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Fractional operators and Applications to fractional martingal

Research Article

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Abstract: In this paper, we use the fractional operators for to give the fractional martingale properties. We give again some examples

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1. Introduction

The theory of derivatives of non-integer order goes back to leibniz, Liouville, Gronwald, Letnikov and Riemann. For many decades afterwards, the theory of fractional derivatives was developed primarily as a theorical field of mathematics see([8], [12], [14]) fractional derivatives provide an excellent instrument for the description of fractional martingal and fractional brownian motion.
In martingal theory, the notion of fractional integrals was first introduced by Chao and Ombe for dyadic martingales. Since then, fractional integrals were defined for more general martingales and studied on various martingals.
The notion of fractional martingales has been introduced in Hu et al. [7] where the authors proved an extension of Levy characterization theorem to the fractional Brownian.
In the paper, they introduce the of fractional martingale as the fractional derivative of order α of a continuous local martingale where $\alpha \in (\frac{-1}{2}, \frac{1}{2})$, and they show that it has a nonzero finite variation of order $\frac{2}{1+2\alpha}$ under some integability assumptions on the quadratic variation of the local martingale.

2. Fractional operators

In 2010, an interesting perspectives to the subject, unifying all mentroned notions of fractional derivates and integrals was introduced in Agrawal et al. [21], Klimek and Lupa [20]. Precisely, authors considered general operators, which by choosing special Kernel, reduce to the standard fractional operators. However, other nonstandard Kernels can also be considered as particular cases.

We present defintions and properties of the one-dimensional fractional integrals and derivatives under consideration.

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Definition 2.1 (left and right Riemann-Liouville fractional integral).
We define the left and the right Riemann-Liouville fractional integrals aI_t^α and ${}_tI_b^\alpha$ of order $\alpha \in R$ $(\alpha > 0)$ by

$$aI_t^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(z)}{(t-z)^{1-\alpha}} dz \quad t \in (a,b]$$

and

$${}_tI_b^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(z)}{(z-t)^{1-\alpha}} dz \quad t \in [a,b)$$

respectively. Here $\Gamma(\alpha)$ denotes Euler's Gamma function. Note that, $aI_t^\alpha[f]$ and ${}_tI_b^\alpha[f]$ are definrd a.e on (a,b) for $f \in L^1(a,b,R)$

Definition 2.2 (left and right Hadamard fractional integral).
Let $0 \le a < b < \infty$.
We define the left -sided and right -sided Hadamard integrals of fractional order $\alpha \in R$ $(\alpha > 0)$ by

$$aI_t^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (ln\frac{t}{z})^{\alpha-1} \frac{f(z)}{z} dz, \quad t > a$$

and

$${}_tI_b^\alpha[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (ln\frac{z}{t})^{\alpha-1} \frac{f(z)}{(z-t)^{1-\alpha}} dz, \quad t < b$$

respectively
We define Riemann-Liouville fractional derivatives for order α
with $0 < \alpha < 1$. A more general definition for any α with $Re(\alpha) > 0$ can be found in Kilbas et al. [1]

Definition 2.3.
We fix $\alpha \in (\frac{-1}{2}, \frac{1}{2})$, if $M = (M_t)_{t\ge 0}$ is a continuous local martingale, the process $M^{(\alpha)} = (M_t^\alpha)_{t\ge 0}$ defined by $M_t^{(\alpha)} = \int_0^t (t-s)^\alpha dM_s$ will be called a fractional martingale provided that the above integral exists.
Notice that $M^{(\alpha)}$ is no longer a martingale.

$$If \quad \alpha \in (0, \frac{1}{2}), \quad then \quad M_t^{(\alpha)} = \int_0^t (t-s)^\alpha dM_s$$

always exists, and $M_t^{(\alpha)} = \Gamma(1+\alpha) I_{0+}^\alpha (M)_t$, where I_{0+}^α is the left-sided fractional integral of order α.
If $\alpha \in (\frac{-1}{2}, 0)$ and M has α'-Holder continuous trajectories on any finite interval for some $\alpha' > -\alpha$, then $M_t^{(\alpha)}$ exists and $M_t^{(\alpha)} = \Gamma(1+\alpha) D_{0^+}^{-\alpha}(M)_t$ where $D_{0^+}^{-\alpha}$ is the left- sided fractional derivative of order $-\alpha$.

