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The Persistence of double predator model with hosts

Research Article

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Abstract: In this paper, the conditions of the occurrence of persistence of a mathematical model consist from three-species Syn-Ecosymbiosis involving different types of ecological interactions is proposed and analyzed. Finally, in order to confirm our obtained analytical results, numerical simulations have been done for a hypothetical set of parameter values

MSC: 92D40

Keywords: Equilibrium point (EQ) • Persistence • Lyapunov function

1. Introduction

The dynamic relationship between predators and their prey has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology due to the universal existence and importance. An important and ubiquitous problem in predator-prey theory and related topics in mathematical ecology, concerns the long term coexistence (or persistence) of species, for example, see [1–8]. Freedman and Waltman [9] considered three-level food webs-two competing predators feeding on a single prey and a single predator feeding on two competing prey species. They obtained criteria for the system to be persist. Saadi [10] proposed and analyzed a prey-predator model with three Syniecological system with Holling type-II functional response, they obtained a set of sufficient and necessary condition, which guarantee the lacal and global stability of this system. In this paper, however, we will establish the conditions of the occurrence of persistence of a mathematical model proposed by Saadi [10].

2. The mathematical model

An ecological model of three-species Syn-Ecosymbiosis, comprising of prey-predator and commensalisms [10]

$$\left.\begin{aligned} \frac{dN_1}{dT} &= r_1 N_1\left(1-\frac{N_1}{k_1}\right)-\frac{a_1 N_1}{b_1+N_1}N_2 + c\,N_1 N_3\\ \frac{dN_2}{dT} &= e_1\frac{a_1 N_1}{b_1+N_1}N_2 + e_2\frac{a_2 N_3}{b_2+N_3}N_2 - d_1 N_2 - d_2 N_2^2\\ \frac{dN_3}{dT} &= r_2 N_3\left(1-\frac{N_3}{k_2}\right)-\frac{a_2 N_3}{b_2+N_3}N_2 \end{aligned}\right\} \tag{1}$$

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where $0 < e_i < 1$; $i = 1,2$. represents the conversion rate.

This model consists of a prey (for example, Great Egret Bird) whose populace thickness at time T indicated by N_1, the predator (for example, Crocodile) whose populace thickness at time T indicated by N_2, the host (for example, Water Buffalo) whose populace thickness at time T indicated by N_3. We assumed that each parameters non-negative and described as given in [10].

Now, for further simplification of the model (1), the following dimensionless variables are used in [10].

$$t = r_1 T \ , \ x = \frac{N_1}{k_1} \ , \ y = \frac{N_2}{k_1} \ , \ z = \frac{c N_3}{r_1} \ ,$$
$$u_1 = \frac{a_1}{r_1} \ , \ u_2 = \frac{b_1}{k_1} \ , \ u_3 = \frac{a_2}{r_1} \ , \ u_4 = \frac{c b_2}{r_1} \ ,$$
$$u_5 = \frac{d_1}{r_1} \ , \ u_6 = \frac{d_2 k_1}{r_1} , u_7 = \frac{r_2}{r_1} \ , \ u_8 = \frac{r_1}{c k_2} \ , \ u_9 = \frac{c a_2 k_1}{r_1^2}$$

We use the dimensional method to reduce the parameters of model (1):

$$\left.\begin{array}{l}
\frac{dx}{dt} = x\left[(1-x) - \frac{u_1 y}{u_2 + x} + z\right] = x f_1(\hbar) \\
\frac{dy}{dt} = y\left[\frac{e_1 u_1 x}{u_2 + x} + \frac{e_2 u_3 z}{u_4 + z} - u_5 - u_6 y\right] = y f_2(\hbar) \\
\frac{dz}{dt} = z\left[u_7(1 - u_8 z) - \frac{u_9 y}{u_4 + z}\right] = z f_3(\hbar.) \\
\text{where } \hbar = (x, y, z)
\end{array}\right\} \tag{2}$$

Here x_0, y_0 *and* z_0 are non-negative. Clearly the function in the right hand f_1 , f_2 & f_3 in the model (2) are continuous, also the partial derivatives on the region $R^3_+ = (\mathcal{A})$ are continuous:

$$\mathcal{A} = \{\hbar \in \mathcal{A} : x_0 \geqslant 0, y_0 \geqslant 0, z_0 \geqslant 0\}$$

Here f_1 , f_2 & f_3 are Lipschitzian on $\mathcal{A}$, then the trajectories of the model (2) existence and uniquence. The uniformly bounded and the existence of (EQ) in model (2) [see [10]].

