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Bracket polynomials of torus links as Fibonacci polynomials

Research Article

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Abstract: In this paper we work the bracket polynomial of $(2,n)$-torus link as a Fibonacci polynomial. We show that the bracket polynomial of $(2,n)$-torus link provides recurrence relation as similar to the Fibonacci polynomial and give its some fundamental properties. We also prove important identities, which are similar to the Fibonacci identities, for the bracket polynomial of $(2,n)$-torus link and prove Fibonacci-like identities of the Jones polynomial of $(2,n)$-torus link as a result of the bracket polynomial. Finally, we observe that the bracket polynomial of $(2,n)$-torus link and therefore its Jones polynomial can be derived from its Alexander-Conway polynomial or classical Fibonacci polynomial.

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1. Introduction

The knot (and link) polynomials are important invariants of the knot. There are very popular knot polynomials in the knot theory [1–8]. The first of these was defined by J. W. Alexander in 1928 [1]. Then, J. Conway described the Alexander polynomial with a skein relation in 1969 [4], known as the Alexander-Conway polynomial. The Alexander-Conway polynomial associated with the oriented link diagram L, $\nabla_L(x)$, is a Laurent polynomial in the variable x. $\nabla_L(x)$ is an ambient isotopy invariant of the link L determined by the following axioms:

$$\nabla_{L_+}(x) - \nabla_{L_-}(x) = x\nabla_{L_0}(x),$$
$$\nabla_{\bigcirc}(x) = 1,$$

where L_+, L_-, L_0 are link diagrams resulting from crossing and smoothing changes on a local region of a specified crossing of the diagram L as drawn in Fig. 1 and $\bigcirc$ is any diagram of the unknot.

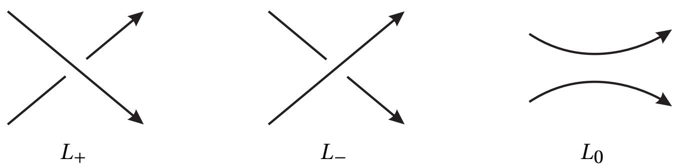

Fig. 1. Skein diagrams

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In 1984, a new polynomial for knots and links was discovered by V. Jones [6]. The Jones polynomial associated with the oriented link diagram L, $V_L(t)$, is a Laurent polynomial in the variable $t^{1/2}$. $V_L(t)$ is an invariant of the ambient isotopy for the link L determined by the following axioms:

$$t^{-1}V_{L_+}(t) - tV_{L_-}(t) = (t^{1/2} - t^{-1/2})V_{L_0}(t), \tag{1}$$

$$V_{\bigcirc}(t) = 1. \tag{2}$$

From the axioms (1) and (2), we have $t^{-1} - t = (t^{1/2} - t^{-1/2})V_{\bigcirc\bigcirc}(t)$ or $\delta = -t^{1/2} - t^{-1/2}$ with $\delta = V_{\bigcirc\bigcirc}(t)$, where L_+, L_-, L_0 are diagrams in Fig. 1 and $\bigcirc\bigcirc$ is trivial link with two components.

In 1987, L. H. Kauffman defined an oriented state model of Jones polynomial [9]. The primary version of this is an invariant of regular isotopy for unoriented knot and link diagram L, called bracket polynomial and by denoted $\langle L\rangle$ and determined by the following axioms:

$$\begin{aligned} \langle D_+\rangle &= A\langle D_0\rangle + A^{-1}\langle D_\infty\rangle,\\ \langle D_-\rangle &= A^{-1}\langle D_0\rangle + A\langle D_\infty\rangle,\\ \langle\bigcirc\rangle = 1, &\quad \langle I\rangle = -A^{-3}\langle I_0\rangle,\\ \delta = -A^2 - A^{-2}, &\quad \langle I'\rangle = -A^3\langle I_0\rangle, \end{aligned} \tag{3}$$

where D_+, D_-, D_0, D_∞, I, I' and I_0 are diagrams drawn in Fig. 2.