We are interested in the variation properties of fractional martingale.

3. Some results for the fractional martingale

Theorem 3.1.
We suppose that the process $M^{(\alpha)}$ has holder continuous trajectories of order γ on any finite interval, for any $\gamma < \frac{1}{2} + \alpha$, we have the result. For

$$\alpha \in \left] \frac{-1}{2} \right[$$

and M martingale local continuous satisfied H then

$$\int_0^t (t-s)^{2\alpha} d\langle M \rangle_s < \infty$$

and M_t^α exists as a integral of Riemann-Stieltjes and

$$M_t^{(\alpha)} = \Gamma(1+\alpha) D_{0^+}^{-\alpha} M_t$$

Proof. Consider

$$Z_t = |M_t| + M_t + sup_{0\leq s\leq u\leq t}\frac{|M_s - M_u|}{|s-u|^{\alpha'}}$$

For n integer $n \geq 1$, we define

$$T_N = \inf\{\ t \geqslant 0, Z_t \geqslant N\}$$

then, T_N is an nondecreasing sequence of stopping times such that $T_N \to \infty$.
For any $s < t$ we can write

$$E\Big(\big|\langle M\rangle_{t\wedge T_N} - \langle M\rangle_{s\wedge T_N}\big|^P\Big) \leqslant C_P E\Big(\big|\langle M\rangle_{\ t\wedge T_N} - \langle M\rangle_{s\wedge T_N}\big|^{2P}\Big) \leqslant C_P N^{2P}\big|t-t'\big|^{2P\alpha'}$$

By Kolmogorov's continuity criterion the sample paths of $\langle M\rangle$ are holding of order δ for $\delta < 2\alpha^{'}$, on any finite interval and the stochastic integral is a Riemann Stieltjes integarl and coincids with $\Gamma(\alpha+1)D_t^{-\alpha}\langle M\rangle_t$ □

3.1. Hypothesis

The continuous local martingale M is of the form :

$$M_t = \int_0^t \xi_s dW_s$$

Where $W = (W_t)_{t\geq 0}$ is a $\mathscr{F}_t$-brownian motion and $\xi = (\xi_t)_{t\geq 0}$is a progressively measurable process such that for all $t \geq 0$

$$\begin{cases} \int_0^t \mathbb{E}\Big(|\xi_s|^{\beta'}\Big)ds < \infty \quad \text{pour} \quad \beta' > \beta, \quad si\alpha < 0; \\ \int_0^t \mathbb{E}\big(\big|\xi_s^2\big|\big)ds < \infty, \quad si\alpha > 0 \end{cases}$$

Under Hypothesis 3.1, the integral appearing in (2) always exists as a RiemannâĂŞStieltjes integral. This is a consequence of Hu et al. [7] and the fact that the trajectories of M are M α'-Holder continuous on finite intervals. Moreover, by Theorem 2.6 and Remark 2.7 of Hu and al. [7] , $M^{(\alpha)}$ exist in $\mathbb{L}^1$ and

$$\langle M^{(\alpha)}\rangle_{\beta.t} = c_\alpha \int_0^t |\xi_s|^\beta ds,$$

$$c_\alpha = C_H K_H^{-\frac{1}{H}}$$

with $K_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(H+\frac{1}{2})\Gamma(2-2H)}\right)^{\frac{1}{2}}$, $C_H = \mathbb{E}(|B_1^H|^{\frac{1}{H}})$ et $H = \frac{1}{2}+\alpha$

Nevertheless we stress the point that under Hypothesis I3.1, the expression of $M^{(\alpha)}$ is given by:

$$M_t^{(\alpha)} = \int_0^t (t-s)^\alpha \xi_s dW_s. \tag{1}$$

Moreover, using Holder inequality we deduces the following relations between the β-variation of $M^{(\alpha)}$and the quadratic variation of the underlying martingale M:

$$\begin{cases} \langle M\rangle_t \leqslant c_\alpha^{-\frac{2}{\beta}}\, t^{\frac{\beta-2}{\beta}}\, \langle M^{(\alpha)}\rangle_{\beta.t}^{\frac{2}{\beta}} \quad \alpha < 0; \\ \langle M^{(\alpha)}\rangle_{\beta.t} \leqslant c_\alpha t^{\frac{2-\beta}{2}}\, \langle M\rangle_t^{\frac{\beta}{2}} \quad \alpha > 0 \end{cases}$$

The first result is a generalization of Bernstein inequality to fractional martingales is stated in the next theorem.