3. The stability analysis of equilibrium points of model

The three-species Syn-Ecosymbiosis model given by the model (2) has five equilibrium points, which are mentioned with their existence conditions in [10] as in the following (EQ):

The (EQ) namely $E_0 = (0,0,0)$ unstable, the first two species (EQ) namely $E_1 = (\bar{x}, \bar{y}, 0)$ is stable under the condition (3a) and (3b):

$$\frac{u_1 \bar{y}}{(u_2 + \bar{x})^2} < 1 \tag{3a}$$

$$u_7 < \frac{u_9 \bar{y}}{u_4} \tag{3b}$$

The second two species (EQ) namely $E_2 = \left(\breve{x}., 0, \breve{z}\right) = \left(\frac{1+u_8}{u_8}, 0, \frac{1}{u_8}\right)$ is stable under the condition:

$$\frac{e_1 u_1 \breve{x}}{u_2 + \breve{x}} + \frac{e_2 u_3 \breve{z}}{u_4 + \breve{z}} < u_5 \tag{4}$$

The third two species (EQ) namely $E_3 = \left(0, \hat{y}, \hat{z}\right)$ is stable under the condition:

$$u_6 \hat{y} + u_7 u_8 \hat{z} > \frac{u_9 \hat{y} \hat{z}}{(u_4 + \hat{z})} \tag{5a}$$

$$u_7 u_8 \hat{z} > \frac{u_9 \hat{y} \hat{z}}{(u_4 + \hat{z})^2} \tag{5b}$$

$$1 + \hat{z} < \frac{u_1 \hat{y}}{u_2} \tag{5c}$$

And the positive (EQ) namely $E_4 = (x^*, .y^*, z^*)$ is stable if the following conditions hold

$$\frac{u_1 y^*}{(u_2 + x^*)^2} > 1 \tag{6a}$$

$$u_7 u_8 > \frac{u_9 y^*}{(u_4 + z^*)^2} \tag{6b}$$

$$S_1 + S_2 + S_3 - S_4 > S_5 \tag{6c}$$

where

$$S_1 = \left(-x^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]-u_6 y^*\right)\left[-u_6 y^* x^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]-\frac{e_1 u_1^2 u_2 x^*}{(u_2+x^*)^3}\right]$$
$$S_2 = \left(-u_6 y^* + z^*\left[-u_7 u_8+\frac{u_9 y^*}{(u_4+z^*)^2}\right]\right)\left(\frac{-e_2 u_3 u_4 z^*}{(u_4+z^*)^3}+u_6 y^* z^*\left[-u_7 u_8+\frac{u_9 y^*}{(u_4+z^*)^2}\right]\right)$$
$$S_3 = x^* z^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]\left[-u_7 u_8+\frac{u_9 y^*}{(u_4+z^*)^2}\right]\left(-x^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]+z^*\left[-u_7 u_8+\frac{u_9 y^*}{(u_4+z^*)}\right]\right)$$
$$S_4 = 2u_6 x^* y^* z^*\left[-1+\frac{u_1 y^*}{(u_2+x^*)^2}\right]\left[-u_7 u_8+\frac{u_9 y_5}{(u_4+z^*)^2}\right]$$
$$S_5 = \frac{e_1 u_1 u_2 u_9 x^* y^* z^*}{(u_2+x^*)^2 (u_2+z^*)}$$

4. The persistence of model

In general, persistence is a global property of a dynamical system; it is not dependence upon interior solution space structure but is dependent upon solution behavior near extinction boundaries (boundary planes). From the biological point of view, persistence of a system means the survival of all population of the system in future time. However, mathematically it means that strictly positive solutions do not have omega limit set on the boundary of the non-negative cone [11]. Accordingly, if the dynamic system does not persist, then the solution have omega limit set on the boundary of the non-negative cone, and hence the dynamic system faces extinction. Now, before examine the persistence of stage structure model given by model (2) by using the method of average Lyapunov function as given in [11], we need to study the global dynamics in the boundary planes xy, xz and yz as shown in the following theorems.