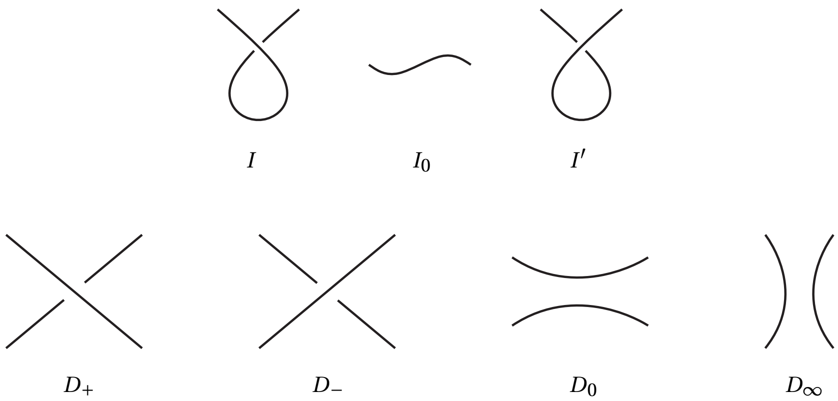

Fig. 2. Crossings and splits

It is possible to create an invariant of ambient isotopy associated with the bracket polynomial $\langle L\rangle$ for oriented diagram L which is called normalized bracket polynomial and denoted by N_L. The normalized bracket polynomial defined as bellow:

$$N_L = (-A^3)^{-\omega(L)}\langle L\rangle, \tag{4}$$

where $\omega(L)$ is the writhe of the oriented link diagram L [9, 10].

The normalized bracket polynomial N_L yields the Jones polynomial V_L [9, 10]. Namely,

$$N_L(t^{-1/4}) = V_L(t). \tag{5}$$

The theory of the Fibonacci numbers and Fibonacci polynomials have important applications in almost every branch of modern science. In literature, there are many studies on the Fibonacci polynomials and its generalizations, see [11–21] and others. It is observed that the Alexander-Conway polynomial of $(2,n)$-torus link is the classical Fibonacci polynomial, see [7, 22]. Also, the relations between the Fibonacci-type polynomials, the recurrence relations and the knot polynomials were studied in [23–26].

This paper is organized as follows: Section 2 includes summary information about the generalized Fibonacci polynomial in [26]. In Section 3, we show that the bracket polynomial of $(2,n)$-torus link provides a recurrence relation as similar to the Fibonacci polynomial and examine its Fibonacci-like properties. We prove identities similar to the identities of Catalan, Cassini and d'Ocagne which are important Fibonacci identities. We also give the properties of the Jones polynomial as results of Fibonacci-like properties of bracket polynomial. We see that the bracket polynomial of $(2,n)$-torus link and therefore its Jones polynomial can be derived from its Alexander-Conway polynomial.

2. A generalization of Fibonacci polynomials

In this section, we give summary information from [26] about the generalized Fibonacci polynomials which will be the basis to study bracket polynomials of $(2,n)$-torus link and their properties.

Definition 2.1.
The generalized Fibonacci polynomials $\{F_n(a,x)\}_{n=0}^{\infty}$ in two variables a and x are defined by the recurrence relation

$$F_n(a,x) = axF_{n-1}(a,x) + a^2F_{n-2}(a,x), \quad n \geq 2 \tag{6}$$

with initial conditions

$$F_0(a,x) = 0 \text{ and } F_1(a,x) = 1.$$

For $a = 1$, we obtain the Fibonacci polynomials in x defined by the recurrence relation

$$f_n(x) = xf_{n-1}(x) + f_{n-2}(x), \quad n \geq 2$$

with initial conditions

$$f_0(x) = 0 \text{ and } f_1(x) = 1.$$

From the equality $\{n\}_{s,t} = t^{(n-1)/2} f(\frac{s}{\sqrt{t}})$ for the relation $\{n\}_{s,t} = s\{n-1\}_{s,t} + t\{n-2\}_{s,t}$ in [11], it is easily to see that there is the following relation between $F_n(a,x)$ and $f_n(x)$

$$F_n(a,x) = a^{n-1} f_n(x). \tag{7}$$

The characteristic equation of the relation (6) is

$$\lambda^2 - ax\lambda - a^2 = 0$$

and the roots are

$$\alpha = \frac{ax + a\sqrt{x^2+4}}{2} \text{ and } \beta = \frac{ax - a\sqrt{x^2+4}}{2}. \tag{8}$$

We have the following relations between a, x and α, β:

$$\alpha + \beta = ax, \quad \alpha\beta = -a^2, \quad \alpha - \beta = a\sqrt{x^2+4}. \tag{9}$$

We now give some fundamental properties of the polynomial $F_n(a,x)$.

