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Transient thermoelastic analysis of a prolate spheroid subjected to radiation conditions

Research Article

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Abstract: The principal aim of this paper is to investigate the thermoelastic responses in a generalised prolate spheroid subjected to the radiation type boundary conditions on the inner and outer curved surfaces. The recent research activity motivates the work carried out on a different spheroid (oblate or prolate) to avoid structural failure due to thermal stress response. The method of integral transformation approach is used to obtain an exact solution of heat conduction equation subjected to the generation of heat within the body. The relations obtained in this paper can be applied to any arbitrary boundary and initial conditions. Some results which are derived using computational tools are accurate enough for the practical purpose were depicted graphically.

MSC: 35B07 • 35G30 • 35K05 • 44A10

Keywords: Prolate spheroid • Transient heat conduction • Thermal stresses • Integral transform

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1. Introduction

A surface of revolution generated on the rotation of an ellipse about its own axis gives the prolate surface as described in the book by Hilbert [1]. A prolate spheroid is "pointy" instead of "squashed," in other words it is one for which the polar radius c is greater than the equatorial radius a, so $c > a$ called "spindle-shaped ellipsoid" by Tietze [2]. A regular egg with the same shape at both ends would approximate a prolate spheroid. There are many textbook examples of the application of spheroidal coordinates. Some of the applications include the gravitational potential of an astronomical body, the solution to Maxwell's equations in electromagnetism, solid mechanics, and the study of various problems in quantum mechanics. A short history of the research work associated with the heat conduction with spheroidal coordinate system insights many novels and remarkable methods introduced by Green-Lame, Mathieu, Niven etc. The mathematical treatment of the problem of the conduction of heat for spheroids in terms of spheroidal wave functions was investigated dates back by Niven which constitute a generalization of the corresponding solution for the sphere. Few highly cited literature on heat conduction in spheroid (prolate or oblate) making use of the spheroidal wave functions were considered by Flammer [3], Abramowitz and Stegun [4], Hodge [5], Meixner et al. [6] and Li et al. [7] in their books. In this regards Gupta [8] has introduced an integral transform applicable to spheroidal wave functions analogous to finite Fourier transform which applies to few boundary value problems relating to spheroids. Ivers [9] proposed the spectral numerical method for solving the heat equation in oblate or prolate

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spheroids. Alassar [10] obtained the analytical solutions of the problem of conduction heat transfer from oblate and prolate spheroids to an infinite medium. Also, Alassar et al. [11] investigated two solutions of the problem, first by making use of the spheroidal wave functions and other by using an implicit finite difference scheme for an unsteady heat conduction from a spheroid (prolate or oblate). Recently, Pei and Ding Pei and Ding [12] used prolate spheroidal wave functions for analyzing the properties of the finite extension Fourier transform. Moore and Cada [13] exploited the orthogonal properties of prolate spheroidal wave functions in the form of a new orthogonal expansion which we have named the Slepian series. Wang [14] reviewed the prolate spheroidal wave functions and their variants from the viewpoint of spectral/spectral-element approximations using such functions as basic functions. Zayed [15] obtained the generalization of the prolate spheroidal wave functions, which are a special case of the spheroidal wave functions; possess a very surprising and unique property. Xiao et al. [16] introduced analogous techniques based on the assumption that the function to be dealt with is band-limited, and use the well-developed apparatus of prolate spheroidal wave functions to construct quadrature, interpolation and differentiation formulae, etc, for band-limited functions. From the above cited it is noted that most of the authors have utilized numerical solution for finding their solutions. Though limited utilisation of analytical solutions mustn't diminish their merit over numerical ones; since exact solutions, if available, provide an insight into the governing physics of the problem, which is often missing in any numerical solution. It was also observed from the previous literature that almost all researchers have considered heat conduction without internal heat source subjected to radiation conditions. Things get further complicated when internal heat generation persists on the object under consideration and further becomes unpredictable when sectional heat supply is also impacted on the body. Both analytical and numerical techniques have proved to be the best methodology to solve such problems. Nonetheless, numerical solutions are preferred and prevalent in practice, due to either non-availability or mathematical complexity of the corresponding exact solutions. Moreover, analysing closed-form solutions to obtain optimal design options for any particular application of interest is relatively simpler. Thus, a new set of an integral transform and its inverse theorem for a heat conduction problem subjected to radiation conditions are needed to obtain an exact solution. To the best of authors' knowledge, no research work has been taken for the thermoelastic analysis of prolate spheroids with internal heat source subjected to prescribed temperature on the curved surface. Owing to the lack of research in this field, the authors have been motivated to conduct this study. In this present paper, the realistic problem of thin spheroids subjected to arbitrary initial temperature is studied. The theoretical calculation has been considered using the dimensional parameter, whereas, graphical calculations are carried out using the dimensionless parameter. The success of this research mainly lies on the new mathematical procedures which present a much more straightforward approach for optimization of the design in terms of material usage and performance in engineering problem, particularly in the determination of thermoelastic behaviour in spheroids engaged as pressure vessels, furnaces, etc.

