204
0.5	0.125	0.124 978 30	0.125 000 0209
0.6	0.144	0.143 982 45	0.144 000 0170
0.7	0.147	0.146 989 34	0.147 000 0107
0.8	0.128	0.127 995 77	0.128 000 0046
0.9	0.081	0.080 999 33	0.081 000 0011
		Maximum Absolute error	4.17444E-14
		CPU time(s)	1.5625

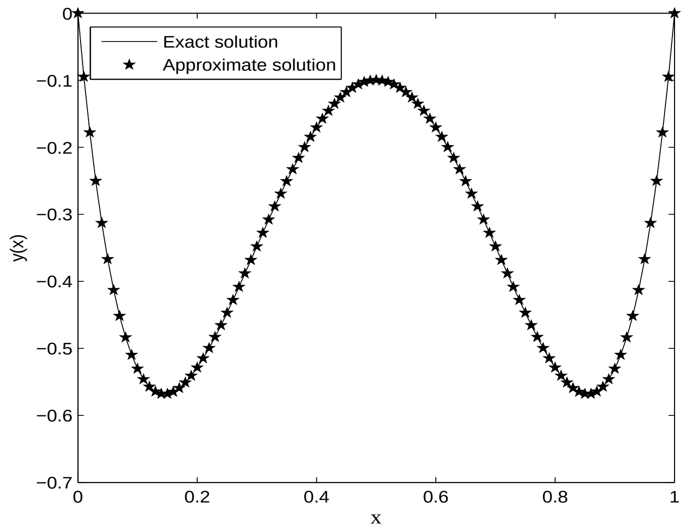

Fig. 5. Comparison of exact and approximate solution for Example 4.5 at $\epsilon = \frac{1}{128}$.

Example 4.5.
Consider the following singularly perturbed boundary value problem(cf. Shah and Abass[15])

$$-\epsilon y''(x) + y(x) = -\cos^2(\pi x) - 2\epsilon\cos(2\pi x),\ x \in [0,1] \tag{30}$$

subject to the boundary conditions

$$y(0) = 0 = y(1). \tag{31}$$

The exact solution of the problem is

$$y(x) = \frac{1+2(1+\pi^2)\epsilon}{(1+e^{\frac{1}{\sqrt{\epsilon}}})(1+4\pi^2\epsilon)}\left(e^{\frac{x}{\sqrt{\epsilon}}} + e^{\frac{1-x}{\sqrt{\epsilon}}}\right) - \frac{1+4\epsilon}{2+8\pi^2\epsilon}\cos(2\pi x) - \frac{1}{2}.$$

The problem is solved by putting $m = 20$ and $\epsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$ and the obtained maximum absolute errors by the present method and those obtained by HWCM method (for $N = 128$) are shown in Table 5. The numerical result for $\epsilon = \frac{1}{128}$ is shown in Fig. 5. These again exhibit the usefulness of the present method.

Table 4. Comparison of results for Example 4.4

x	Exact solution	B-Spline solution($n = 20$)	Bernstein polynomial method
0.1	0.009	0.009 000 02	0.009 000 0031
0.2	0.032	0.032 000 01	0.032 000 0174
0.3	0.063	0.063 000 25	0.063 000 0389
0.4	0.096	0.096 000 40	0.096 000 0571
0.5	0.125	0.125 000 45	0.125 000 0636
0.6	0.144	0.144 000 43	0.144 000 0559
0.7	0.147	0.147 000 30	0.147 000 0376
0.8	0.128	0.128 000 13	0.128 000 0169
0.9	0.081	0.081 000 00	0.081 000 0029
		Maximum Absolute error	2.39808E-14
		CPU time(s)	0.328125

Table 5. Comparison of results for Example 4.5

x	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$	$\epsilon = \frac{1}{256}$
HWCM ($N = 128$)	4.234E-07	1.120E-07	1.321E-08	9.541E-08	2.905E-07
Our method	2.688E-09	5.418E-09	1.031E-08	2.566E-09	2.330E-09
CPU time(s)	0.0312	0.0468	0.0312	0.0468	0.0468