Theorem 3.2.
We assume Hypothesis 3.1. We denote

$$C_t = 2 + 2^{\frac{1}{2}} t^2$$

For any positive function $t \mapsto v_t$ any $L \geqslant 1$ the following exponential inequalities hold

i. *When $\alpha < 0$ we have,*

$$P\left(\sup_{0\leqslant s\leqslant t}|M_s^{(\alpha)}| \geqslant Lc_1 t^{\frac{\beta'-\beta}{2\beta\beta'}} v_t^{\frac{1}{2}}, \left(\int_0^t |\xi_\tau|^{\beta'}d\tau\right)^{\frac{2}{\beta'}} \leqslant v_t\right) \leqslant C_t \exp\left\{-\frac{k^2L^2}{t^{\frac{\beta'-\beta}{\beta\beta'}}}\right\} \tag{2}$$

$$k^2 = 4\pi\left(\frac{\beta\beta'}{(\beta'-\beta)}\right)^3.$$

ii. when $\alpha > 0$, for any $\varepsilon \in (0, \alpha)$ it holds that,

$$P\left(\sup_{0\leqslant s\leqslant t} |M_s^{(\alpha)}| \geqslant L2^6 k t^{\alpha-\varepsilon} v_t^{\frac{1}{2}}, \int_0^t |\xi_\tau|^2 d\tau \leqslant v_t\right) \leqslant C_t \exp\left\{-\frac{k^2 L^2}{t^{2(t-\alpha)}}\right\} \tag{3}$$

with $k = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \varepsilon^{-\frac{3}{2}}$

iii. If we assume that the process ξ is bounded by c_∞, almost-surely, then for any $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$ and for any $\varepsilon \in \left(0, \frac{1}{2} + \alpha\right)$

$$P\left(\sup_{0\leqslant s\leqslant t} |M_s^{(\alpha)}| \geqslant L2^6 k c_\infty t^{\frac{1}{2}+\alpha-\varepsilon}\right) \leqslant C_t \exp\left\{-\frac{k^2 L^2}{t^{2\alpha-}}\right\} \tag{4}$$

with $k = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \varepsilon^{-\frac{3}{2}}$.

Proof. With

$$\Psi(x) = exp\left(\frac{-x^2}{4}\right)$$

and p is a function non negative , continue in $(0, t)$ so that $p(0) = 0$, in section 3.1 indicates as following: for any $0 \leqslant r \leqslant s \leqslant t$ we have

$$M_s^{(\alpha)} - M_r^{(\alpha)}| \leqslant 8 \int_0^{|s-r|} \Psi^{-1}\left(\frac{4B}{y^2}\right) dp(y) \tag{5}$$

provided that

$$B := \int_0^t \int_0^t \Psi\left(\frac{M_s^{(\alpha)} - M_r^{(\alpha)}}{p(|s-r|)}\right) dsdr < \infty$$

the function will be chosen after. We denote that the function Ψ^{-1} is define by $u \geqslant \Psi(0)$.
Since

$$\Psi^{-1}(u) = sup\{v; \Psi(v) \leqslant u\}.$$

with a function define by ln^+

$$ln^+(z) = max(ln(z), 0)$$

For inequality $z > 0$,

$$\frac{B}{y^2} \leqslant exp\left(ln^+\left(\frac{B}{y^2}\right)\right) \quad \Rightarrow \quad \Psi^{-1}\left(\frac{B}{y^2}\right) \leqslant 2\left(ln^+\left(\frac{B}{y^2}\right)\right)^{\frac{1}{2}}$$

and

$$\left(ln^+\left(\frac{B}{y^2}\right)\right)^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}}\left(ln^+\left(\frac{B}{y^2}\right)\right)^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}}\left\{\left(ln^+(B)\right)^{\frac{1}{2}} + \left(ln^+(\frac{1}{y^2})\right)^{\frac{1}{2}}\right\}.$$

$$\left(ln^+\left(\frac{B}{y^2}\right)\right)^{\frac{1}{2}} \leqslant 2^{\frac{1}{2}}\left\{\left(ln^+(B)\right)^{\frac{1}{2}} + \left(ln^+(y^{-2})\right)^{\frac{1}{2}}\right\}.$$