Theorem 4.1.
Suppose that the (EQ) E_1 of the model (2) is locally stable in the Interior R_+^2 (I) and the following conditions hold

$$\frac{u_1}{(u_2+x)^2} < \frac{1}{y} + \frac{u_6}{x} \tag{7}$$

Then the (EQ) E_1 of the model (2) is globally stable in the (I) of $xy-$plane.

Proof. We will proof the theorem in (I). For any initial value clearly in the (I) of $xy-$ plane, model (2) reduces to the following sub model

$$\begin{aligned}\frac{dx}{dt} &= x - x^2 - \frac{u_1 xy}{u_2+x} = h_1(x,y)\\ \frac{dy}{dt} &= \frac{e_1 u_1 xy}{u_2+x} - u_5 y - u_6 y^2 = h_2(x,y)\end{aligned} \tag{8}$$

Assume that $H(x,y) = \frac{1}{xy}$. Clearly, Clearly, is a function and is a positive for all. is a C^1 function and is a positive for all. Further

$$\begin{aligned}\Delta(x,y) &= \frac{\partial}{\partial x}(Hh_1) + \frac{\partial}{\partial y}(Hh_2)\\ &= \frac{u_1}{(u_2+x)^2} - \frac{1}{y} - \frac{u_6}{x}\end{aligned}$$

Then by using condition (7) we get $\Delta(x,y) < 0$. Thus E_1 is a globally stable in the (I) of $xy-$plane. □

Theorem 4.2.
Suppose that the (EQ) $E_@$ of the model (2) is locally stable in the (I) then it is a globally stable in the (I) of $xz-$plane.

Proof. We will proof the theorem in the (I). For any initial value clearly in the (I) of $xz-$ plane, model (2) reduces to the following sub model

$$\begin{aligned}\frac{dx}{dt} &= x - x^2 + xz = h_1(x,z)\\ \frac{dy}{dt} &= u_7 z - u_8 z^2 = h_2(x,z)\end{aligned} \tag{9}$$

Obviously E_2 represents the positive (EQ) of sub model (2) in the (I) of $xz-$plane. Assume that $H(x,z) = \frac{1}{xz}$. Clearly, $H(x,z)$ is a C^1 function and is a positive for all.
Further

$$\begin{aligned}\Delta(x,z) &= \frac{\partial}{\partial x}(Hh_1) + \frac{\partial}{\partial z}(Hh_2)\\ &= -\frac{1}{z} - \frac{u_8}{z}.\end{aligned}$$

Not that $\Delta(x,z)$ does not change sign and is not identically zero in the (I) of the $xz-$plane. Then according to "Bendixson-Dualic criterion" sub model (9) has no periodic dynamic in the interior of positive quadrant of $xz-$plane. Further, since E_2 is the only positive equilibrium point of sub model (9) in the interior of positive quadrant of $xz-$plane. Hence according to poincare-Bendixson theorem E_2 is a globally stable in the interior of positive quadrant. □

Theorem 4.3.
Suppose that the (EQ) E_3 of the model (2) is locally stable in the (I) then it is a globally stable in the (I) of $yz-$plane.

The proof of this theorem is similar to Theorem 4.2.

Theorem 4.4.
Assume that there are no periodic dynamics of model (2) in the boundary of the solution. Further, if the following conditions are hold:

$$u_7 > \frac{u_9\,\bar{y}}{u_4} \tag{10a}$$

$$\frac{e_1 u_1 \breve{x}}{u_2+\breve{x}} + \frac{e_2 u_3 \breve{z}}{u_4+\breve{z}} > u_5 \tag{10b}$$

$$1+\hat{z} > \frac{u_1\,\hat{y}}{u_2} \tag{10c}$$

Then model (2) is uniformly persists.