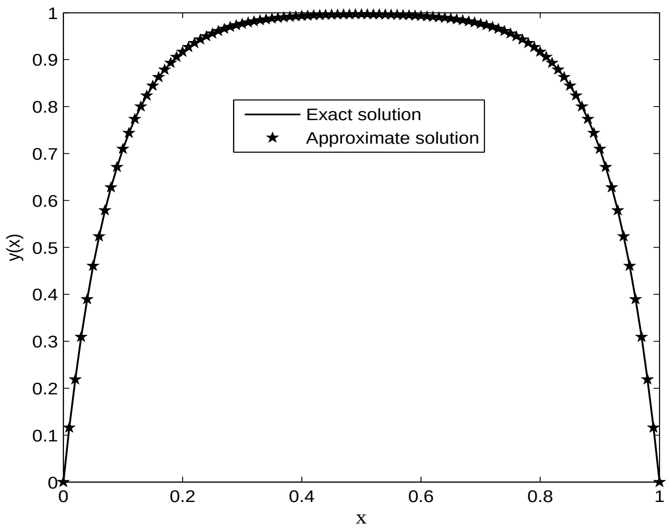

Fig. 6. Comparison of exact and approximate solution for Example 4.6 at $\epsilon = \frac{1}{128}$.

Example 4.6.
Let us consider the following singularly perturbed boundary value problem(cf. Shah and Abass[15])

$$-\epsilon y''(x) + y(x) = 1 + 2\sqrt{\epsilon}\left(e^{-\frac{x}{\sqrt{\epsilon}}} + e^{\frac{x-1}{\sqrt{\epsilon}}}\right), \; x \in [0,1] \tag{32}$$

subject to the boundary conditions

$$y(0) = 0 = y(1). \tag{33}$$

The exact solution of the problem is

$$y(x) = 1 - (1-x)e^{-\frac{x}{\sqrt{\epsilon}}} - xe^{\frac{x-1}{\sqrt{\epsilon}}}.$$

We implement our method for $m = 20$ and $\epsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$ and the maximum absolute errors by the present method and HWCM method (for $N = 128$) are shown in Table 6. The numerical results obtained by our method for $\epsilon = \frac{1}{128}$ and from the exact solution are compared in Fig. 6 and this shows that the present method is quite accurate.

Example 4.7.
We consider the singularly perturbed boundary value problem which is also discussed by Shah and Abass[15]

$$-\epsilon y''(x) + 4y(x) = f(x) \tag{34}$$

with the boundary conditions

$$y(0) = 0 = y(1), \tag{35}$$

Table 6. Comparison of results for Example 4.6

x	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$
HWCM ($N=128$)	6.292E-08	1.410E-07	3.370E-07	6.819E-07	1.695E-07
Our method	7.361E-09	5.165E-08	6.028E-10	2.128E-08	2.148E-08
CPU time(s)	0.0312	0.0468	0.0312	0.0312	0.0468

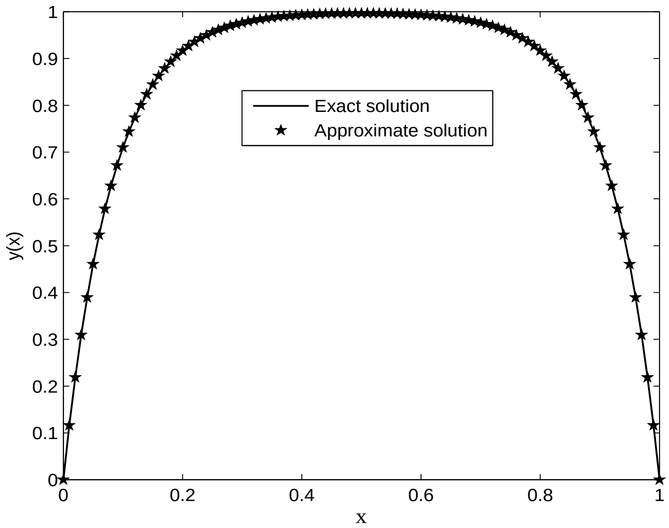

Fig. 7. Comparison of exact and approximate solution for Example 4.7 at $\epsilon = \frac{1}{128}$.