Hence,

$$\Psi^{-1}\left(\frac{B}{y^2}\right) \leqslant 2 \times 2^{\frac{1}{2}}\left\{\left(ln^+(B)\right)^{\frac{1}{2}} + \left(ln^+(y^{-2})\right)^{\frac{1}{2}}\right\}$$

Since $M_0^{(\alpha)} = 0$, we deduce from (5) that

$$|M_s^{(\alpha)} - M_r^{(\alpha)}| \leqslant 8 \int_0^{|s-r|} 2 \times 2^{\frac{1}{2}}\left\{\left(ln^+(B)\right)^{\frac{1}{2}} + \left(ln^+(y^{-2})\right)^{\frac{1}{2}}\right\} dp(y)$$

$$|M_s^{(\alpha)} - M_r^{(\alpha)}| \leqslant 2^3 \int_0^{|s-r|} 2 \times 2^{\frac{1}{2}}\left\{\left(ln^+(B)\right)^{\frac{1}{2}} + \left(ln^+(y^{-2})\right)^{\frac{1}{2}}\right\} dp(y)$$

$$|M_s^{(\alpha)} - M_r^{(\alpha)}| \leqslant 2^{\frac{9}{2}} \int_0^{|s-r|} \left\{\left(ln^+(B)\right)^{\frac{1}{2}} + \left(ln^+(y^{-2})\right)^{\frac{1}{2}}\right\} dp(y)$$

$$0\leqslant s\leqslant tsup\left|M_s^{(\alpha)}\right|\leqslant 2^{\frac{9}{2}}\int_0^t\left\{\left(ln^+(B)\right)^{\frac{1}{2}}+\left(ln^+(y^{-2})\right)^{\frac{1}{2}}\right\}dp(y) \tag{6}$$

We fix t and for all $0\leqslant r<s<t$ we write

$$M_s^{(\alpha)}-M_r^{(\alpha)}=\int_0^s g_{s.r}(\tau)\,dW_\tau$$

with

$$g_{s.r}(\tau)=\xi_\tau(s-\tau)^\alpha 1_{\{r<\tau\leqslant s\}+\xi_\tau((s-\tau)^\alpha-(r-\tau)^\alpha)1_{\{\tau\leqslant r\}}}.$$

We denote in first that

$$\left|g_{s.r}(\tau)\right|^2d\tau=\int_r^s(s-\tau)^{2\alpha}|\xi_\tau|^2d\tau+\int_0^r\left((s-\tau)^\alpha-(r-\tau)^\alpha\right)^2|\xi_\tau|^2d\tau \tag{7}$$

Indeed,

$$\begin{aligned}\int_0^t\left|g_{s.r}(\tau)\right|^2d\tau&=\int_0^t|\xi_\tau(s-\tau)^\alpha 1\\&=\int_r^s\left|\xi_\tau(s-\tau)^\alpha\right|^2d\tau+\int_0^r\left|\xi_\tau\left((s-\tau)^\alpha-(r-\tau)^\alpha\right)\right|^2d\tau\\&=\int_r^s|\xi_\tau|^2(s-\tau)^{2\alpha}d\tau+\int_0^r|\xi_\tau|^2\left((s-\tau)^\alpha-(r-\tau)^\alpha\right)^2d\tau\end{aligned}$$

In order to have some estimates of the quantity $\int_0^t\left|g_{s.r}(\tau)\right|^2d\tau$ we treat different cases according to the sign of and the assumption we make in the α process xi

Case 1: We suppose the precedent hypothesis and $\alpha>0$ with $\beta'>\beta$ of the hypothesis, we denote $p=\frac{\beta'}{2}>1$ with

$$\frac{1}{p}+\frac{1}{q}=1=\frac{\beta'-2}{\beta'}$$

Either $\varepsilon>0$ which will be fixed. We use the lemma 2.3.1 in [7] for writing

$$\int_0^t\left|g_{s.r}(\tau)\right|^2d\tau=\int_r^s(s-\tau)^{2\alpha}|\xi_\tau|^2d\tau+\int_0^r\left((s-\tau)^\alpha-(r-\tau)^\alpha\right)^2|\xi_\tau|^2d\tau$$