Proof. Consider the following average Lyapunov function $\delta(x,y,z) = x^{P_1}\,y^{P_2}\,z^{P_3}$, where each P_i , $i=1,2,3$ is assumed to be positive, obviously $\delta(x,y,z)$ is continuously differentiable positive function defined in since.

$$\begin{aligned}\Psi(x,y,z) &= \frac{\delta'(x,y,z)}{\delta(x,y,z)}\\ &= P_1\left[1-x-\frac{u_1\,y}{u_2+x}+z\right] + P_2\left[\frac{e_1 u_1 x}{u_2+x}+\frac{e_2 u_3 z}{u_4+z}-u_5-u_6\,y\right] + P_3\left[u_7\,(1-u_8 z)-\frac{u_9\,y}{u_4+z}\right]\end{aligned}$$

1. For $E_1 = (\bar{x},\bar{y},0)$ we have

$$\Psi(E_1) = P_3\left(u_7-\frac{u_9\,\bar{y}}{u_4}\right)$$

By using the condition (10a) we get $\Psi(E_1) > 0$.

2. For $E_2 = (\breve{x},0,\breve{z})$ we have

$$\Psi(E_2) = P_2\left(\frac{e_1 u_1\,\breve{x}}{u_2+\breve{x}}+\frac{e_2 u_3\,\breve{z}}{u_4+\breve{z}}-u_5\right)$$

By using the condition (10b) we get $\Psi(E_2) > 0$.

3. For $E_3 = (0,\hat{y},\hat{z})$ we have

$$\Psi(E_3) = P_1\left(1-\frac{u_1\hat{y}}{u_2}+\hat{z}\right)$$

By using the condition (10c) we get $\Psi(E_3) > 0$.

Hence model (2) is uniformly persists. □

5. Numerical simulation

We study numerically of model (2) in this section. Knowing the dynamical behavior of model (2) by use various. Sets of parameter and various sets of initial .The influence of varying the value of all parameter on model(2) are studied to analysis the dynamical behavior for the model proposed and obtained analytical results . It is notes that, for the following set of hypothetical parameters that satisfies stability conditions of positive (EQ), model (2) has a globally stable positive (EQ) as shown in Fig. 1. Note that, from now onward the blue, green and red colors are used to describing the trajectories of x, $y$and z.

$$\begin{aligned} &u_1 = 0.6,\ u_2 = 0.25\,,\ u_3 = 3.9\,,\ u_4 = 0.05\,,\ u_5 = 2\,,\ u_6 = 0.5\,,\ u_7 = 2 \\ &u_8 = 0.75\,,\ u_9 = 0.8\,,\ e_1 = 0.5\,,\ e_2 = 0.5\,. \end{aligned} \tag{11}$$

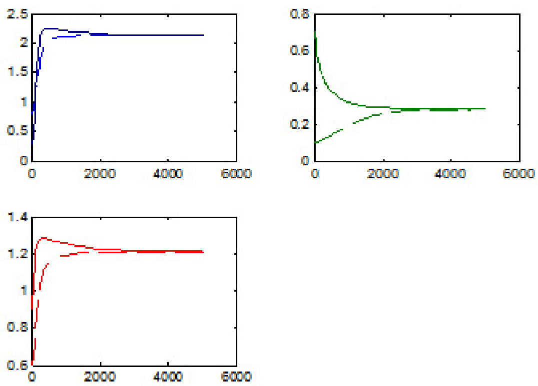

Fig. 1. Time arrangement of the arrangement of model (2) that started from two different initial points (0.8, 0.7, 0.6) and (0.9, 0.3, 0.1) for the data given by Eq. (11). (a) directions of x (b) directions of y (c) directions of z.

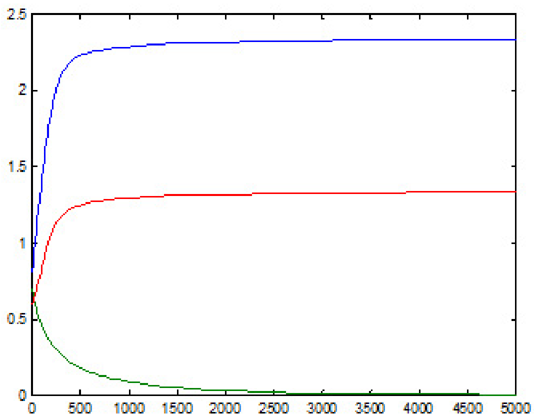

Fig. 2. The trajectory approaches to E_2.