Proposition 2.1.
The generating function

$$g_F(\lambda) = \sum_{n=0}^{\infty} F_n(a,x)\lambda^n$$

of the sequence $\{F_n(a,x)\}$ *is given by*

$$g_F(\lambda) = \frac{\lambda}{1 - ax\lambda - a^2\lambda^2}.$$

Proposition 2.2.
For $n \geq 0$*, the Binet's formula of* $F_n(a,x)$ *is*

$$F_n(a,x) = \frac{\alpha^n - \beta^n}{\alpha - \beta}. \tag{10}$$

Proposition 2.3.
For $n \geq 1$*, the explicit formula of* $F_n(a,x)$ *is given by*

$$F_n(a,x) = \sum_{k=0}^{\lfloor n-1/2 \rfloor} \binom{n-k-1}{k} a^{n-1} x^{n-2k-1}. \tag{11}$$

Theorem 2.1 (Generalized Catalan Identity).
Let F_n be the generalized Fibonacci polynomials. Then, for $m \geq n \geq 1$,

$$F_n F_m - F_{n-r} F_{m+r} = (-1)^{n-r} a^{2n-2r} F_{m-n+r} F_r. \tag{12}$$

Theorem 2.2 (Generalized Cassini Identity).
For $n \geq 1$,

$$F_{n+1} F_{n-1} - F_n^2 = (-1)^n a^{2n-2}. \tag{13}$$

Theorem 2.3 (Generalized d'Ocagne Identity).
For $m \geq n \geq 1$,

$$F_{n+1} F_m - F_n F_{m+1} = (-1)^n a^{2n} F_{m-n}. \tag{14}$$

3. Bracket polynomials of $(2, n)$-torus links

Let L_n be an unoriented diagram for $(2, n)$-torus link, see Fig. 3. For simplicity , let $P_n(A)$ denote the bracket polynomial $P_{L_n}(A)$.

Theorem 3.1.
The bracket polynomial of $(2, n)$-torus link provides the following recurrence relation:

$$P_n(A) = (A - A^{-3}) P_{n-1}(A) + A^{-2} P_{n-2}(A), \quad n \geq 2. \tag{15}$$

Proof. Let us apply the skein operation to a designated crossing of unoriented diagram of $(2, n)$-torus link, L_n, drawn in Fig. 3. If we apply the equality (3) to the designated crossing of the L_n, we have

$$P_n(A) = A P_{n-1}(A) + A^{-1}(-A^3)^{n-1}. \tag{16}$$

Similarly we get

$$P_{n-1}(A) = A P_{n-2}(A) + A^{-1}(-A^3)^{n-2}. \tag{17}$$

After the multiplication both sides of the equality (17) with $-A^3$ and we check it with the equation (16) we obtain the equality (15) for $n \geq 2$. So the proof is complete. □

Fig. 3. $(2, n)$-Torus Link

Now, let's revise the Theorem 3.1 as a recurrence relation with initial conditions.

Definition 3.1.
The bracket polynomials $\{P_n(A)\}_{n=0}^{\infty}$ for the unoriented diagrams of $(2, n)$-torus links are defined by the recurrence relation

$$P_n(A) = (A - A^{-3}) P_{n-1}(A) + A^{-2} P_{n-2}(A), \quad n \geq 2. \tag{18}$$

with initial conditions

$$P_0(A) = -A^2 - A^{-2} \quad \text{and} \quad P_1(A) = -A^3. \tag{19}$$

Then, $P_n(A)$ is a generalized Fibonacci polynomial.
The characteristic equation of the recurrence relation (18) is

$$\lambda^2 - (A - A^{-3})\lambda - A^{-2} = 0$$

and the roots are

$$\alpha = A \quad \text{and} \quad \beta = -A^{-3}. \tag{20}$$

By using (18) and (19), we give the following table for the bracket polynomials of $(2, n)$-torus links.

n	$P_n(A)$
0	$-A^2 - A^{-2}$
1	$-A^3$
2	$-A^4 - A^{-4}$
3	$-A^5 - A^{-3} + A^{-7}$
4	$-A^6 - A^{-2} + A^{-6} - A^{-10}$
5	$-A^7 - A^{-1} + A^{-5} - A^{-9} + A^{-13}$
$\vdots$	$\vdots$

We now prove some properties of the polynomial $P_n(A)$.