where

$$f(x) = 4 + 2\sqrt{\epsilon}\left(e^{-\frac{x}{\sqrt{\epsilon}}} + e^{\frac{x-1}{\sqrt{\epsilon}}}\right) - 3(1-x)e^{-\frac{x}{\sqrt{\epsilon}}} - 3xe^{\frac{x-1}{\sqrt{\epsilon}}}.$$

The exact solution of the problem is

$$y(x) = 1 - (1-x)e^{-\frac{x}{\sqrt{\epsilon}}} - xe^{\frac{x-1}{\sqrt{\epsilon}}}.$$

We implement our method for $m = 20$ and $\epsilon = \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}, \frac{1}{256}$ and the maximum absolute error obtained by the proposed method and by HWCM method (for $N = 128$) are shown in Table 7. The numerical result for $\epsilon = \frac{1}{128}$ is shown in Fig. 7. These exhibit the usefulness of the present method.

Example 4.8.
We consider the following singularly perturbed two-point boundary value problem which is also discussed by Shah and Abass[15]

$$-\epsilon y''(x) + [1 + x(1-x)]y(x) = f(x) \tag{36}$$

subject to the boundary conditions

$$y(0) = 0 = y(1), \tag{37}$$

where

$$f(x) = 1 + x(1-x) + [2\sqrt{\epsilon} - x^2(1-x)]e^{\frac{x-1}{\sqrt{\epsilon}}} + [2\sqrt{\epsilon} - x(1-x)^2]e^{-\frac{x}{\sqrt{\epsilon}}}.$$

The exact solution of the problem is

$$y(x) = 1 + (x-1)e^{-\frac{x}{\sqrt{\epsilon}}} - xe^{\frac{x-1}{\sqrt{\epsilon}}}.$$

Table 7. Comparison of results for Example 4.7

x	$\epsilon=\frac{1}{16}$	$\epsilon=\frac{1}{32}$	$\epsilon=\frac{1}{64}$	$\epsilon=\frac{1}{128}$	$\epsilon=\frac{1}{256}$
HWCM ($N=128$)	6.292E-09	1.410E-09	9.370E-08	3.810E-09	1.690E-09
Our method	9.930E-10	8.750E-10	9.284E-09	2.212E-10	1.521E-10
CPU time(s)	0.0312	0.0312	0.0312	0.0468	0.0468

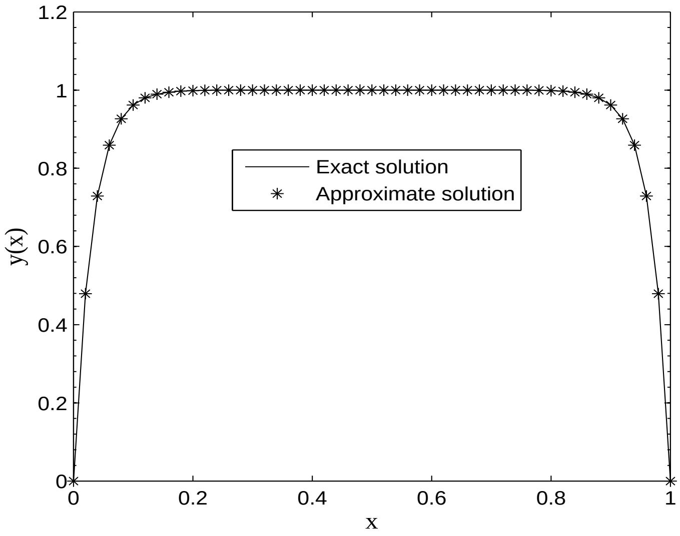

Fig. 8. Comparison of exact and approximate solution for Example 4.8 at $\epsilon = 1.0E-03$.

Table 8 shows the comparison between the maximum absolute errors of the present method and those obtained by HWCM method (for $N = 128$) for $m = 15$ and $m = 20$ and $\epsilon = 1.0E-1,\ 1.0E-2,\ 1.0E-3,\ 1.0E-4$. Comparison between an exact and approximate solution is shown in Fig. 8 for $\epsilon = 1.0E-3$. These show that the present method is quite correct.