Since

$$\left|g_{s.r}(\tau)\right|^2d\tau\leqslant(s-r)^{2\varepsilon}\left[\int_r^s(s-\tau)^{2\alpha-2\varepsilon}\xi_\tau^2d\tau+\int_0^r(r-\tau)^{2\alpha-2\varepsilon}\xi_\tau^2d\tau\right] \tag{8}$$

Putting,

$$\begin{aligned}I_1&=\int_r^s(s-\tau)^{2\alpha-2\varepsilon}\xi_\tau^2d\tau\\I_2&=\int_0^r(r-\tau)^{2\alpha-2\varepsilon}\xi_\tau^2d\tau,\end{aligned}$$

We have

$$\left|g_{s.r}(\tau)\right|^2d\tau\leqslant(s-r)^{2\varepsilon}\left[I_1+I_2\right] \tag{9}$$

Now we choose ϵ such as $1+2\alpha q-2\varepsilon q>0$. We notice that the choice of a is always possible. Indeed,

$$\begin{aligned}1+2\alpha q-2\varepsilon q&=1+2\alpha\left(\frac{\beta'}{\beta'-2}\right)-2\varepsilon\left(\frac{\beta'}{\beta'-2}\right)\\&=\frac{\left(\beta'-2\right)+2\alpha\beta'-2\varepsilon\beta'}{\beta'-2}\\&=\frac{\left(\beta'-2\right)+2\alpha\beta'-2\varepsilon\beta}{\beta-2}\end{aligned}$$

and then choosing ϵ in the form $\varepsilon = \frac{a(\beta'-\beta)}{\beta\beta'}$.
For $a \in (0,1)$,

$$1+2\alpha q-2\varepsilon q = \frac{2\left(\beta'-\beta\right)}{\beta\left(\beta'-2\right)} - 2\left[a\left(\frac{\beta'-\beta}{\beta\beta'}\right)\times\frac{\beta\beta'}{\beta\left(\beta'-2\right)}\right] = \frac{2\left(\beta'-\beta\right)}{\beta\left(\beta'-2\right)} - \frac{2a\left(\beta'-\beta\right)}{\beta\left(\beta'-2\right)}$$

we notice that

$$0 < 1+2\alpha q-2\varepsilon q < \frac{2\left(\beta'-\beta\right)}{\beta\left(\beta'-2\right)}$$

We choose $a = \frac{1}{2}$, henceforth ϵ is fixed as

$$\varepsilon = \frac{1}{2}\left(\frac{\beta'-\beta}{\beta\beta'}\right)$$

$$I_1 = \int_r^s (s-\tau)^{2\alpha-2\varepsilon}\xi_\tau^2 d\tau$$

By Holder inequality we have

$$I_1 \leqslant \left(\int_r^s (s-\tau)^{(2\alpha-2\varepsilon)q} d_\tau\right)^{\frac{1}{q}} \left(\int_r^s \xi_\tau^{2p} d\tau\right)^{\frac{1}{p}}$$

or

$$\left(\int_r^s \xi_\tau^{2p} d\tau\right)^{\frac{1}{p}} = \|\xi\|^2_{L^{2p}(0,t)}$$

That is,

$$\begin{aligned}
\left(\int_r^s (s-\tau)^{(2\alpha-2\varepsilon)q} d_\tau\right)^{\frac{1}{q}} &= \left(\left[\frac{(s-\tau)^{(2\alpha-2\varepsilon)q+1}}{(2\alpha-2\varepsilon)\,q+1}\right]_r^s\right)^{\frac{1}{q}} \\
&= \left(\frac{-(s-r)^{(2\alpha-2\varepsilon)q+1}}{(2\alpha-2\varepsilon)\,q+1}\right)^{\frac{1}{q}} \\
&= \frac{(s-r)^{2\alpha-2\varepsilon+1/q}}{\left((-2\alpha+2\varepsilon)\,q-1\right)^{\frac{1}{q}}} \\
&\leqslant (s-r)^{2\alpha-2\varepsilon+1/q} \times \frac{1}{\left((2\varepsilon)\,q\right)^{\frac{1}{q}}}
\end{aligned}$$