Proposition 3.1.
The generating function

$$G_P(\lambda) = \sum_{n=0}^{\infty} P_n(A)\lambda^n$$

of the sequence $\{P_n(A)\}$ *is given by*

$$G_P(\lambda) = \frac{A^{-5}\lambda + A^{-2} + A^2}{A^{-2}\lambda^2 + (A - A^{-3})\lambda - 1}. \tag{21}$$

Proof. The generating function of $P_n(A)$ has the following form:

$$G_P(\lambda) = P_0(A) + P_1(A)\lambda + P_2(A)\lambda^2 + \cdots$$

After the multiplications $-(A - A^{-3})\lambda G_P(\lambda)$ and $-A^{-2}\lambda^2 G_P(\lambda)$, we have

$$\begin{aligned}(1 - (A - A^{-3})\lambda - A^{-2}\lambda^2)G_P(\lambda) &= P_0(A) + \big(P_1(A) - (A - A^{-3})P_0(A)\big)\lambda \\ &= A^{-2}\lambda^2 + (A - A^{-3})\lambda - 1.\end{aligned}$$

Hence, we get the generating function of $P_n(A)$ as (21). □

Proposition 3.2.
The Binet's formula for the bracket polynomial sequence $\{P_n(A)\}$ *is given by*

$$P_n(A) = c_1\alpha^n + c_2\beta^n,$$

where

$$c_1 = \frac{2A^3 - A^{-1}}{A + A^{-3}} \quad \textit{and} \quad c_2 = \frac{-A^{-5} - A^{-1} - 3A^3}{A + A^{-3}}.$$

Proof. According to the initial conditions (19)

$$P_0(A) = c_1 + c_2 = -A^2 - A^{-2}$$

and

$$P_1(A) = A^{-3}c_1 - Ac_2 = A^{-3}.$$

If these two equations are solved together, we get c_1 and c_2. □

Proposition 3.3.
For $n \geq 2$, the explicit formula for the bracket polynomial of $(2,n)$-torus link is given by

$$P_n(A) = \sum_{k=0}^{\lfloor n-1/2 \rfloor} (-1)^k \binom{n-k-1}{k} (-A^{-2})^k (A - A^{-3})^{n-2k-1}. \tag{22}$$

Proof. In the same way as the proof of Proposition 2.3, by using identity 1.60 in [27] and considering the Eqs. (9) and Binet formula we get the explicit formula. □

Theorem 3.2.
The relation between the bracket polynomial $P_n(A)$ of $(2,n)$-torus link and generalized Fibonacci polynomial $F_n(a,x)$ is

$$P_n(A) = \left(F_n(A^4, A^2 - A^{-2}) + (-A^6 - A^{10})F_{n-1}(A^4, A^2 - A^{-2})\right)(-A^3)^{-n} \tag{23}$$

with $a = A^4$ and $x = A^2 - A^{-2}$.

Proof. Proof follows from the equalities (11) and (22). □

Theorem 3.3 (Catalan-like Identity).
Let P_n be the bracket polynomial of $(2,n)$-torus link. Then, for $m \geq n \geq 1$,

$$P_nP_m - P_{n-r}P_{m+r} = (-1)^{n-r+1}A^{5n-3m-8r}(A^4 + A^8 + A^{12})F_{m-n+r}F_r.$$

Proof. From Theorem 3.2, we have

$$\begin{aligned} P_nP_m - P_{n-r}P_{m+r} &= \left[\left(F_n + (-A^6 - A^{10})F_{n-1}\right)\left(F_m + (-A^6 - A^{10})F_{m-1}\right)\right](-A^3)^{-n-m} \\ &\quad - \left[\left(F_{n-r} + (-A^6 - A^{10})F_{n-r-1}\right)\left(F_{m+r} + (-A^6 - A^{10})F_{m+r-1}\right)\right](-A^3)^{-n-m} \\ &= \big[F_nF_m - F_{n-r}F_{m+r} + (-A^6 - A^{10})F_nF_{m-1} + (-A^6 - A^{10})F_{n-1}F_m \\ &\quad + (-A^6 - A^{10})^2F_{n-1}F_{m-1} - (-A^6 - A^{10})F_{n-r}F_{m+r-1} \\ &\quad - (-A^6 - A^{10})F_{n-r-1}F_{m+r} - (-A^6 - A^{10})^2F_{n-r-1}F_{m+r-1}\big](-A^3)^{-n-m}. \end{aligned}$$