2. Formulation of the problem

For a spheroid structure with isotropic property with coordinate origin is at the center, the equation of energy conservation in three-dimensional curve linear coordinates can be given as

$$\frac{\lambda}{h_\xi h_\eta h_\varphi}\left[\frac{\partial}{\partial \xi}\left(\frac{h_\eta h_\varphi}{h_\xi}\frac{\partial T}{\partial \xi}\right)+\frac{\partial}{\partial \eta}\left(\frac{h_\xi h_\varphi}{h_\eta}\frac{\partial T}{\partial \eta}\right)+\frac{\partial}{\partial \varphi}\left(\frac{h_\xi h_\eta}{h_\varphi}\frac{\partial T}{\partial \varphi}\right)\right]+C_0 T=\rho C_v\frac{\partial T}{\partial t}+\Theta \tag{1}$$

in which (ξ,η,φ) are the prolate spheroidal coordinate variables, $T = T(\xi,\eta,\varphi,t)$ is the temperature function, $\Theta = \Theta(\xi,\eta,\varphi,t)$ is the source of heat generation depending upon spheroidal coordinates, $C_0 = C_0(x, y, z)$ is in the rectangular coordinate, $\kappa = \lambda/\rho C_v$ representing thermal diffusivity in which λ is the thermal conductivity of the material, ρ is the density, and C_v is the calorific capacity, respectively. The above heat conduction equation (1) could be used for spherical or cylindrical coordinate system after defining appropriate scale factors (metric coefficients) h_ξ, h_η and h_φ [17]. For the present problems, it is convenient to use a prolate spheroidal coordinate system that can be established by revolving the two-dimensional elliptic coordinate system about the major axes of the ellipse [3]. The scale factors (metric coefficients) h_ξ, h_η and h_φ, for the prolate spheroidal coordinates system about rectangular coordinates (x, y, z) by the transformation can be taken as

$$x = c\sinh\xi\,\sin\eta\cos\varphi,\ y = c\sinh\xi\,\sin\eta\sin\varphi,\ z = c\cosh\xi\,\cos\eta \tag{2}$$

alternatively, other equivalent transformation [17] from Eq. (2) as

$$x = c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\cos\varphi,\ y = c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2}\sin\varphi,\ z = c\xi\eta \tag{3}$$

with $-1\le\eta\le 1,\ \xi>1,\ 0\le\varphi\le 2\pi,\ c>0$

Moreover, the prolate spheroidal coordinate scale factors are given as

$$h_\xi=\left|\frac{\partial r}{\partial \xi}\right|=c\frac{(\xi^2-\eta^2)^{1/2}}{(\xi^2-1)^{1/2}},h_\eta=\left|\frac{\partial r}{\partial \eta}\right|=c\frac{(\xi^2-\eta^2)^{1/2}}{(1-\eta^2)^{1/2}},h_\varphi=\left|\frac{\partial r}{\partial \varphi}\right|=c(\xi^2-1)^{1/2}(1-\eta^2)^{1/2} \tag{4}$$

in which r is the Cartesian position vector of $r = xi + yj + zk$, $(\hat{\xi}, \hat{\eta}, \hat{\varphi})$ is the unit vectors defined as $[(\partial r/\partial\xi)/h_\xi, (\partial r/\partial\eta)/h_\eta, (\partial r/\partial\varphi)/h_\varphi)]$, c is the positive constant represents the semi-focal length,and here the metric coefficients are used to transform the governing equations from rectangular coordinates into the prolate spheroidal coordinate system. A prolate spheroid is generated by revolving an ellipse around its major axis, while an oblate spheroid results from revolving the ellipse around its minor axis. The semi-major axis from inner long diameter $2a_i$ to outer long diameter $2a_o$, whereas semi-minor axis moves from the inner short diameter $2b_i$ to the outer short diameter $2b_o$. The length $2c$ is the distance between their common foci as shown in the geometrical configuration described in Fig. 1, which can be defined as $2c = 2\sqrt{a_o^2 - b_o^2} = 2\sqrt{a_i^2 - b_i^2}$. The surface of the spheroid, whether prolate or oblate is defined as $\xi \in [\xi_i, \xi_o]$, and that can be related to the axis ratio as the inner boundary $a = \tanh^{-1}(b_i/a_i)$ and the outer boundary $b = \tanh^{-1}(b_o/a_o)$. The limiting case for both coordinate, as $(a, b) \to \infty$, is a sphere. On the other limiting case as $(a, b) \to 0$, the oblate spheroid becomes flat circular disk while the prolate spheroid becomes infinitely thin needle. Utilizing the metric coefficients (4) in Eq. (1) can be specialised for prolate spheroidal coordinate, the