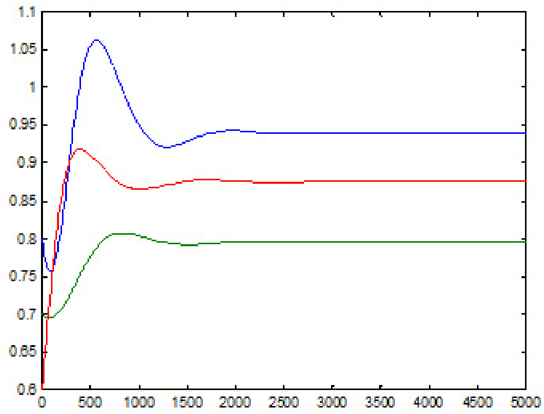

Fig. 3. The trajectory approaches to E_4.

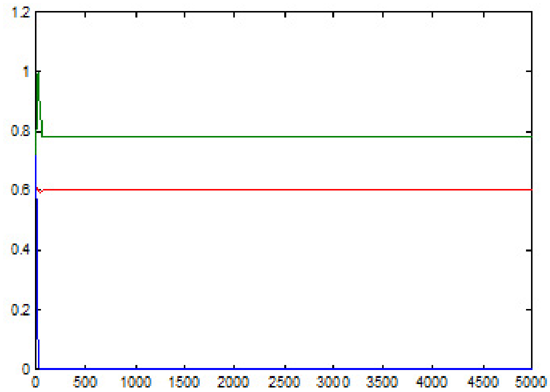

Fig. 4. The trajectory approaches to E_3.

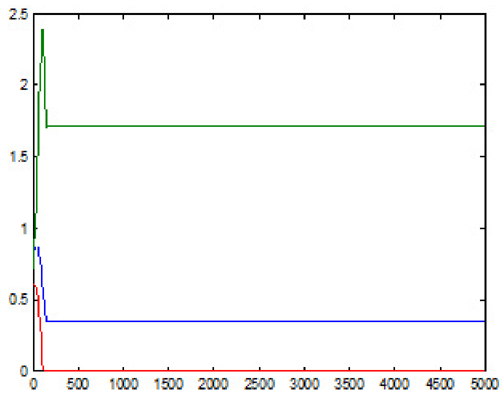

Fig. 5. The trajectory approaches to E_1.

It is note that for the data as given in Eq. (11) with $u_1 \leqslant 0.1$, the trajectory of model (2) approaches to $E_2 = (\breve{x}, 0, \breve{z})$ in the $xz-$plane as shown in Fig. 2

For the parameters values given in Eq. (11) with $0.2 \leqslant u_1 \leqslant 1.5$, it is note that the trajectory of model (2) approaches to $E_4 = (x^{*\cdot}, y^{*\cdot}, z^{*\cdot})$ in the $xyz-$space as shown in Fig. 3.

For the parameters values given in Eq. (11) with $5 \leqslant u_1 \leqslant 8.8$, it is note that, the trajectory of model (2) approaches to $E_3 = (0, \hat{y}, \hat{z})$ in the $yz-$plane as shown in Fig. 4.

For the parameters values given in Eq. (11) with $u_5 = 0.01$, it is note that the trajectory of model (2) approaches to $E_1 = (\bar{x}, \bar{y}, 0)$ in the $xy-$plane as shown in Fig. 5.

Keeping the above in view we will summarize our obtained numerical results in the form of Table 1 as shown below.

Table 1. Numerical behavior of model (2) as changing in a specific parameter conservation different parameters settled as in Eq. (11)