$$\left(\int_r^s (s-\tau)^{(2\alpha-2\varepsilon)q} d_\tau\right)^{\frac{1}{q}} \leqslant (s-r)^{2\alpha-2\varepsilon+\frac{1}{q}} \times \frac{1}{\left(\left(2\times\frac{\beta'-\beta}{2\beta\beta'}\right)\frac{\beta'}{\beta'-2}\right)^{\frac{\beta'-2}{\beta}}}$$

$$\left(\int_r^s (s-\tau)^{(2\alpha-2\varepsilon)q} d_\tau\right)^{\frac{1}{q}} \leqslant (s-r)^{2\alpha-2\varepsilon+\frac{1}{q}} \times \left(\frac{1}{\frac{\beta'-\beta}{\beta(\beta'-2)}}\right)^{\frac{\beta'-2}{\beta}}$$

We have

$$\left(\int_r^s (s-\tau)^{(2\alpha-2\varepsilon)q} d_\tau\right)^{\frac{1}{q}} \times \|\xi\|^2_{L^{2p}(0,t)} \leqslant (s-r)^{2\alpha-2\varepsilon+\frac{1}{q}} \times \left[\frac{\beta\left(\beta'-2\right)}{\beta'-\beta}\right]^{\frac{\beta-2}{\beta}} \times \|\xi\|^2_{L^{2p}(0,t)}$$

then

$$I_1 \leqslant (s-r)^{2\alpha-2\varepsilon+\frac{1}{q}} \times \left[\frac{\beta\left(\beta'-2\right)}{\beta'-\beta}\right]^{\frac{\beta'-2}{\beta}} \|\xi\|^2_{L^{2p}(0,t)}\, C_{\beta.\beta'}(s-r)^{2\alpha-2\varepsilon+1/q}\,\|\xi\|^2_{L^{2p}(0,t)} \tag{10}$$

with

$$C_{\beta.\beta'} = \left[\frac{\beta\left(\beta,-2\right)}{\beta'-\beta}\right]^{\frac{\beta'-2}{\beta}}$$

the same way, we get the following estimation for I_2 :

$$I_2 \leqslant C_{\beta.\beta'} r^{2\alpha-2\varepsilon+\frac{1}{q}} \|\xi\|^2_{L^{2p}(0,t)} \tag{11}$$

$$\begin{aligned}\int_0^t \left|g_{s.r}(\tau)\right|^2 d\tau &\leqslant (s-r)^{2\varepsilon}\left[I_1+I_2\right]\\ &\leqslant (s-r)^{2\varepsilon}[C_{\beta.\beta'}(s-r)^{2\alpha-2\varepsilon+1/q}\|\xi\|^2_{L^{2p}(0,t)} + C_{\beta.\beta'} r^{2\alpha-2\varepsilon+\frac{1}{q}}\|\xi\|\\ &\leqslant (s-r)^{2\varepsilon}[C_{\beta.\beta'} t^{2\alpha-2\varepsilon+\frac{1}{q}}\|\xi\|^2_{L^{2p}(0,t)} + C_{\beta.\beta'} t^{2\alpha-2\varepsilon+\frac{1}{q}}\|\xi\|\end{aligned}$$

$$\left|g_{s.r}(\tau)\right|^2 d\tau \leqslant 2C_{\beta.\beta'}(s-r)^{2\varepsilon} t^{2\alpha-2\varepsilon+\frac{1}{q}}\|\xi\|^2_{L^{2p}(0,t)} \tag{12}$$

In the evenement $\mathscr{A}_t = \left\{\left(\int_0^t \xi_\tau^{2p} d\tau\right)^{\frac{1}{q}} \leqslant v_t\right\}$, We have

$$\int_0^t \left|g_{s.r}(\tau)\right|^2 d\tau \leqslant 2C_{\beta.\beta'}(s-r)^{2\varepsilon} t^{2\alpha-2\varepsilon+\frac{1}{q}} v_t.$$

Now we have denote that the function p can be define by,

$$p(y) = \left(2C_{\beta.\beta'}(y)^{2\varepsilon} t^{2\alpha-2\varepsilon+\frac{1}{q}} v_t\right)^{\frac{1}{2}}$$

with $y = s-r$

$$p(y) = \left(2C_{\beta.\beta'}\right)^{\frac{1}{2}} t^{\alpha-\varepsilon+\frac{1}{2q}} v_t^{\frac{1}{2}} y^{\varepsilon} \tag{13}$$

and for $r<s$ fixed, we consider a martingale $M=(M_u)_{0\leqslant u\leqslant t}$ defined by

$$M_u = \int_0^u \frac{g_{s.r}(\tau)}{p(s-r)} dW_\tau \tag{14}$$

The quadratic variation satisfed any all $0 \leqslant u \leqslant t$

$$\langle M\rangle_u \leqslant \int_0^t \frac{\left|g_{s.r}(s)\right|^2}{p(s-r)} ds \leqslant 1$$

almost surely in $\mathscr{B}_t$.