From Theorem 2.1,

$$\begin{aligned} P_nP_m - P_{n-r}P_{m+r} &= \big[(-1)^{n-r}a^{2n-2r}F_r(F_{m-n+r} + (-A^6 - A^{10})F_{m-n+r-1} \\ &\quad - (-A^6 - A^{10})A^{-8}F_{m-n+r+1} - (-A^6 - A^{10})^2A^{-8}F_{m-n+r})\big](-A^3)^{-n-m}. \end{aligned}$$

From the Eq. (6), since

$$F_{m-n+r+1} = axF_{m-n+r} + a^2F_{m-n+r-1},$$

we obtain

$$F_{m-n+r+1} = (A^6 - A^2)F_{m-n+r} + A^8F_{m-n+r-1}$$

with $a = A^4$ and $x = A^2 - A^{-2}$. Hence, we get the identity in the theorem. □

Theorem 3.4 (Cassini-like Identity).
Let P_n be the bracket polynomial of $(2,n)$-torus link. Then, for $n \geq 2$,

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^nA^{2n-8}(1 + A^4 + 2A^8 + A^{12}).$$

Proof. From Theorem 3.2, we have

$$\begin{aligned} P_{n+1}P_{n-1} - P_n^2 &= \left(\frac{F_{n+1} + (-A^6 - A^{10})F_n}{(-A^3)^{n+1}}\right)\left(\frac{F_{n-1} + (-A^6 - A^{10})F_{n-2}}{(-A^3)^{n-1}}\right) \\ &\quad - \left(\frac{F_n + (-A^6 - A^{10})F_{n-1}}{(-A^3)^n}\right)^2 \\ &= \frac{\begin{array}{r} F_{n+1}F_{n-1} - F_n^2 + (-A^6 - A^{10})^2(F_nF_{n-2} - F_{n-1}^2) \\ + (-A^6 - A^{10})(F_{n+1}F_{n-2} - F_nF_{n-1}) \end{array}}{(-A^3)^{2n}}. \end{aligned}$$

Since

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n a^{2n-2}$$

and

$$F_nF_{n-2} - F_{n-1}^2 = (-1)^n a^{2n-4}$$

by the identity (13) and

$$F_{n+1}F_{n-2} - F_nF_{n-1} = 0$$

by the take $m = n-2$ in identity (12), we obtain

$$P_{n+1}P_{n-1} - P_n^2 = (-1)^n A^{2n-8}(1 + A^4 + 2A^8 + A^{12})$$

with $a = A^4$ and $x = A^2 - A^{-2}$. □

Theorem 3.5 (d'Ocagne-like Identity).
Let P_n be the bracket polynomial of $(2,n)$-torus link. Then, for $m \geq n \geq 2$,

$$P_{n+1}P_m - P_nP_{m+1} = (-1)^{n+1}A^{5n-3m+1}(1 + A^4 + A^8)F_{m-n}.$$

Proof. From Theorem 3.2, we have

$$\begin{aligned}P_{n+1}P_m - P_nP_{m+1} &= \left(\frac{F_{n+1} + (-A^6 - A^{10})F_n}{(-A^3)^{n+1}}\right)\left(\frac{F_m + (-A^6 - A^{10})F_{m-1}}{(-A^3)^m}\right)\\ &\quad - \left(\frac{F_n + (-A^6 - A^{10})F_{n-1}}{(-A^3)^n}\right)\left(\frac{F_{m+1} + (-A^6 - A^{10})F_m}{(-A^3)^{m+1}}\right).\end{aligned}$$

From the identity (14) with $a = A^4$,

$$F_{n+1}F_m - F_nF_{m+1} = (-1)^n A^{8n}F_{m-n}$$

and

$$(-A^6 - A^{10})^2(F_nF_{m-1} - F_{n-1}F_m) = (-1)^{n-1}A^{8n-8}(-A^6 - A^{10})^2F_{m-n}.$$

By taking $n = n+1$, $m = m-1$, $r = 2$ in the identity (12), we have

$$(-A^6 - A^{10})(F_{n+1}F_{m-1} - F_{n-1}F_{m+1}) = (-1)^{n-1}A^{8n-8}(A^6 - A^2)(-A^6 - A^{10})F_{m-n}.$$

Thus, we obtain the identity in the theorem. □

Corollary 3.1.
Let N_n be the normalized bracket polynomial of $(2,n)$-torus link. Then,

(1) $N_n = F_n(A^4, A^2 - A^{-2}) + (-A^6 - A^{10})F_{n-1}(A^4, A^2 - A^{-2})$.