Parameters varied in system (2)	Numerical behavior of system (2)
$.1.6 \leqslant u_1 \leqslant 4.9$	Approaches to periodic.
$u_1 \geqslant 8.9$	Approaches to stable point in (I).
$0.01 \leqslant u_2 \leqslant 6$	Approaches to stable point in (I).
$u_2 \geqslant 7$	Approaches to stable point in $xz-$plane.
$0.01 \leqslant u_3 \leqslant 3.5$	Approaches to stable point in $xz-$plane.
$3.6 \leqslant u_3 \leqslant 4.8$	Approaches to stable point in (I).
$4.9 \leqslant u_3 \leqslant 12$	Approaches to periodic.
$u_3 \geqslant 12.1$	Approaches to stable point in $xy-$plane.
$u_4 = 0.01$	Approaches to stable point in $xz-$plane.
$0.02 \leqslant u_4 \leqslant 0.22$	Approaches to stable point in (I).
$u_4 \geqslant 0.3$	Approaches to stable point in $xz-$plane.
$0.5 \leqslant u_5 \leqslant 1.5$	Approaches to periodic.
$1.6 \leqslant u_5 \leqslant 2.1$	Approaches to stable point in (I).
$u_5 \geqslant 2.2$	Approaches to stable point in $xz-$plane.
$0.01 \leqslant u_6 \leqslant 13.6$	Approaches to stable point in (I).
$u_6 \geqslant 13.7$	Approaches to stable point in $xz-$plane.
$0.03 \leqslant u_7 \leqslant 0.4$	Approaches to stable point in $xz-$plane.
$u_7 \geqslant 0.41$	Approaches to stable point in (I).
$u_8 = 0.01$	Approaches to stable point in $xz-$plane.
$0.02 \leqslant u_8 \leqslant 2.8$	Approaches to stable point in (I).
$u_8 \geqslant 2.9$	Approaches to stable point in $xz-$plane.
$0.01 \leqslant u_9 \leqslant 2.2$	Approaches to stable point in (I).
$2.3 \leqslant u_9 \leqslant 4$	Approaches to stable point in $xz-$plane.
$u_9 \geqslant 4.1$	Approaches to stable point in $xy-$plane
$0.01 \leqslant e_1 \leqslant 0.1$	Approaches to stable point in $xz-$plane.
$0.2 \leqslant e_1 \leqslant 1$	Approaches to stable point in (I).
$0.01 \leqslant e_2 \leqslant 0.4$	Approaches to stable point in $xz-$plane.
$0.5 \leqslant e_2 \leqslant 0.6$	Approaches to stable point in (I).
$0.7 \leqslant e_2 \leqslant 1$	Approaches to periodic.

6. Conclusion and discussion

In this paper, a mathematical model consisting of three mathematical equations was studied. The system has five fixed points that are acceptable from the biological point of view. Local and global stability was also discussed in [10]. In this paper the existence (persistence) was studied as in theory 1, theory 2, theory 3 and theory 4. Finally, the proposed model was studied numerically. And the following results are obtained:

- For the set of hypothetical parameters values given Eq. (11), the model (2) approaches to globally stable positive (EQ).
- For the set of data by Eq. (11), model(2) has a globally stable positive point in the (I) . However as the attack rate u_1 decreases then the predator species will faces extinction and the solution of model (2) approaches to $E_2 = \left(\breve{x}, 0, \breve{z}\right)$ in the first quadrant $xz-$plane .while increasing u_1 will causes destabilizing of model (2) and the point in the (I). it is watched that the conversion rate parameter e_1 and the intrinsic growth rate.
- As the half saturation rate u_2 decreases conservation the rest of parameters as in Eq. (11) then again the solution of model (2) access stable positive point in the (I). Otherwise the systems still have approaches to $E_2 = (\breve{x}, 0, \breve{z})$ in

the first quadrant $xz-$plane. It is observed that the half saturation rate parameter u_4 and the carrying capacity rate u_8.

- As the attack rate u_3 decreases conservation the rest of parameters as in Eq. (11) then again the solution of model (2) access to $E_2 = (\breve{x}, 0, \breve{z})$ the fist quadrant $xz-$plane. Otherwise the systems still have approaches to $E_1 = (\bar{x}, \bar{y}, 0)$ in the first quadrant $xy-$plane.
- As the natural death u_5 decreases conservation the rest of parameters as in Eq. (11) then again the solution of model (2) access to $E_1 = (\bar{x}, \bar{y}, 0)$ the first quadrant$xy-$plane. Otherwise the models still have approaches to $E_2 = (\breve{x}, 0, \breve{z})$ in the first quadrant $xz-$plane.
- As the attack rate u_9 decreases conservation the rest of parameters as in Eq. (11) then again the solution of model (2) access stable positive point in the (I). Otherwise the models still have approaches to $E_1 = (\bar{x}, \bar{y}, 0)$ in the first quadrant $xy-$ plane.
- As the conversion rate e_2 decreases conservation the rest of parameters as in Eq. (11) then again the solution of model (2) access to $E_2 = (\breve{x}, 0, \breve{z})$ the first quadrant$xz-$plane. While increasing e_2 will causes destabilizing of model (2) and the solution approaches to a limit cycle in (I).

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