Let W be a Brownian motion associated with the martingale M such that $M_u = W_u$. We have

$$\mathbb{E}\left[\psi\left(\frac{M_s^{(\alpha)}-M_r^{(\alpha)}}{p(|s-r|)}\right)1_{\mathscr{A}_t}\right] = \mathbb{E}\left[exp\left(\frac{M_t^2}{4}\right)1_{\mathscr{A}_t}\right]$$

indeed, $\psi(x) = exp\left(\frac{-x^2}{4}\right)$

$$\mathbb{E}\left(\psi(x)\right) = \mathbb{E}\left(exp\left(\frac{-x^2}{4}\right)\right)$$

$$\mathbb{E}\left(\psi(x)1_{\mathscr{A}_t}\right) = \mathbb{E}\left(exp\left(\frac{-x^2}{4}\right)1_{\mathscr{A}_t}\right)$$

Since for any $0 \leqslant r < s < t$ we have $M_s^{(\alpha)} - M_r^{(\alpha)} = \int_0^s g_{s.r}(\tau)\, dW_\tau$, we obtain

$$\frac{M_s^{(\alpha)}-M_r^{(\alpha)}}{p(|s-r|)} = \int_0^t \frac{g_{s.r}(\tau)}{p(|s-r|)} dW_\tau = M_t$$

For any $0 \leqslant r < s < t$,

$$\mathbb{E}\left[\psi\left(\frac{M_s^{(\alpha)}-M_r^{(\alpha)}}{p(|s-r|)}\right)1_{\mathscr{A}_t}\right] = \mathbb{E}\left[exp\left(\frac{M_t^2}{4}\right)1_{\mathscr{A}_t}\right]$$

Since

$$M_t = W_t \Leftrightarrow M_t^2 = W_t^2 \quad \Rightarrow \quad M_t^2 \leqslant \underset{0\leqslant r\leqslant 1}{sup} |W_r|^2$$

Therefore,

$$\mathbb{E}\left[\Psi\left(\frac{M_s^{(\alpha)} - M_r^{(\alpha)}}{p(|s-r|)}\right)1_{\mathscr{A}_t}\right] \leqslant \left[exp\left(\frac{1}{4}\underset{0\leqslant r\leqslant 1}{sup}|W_r|^2\right)\right] \leqslant 2^{\frac{1}{2}}(1)^2 \leqslant 2^{\frac{1}{2}}.$$

$$\mathbb{E}\left(B1_{\mathscr{A}_t}\right) \leqslant 2^{\frac{1}{2}} t^2$$

That is,

$$\mathbb{E}\left(exp\left\{1_{\mathscr{A}_t} ln^+(B)\right\}\right) \leqslant 1 + \mathbb{E}\left(1_{\mathscr{A}_t} exp\left\{ln^+(B)\right\}\right) \leqslant 2 + 2^{\frac{1}{2}} t^2 := C_t.$$

$$\left(ln^+\left(y^{-2}\right)\right)^{\frac{1}{2}} y^{\varepsilon-1} dy \leqslant 2^{\frac{1}{2}} \int_0^{+\infty} z^{\frac{1}{2}} e^{(-\varepsilon z)} dz = \frac{1}{\varepsilon}\sqrt{\frac{\pi}{2\varepsilon}} := k, \tag{15}$$

indeed,

$$\int_0^t \left(ln^+\left(y^{-2}\right)\right)^{\frac{1}{2}} y^{\varepsilon-1} dy \leqslant 2^{\frac{1}{2}} \int_0^{+\infty} z^{\frac{1}{2}} e^{(-\varepsilon z)} dz$$

The following integral

$$\int_0^{+\infty} z^{\frac{1}{2}} e^{(-\varepsilon z)} dz = \frac{1}{\varepsilon\sqrt{\varepsilon}}\frac{\sqrt{\pi}}{2}$$