Table 8. Comparison of results for Example 4.8

x	$\epsilon=1.0E-01$	$\epsilon=1.0E-02$	$\epsilon=1.0E-03$	$\epsilon=1.0E-04$
HWCM ($N=128$)	6.292E-08	1.410E-07	3.370E-07	6.810E-07
Our method	3.392E-11 (m=15)	6.746E-09 (m=15)	2.638E-09 (m=20)	1.632E-08 (m=20)
CPU time(s)	0.0156	0.0156	0.0312	0.0312

Example 4.9.
Let us consider the singularly perturbed boundary value problem (cf. Shah and Abass[15])

$$-\epsilon y''(x) - y(x) = 0 \tag{38}$$

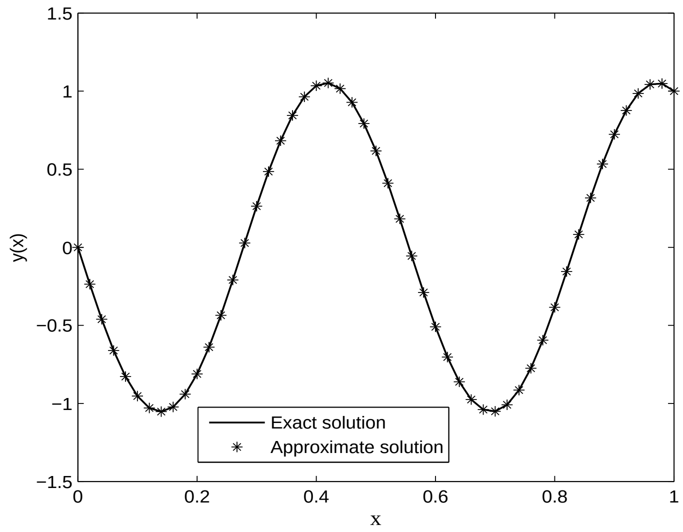

Fig. 9. Comparison of exact and approximate solution for Example 4.9 at $\epsilon = \frac{1}{128}$.

with the boundary conditions

$$y(0) = 0, \; y(1) = 1, \tag{39}$$

whose exact solution is given by

$$y(x) = \frac{\sin(\frac{x}{\sqrt{\epsilon}})}{\sin(\frac{1}{\sqrt{\epsilon}})}, \; \epsilon \neq (n\pi)^{-2}.$$

The maximum absolute errors by the present method and those obtained by HWCM method (for $N = 128$) are presented in Table 9 for $m = 20$ and $\epsilon = \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \frac{1}{128}$ respectively. Fig. 9 shows the comparison between our solution and exact solution for $\epsilon = \frac{1}{128}$. These exhibit the usefulness of the present method.

Table 9. Comparison of results for Example 4.9

x	$\epsilon = \frac{1}{8}$	$\epsilon = \frac{1}{16}$	$\epsilon = \frac{1}{32}$	$\epsilon = \frac{1}{64}$	$\epsilon = \frac{1}{128}$
HWCM ($N = 128$)	8.92E-05	9.10E-06	6.17E-05	6.46E-05	1.80E-04
Our method	2.726E-09	2.352E-10	3.559E-09	2.881E-09	5.324E-08
CPU time(s)	0.0156	0.0312	0.0312	0.0312	0.0468

5. Conclusion

In this paper, we have presented a simple numerical algorithm to solve boundary value problems of two forms, i.e. singular linear or non-linear higher order boundary value problems in section 3.1 and self adjoint singularly perturbed boundary value problems in section 3.2. Here we approximate the unknown function in terms of truncated series involving Bernstein polynomials. The properties of Bernstein polynomials were used to reduce the boundary value problems to the solution of algebraic equations by avoiding the appearance of ill-conditioned matrices or complicated integrations.By adopting a few terms for the truncated series we observed a great accuracy between the exact and the approximate solutions computed numerically.

Thus, the proposed method can be utilized as a powerful solver for the solution of singular and singularly perturbed boundary value problems. As compared to the other methods, the present method strikes as being more convenient and simple in principle and for computer algorithms.

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