(2) Catalan-like Identity:
$N_nN_m - N_{n-r}N_{m+r} = (-1)^{n-r+1}A^{8n-8r}(A^4 + A^8 + A^{12})F_{m-n+r}F_r$.

(3) Cassini-like Identity:
$N_{n+1}N_{n-1} - N_n^2 = (-1)^n A^{8n-8}\left(1 + A^4 + 2A^8 + A^{12}\right)$.

(4) d'Ocagne-like Identity:
$N_{n+1}N_m - N_nN_{m+1} = (-1)^{n+1}A^{8n}(A^4 + A^8 + A^{12})F_{m-n}$.

Proof. Proof follows from the equality (4) and the Theorem 3.2, 3.3, 3.4 and 3.5. □

Corollary 3.2.
Let V_n be the Jones polynomial of $(2,n)$-torus link. Then,

(1) $V_n = F_n(t^{-1}, t^{-1/2} - t^{1/2}) + (-t^{-3/2} - t^{-5/2})F_{n-1}(t^{-1}, t^{-1/2} - t^{1/2})$.

(2) *Catalan-like Identity:*
$V_n V_m - V_{n-r} V_{m+r} = (-1)^{n-r+1} t^{2r+2n} (t^{-1} + t^{-2} + t^{-3}) F_{m-n+r} F_r.$

(3) *Cassini-like Identity:*
$V_{n+1} V_{n-1} - V_n^2 = (-1)^n t^{2-2n} \left(1 + t^{-1} + 2t^{-2} + t^{-3}\right).$

(4) *d'Ocagne-like Identity:*
$V_{n+1} V_m - V_n V_{m+1} = (-1)^{n+1} t^{-2n} (t^{-1} + t^{-2} + t^{-3}) F_{m-n}.$

Proof. Proof follows from the relation (5) and Corollary 3.1. □

Theorem 3.6.
The relation between the bracket polynomial $P_n(A)$ of $(2,n)$-torus link and its Alexander-Conway polynomial $\nabla_n(x)$ is

$$P_n(A) = (-1)^n \left(A^{n-4} \nabla_n (A^2 - A^{-2}) - A^n (A^2 + A^{-2}) \nabla_{n-1} (A^2 - A^{-2})\right)$$

with $x = A^2 - A^{-2}$.

Proof. From The Theorem 3.2, the relation between the bracket polynomial $P_n(A)$ of $(2,n)$-torus link and the generalized Fibonacci polynomial $F_n(a,x)$ is

$$P_n(A) = \left(F_n(A^4, A^2 - A^{-2}) + (-A^6 - A^{10}) F_{n-1}(A^4, A^2 - A^{-2})\right) (-A^3)^{-n}$$

with $a = A^4$ and $x = A^2 - A^{-2}$. If we use the equality (7), we have

$$P_n(A) = (-1)^n \left(A^{n-4} f_n(A^2 - A^{-2}) - A^n (A^2 + A^{-2}) f_{n-1}(A^2 - A^{-2})\right).$$

Since the Alexander-Conway polynomial is a Fibonacci polynomial, proof follows from the equality $\nabla_n(x) = f_n(x)$. □

Corollary 3.3.
The relation between the normalized bracket polynomial $N_n(A)$ of $(2,n)$-torus link and its Alexander-Conway polynomial $\nabla_n(x)$ is

$$N_n(A) = A^n \left(A^{-4} \nabla_n (A^2 - A^{-2}) - (A^2 + A^{-2}) \nabla_{n-1} (A^2 - A^{-2})\right)$$

with $x = A^2 - A^{-2}$.

Proof. Proof follows from the equality (4) and the Theorem 3.6. □

Corollary 3.4.
The relation between the Jones polynomial $V_n(t)$ of $(2,n)$-torus link and its Alexander-Conway polynomial $\nabla_n(x)$ is

$$V_n(t) = t^{-n/4} \left(t \nabla_n (t^{-1/2} - t^{1/2}) - (t^{1/2} + t^{-1/2}) \nabla_{n-1} (t^{-1/2} - t^{1/2})\right).$$

Proof. By using the equalities (5) and the Corollary 3.3, we reach the proof. □

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