Now we finish the proof with Chebyshev exponentielle inequality. For $L \geqslant 1$ we have

$$\mathbb{P}\left(\sup_{0\leqslant s\leqslant t}\left|M_s^{(\alpha)}\right| \geqslant 2Lkc_1(t), \mathscr{A}_t\right) \leqslant \mathbb{P}\left(\left\{\ln^+(B) \geqslant \frac{1}{t^{2\varepsilon}}\left(\frac{2Lkc_1(t)}{c_1(t)} - k\right)^2\right\} \cap \mathscr{A}_t\right)$$
$$\leqslant \mathbb{E}\left[\exp\left(1_{\mathscr{A}_t}\ln^+(B)\right)\right] \exp\left\{\left(\frac{-4k^2L^2 + 4K^2L - K^2}{t^{2\varepsilon}}\right)\right\}$$

then

$$\mathbb{P}\left(\underset{0\leqslant s\leqslant t}{sup}\left|M_s^{(\alpha)}\right| \geqslant 2Lkc_1(t), \mathscr{A}_t\right) \leqslant C_t exp\left\{-\frac{k^2L^2}{t^{2\varepsilon}}\right\}$$

Case 2: we suppose the precedent inequality $\alpha > 0$.

By the hypothesis, $\int_0^t \xi_\tau^2 d\tau$ exist almost sure. Using (7)

$$\int_0^t \left|g_{s.r}(\tau)\right|^2 d\tau \leqslant (s-r)^{2\alpha}\int_r^s \xi_\tau^2 d\tau + (s-r)^{2\varepsilon}\int_0^r (r-\tau)^{2\alpha-2\varepsilon}\xi_\tau^2 d\tau$$

Since $\int_0^r (r-\tau)^{2\alpha-2\varepsilon}\xi_\tau^2 d\tau \leqslant \int_0^r r^{2\alpha-2\varepsilon}\xi_\tau^2 d\tau$, then

$$\int_0^t \left|g_{s.r}(\tau)\right|^2 d\tau \leqslant (s-r)^{2\alpha}\int_r^s \xi_\tau^2 d\tau + (s-r)^{2\varepsilon}\int_0^r r^{2\alpha-2\varepsilon}\xi_\tau^2 d\tau$$
$$\leqslant (s-r)^{2\alpha}\int_r^s \xi_\tau^2 d\tau + (s-r)^{2\varepsilon} r^{2\alpha-2\varepsilon}\int_0^r \xi_\tau^2 d\tau 2(s-r)^{2\varepsilon} t^{2\alpha-2\varepsilon}\int_0^t \xi_\tau^2 d\tau \tag{16}$$

with $0 < \varepsilon < \alpha$.

The rest of the proof is similar with the following modifications. We use the martingale M defined by [14] with the new function p defined by

$$p(y) = 2^{\frac{1}{2}} t^{\alpha-\varepsilon} v_t^{\frac{1}{2}} y^{\varepsilon}.$$

In the evenement $\left\{\int_0^t \xi_\tau^2 d\tau \leqslant v_t\right\}$ M has borned quadratic variation by 1. The inequality (16) is replaced by

$$\underset{0\leqslant r\leqslant t}{sup}\left|M_s^{(\alpha)}\right| \leqslant \left[\left(ln^+(B)\right)^{\frac{1}{2}} t^{\varepsilon} + k\right] \times 32 t^{\alpha-\varepsilon} v_t^{\frac{1}{2}}$$

where k is define in (4). The rest of proofis similar.

Case 3: We suppose that ε is borned.

When $\alpha \in \left(-\frac{1}{2}, 0\right)$, then it exist $\varepsilon > 0$ such that $\varepsilon > 0$. Then in (1), we see that

$$\left|g_{s.r}(\tau)\right|^2 d\tau \leqslant 2c_\infty^2 (t-r)^{2\varepsilon} t^{1+2\alpha-2\varepsilon} \tag{17}$$

when $\alpha > 0$.

The rest of the proof is similar to the previous case with the function p define by

$$p(y) = (2c_\infty)^{\frac{1}{2}} t^{\frac{1}{2}} + \alpha - \varepsilon y^{\varepsilon}$$

□

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