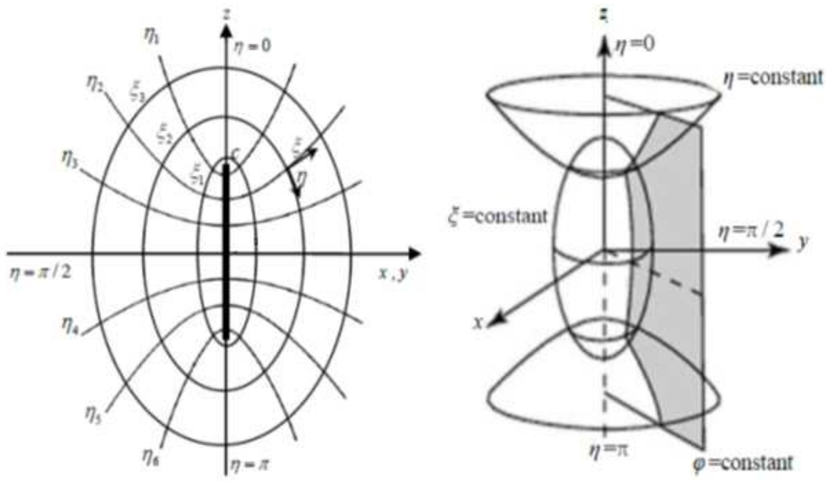

Fig. 1. The prolate spheroidal coordinate system

following equation governs the transient heat conduction equation as

$$\lambda\nabla_1^2 + C_0(x,y,z)\,T = \rho\,C_v\frac{\partial T}{\partial t} + \Theta \tag{5}$$

in which

$$\nabla_1^2 = \frac{1}{c^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right] + \frac{\partial}{\partial\varphi}\left[(\xi^2-\eta^2)\frac{\partial}{\partial\varphi}\right]\right\}$$

alternatively, other equivalent transformation as

$$\nabla_1^2 = \frac{1}{(\sinh^2\xi+\sin^2\eta)\sinh\xi\sin\eta}\left\{\frac{\partial}{\partial\xi}\left[\sinh\xi\sin\eta\frac{\partial}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\sinh\xi\sin\eta\frac{\partial}{\partial\eta}\right] + \frac{\sinh^2\xi+\sin^2\eta}{\sinh\xi\sin\eta}\frac{\partial^2}{\partial\varphi^2}\right\}$$

In a two-dimensional prolate orthogonal coordinate system, the coordinate φ is eliminated because of its rotational symmetry around the z-axis. Then, Eq. (5) takes the following form

$$\lambda\nabla^2 + C_0(x,y,z)\,\theta = \rho\,C_v\frac{\partial\theta}{\partial t} + Q \tag{6}$$

in which $\theta(\xi,\eta,t)$ is the two-dimensional form of temperature distribution,$Q(\xi,\eta,t)$is the two-dimensional form of internal heat source, $C_0(x,y,z)$ in prolate spheroidal coordinate is given as $-\lambda(N^2-1/4)/c^2\xi^2\eta^2$, and the Laplacian ∇^2 can be written as

$$\nabla^2 = \frac{1}{c^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial\theta}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial\theta}{\partial\eta}\right]\right\}$$

alternatively, other equivalent transformation as

$$\nabla^2 = \frac{1}{(\sinh^2\xi+\sin^2\eta)\sinh\xi\sin\eta}\left\{\frac{\partial}{\partial\xi}\left[\sinh\xi\sin\eta\frac{\partial}{\partial\xi}\right] + \frac{\partial}{\partial\eta}\left[\sinh\xi\sin\eta\frac{\partial}{\partial\eta}\right]\right\}$$

2.1. Heat conduction:

The differential equation describes the two-dimensional governing equation of temperature distribution in a prolate confocal spheroid can be obtained by substituting the value of $C_0(x, y, z)$ in Eq. (6) as follows

$$\frac{1}{c^2(\xi^2-\eta^2)}\left\{\frac{\partial}{\partial\xi}\left[(\xi^2-1)\frac{\partial}{\partial\xi}\right]+\frac{\partial}{\partial\eta}\left[(1-\eta^2)\frac{\partial}{\partial\eta}\right]+\left(N^2-\frac{1}{4}\right)\left(\frac{1}{\eta^2}-\frac{1}{\xi^2}\right)\right\}\theta=\frac{1}{\kappa}\frac{\partial\theta}{\partial t}+\frac{Q}{\lambda} \tag{7}$$

subjected to the initial and boundary conditions

$$\left.\begin{array}{l}\text{(i)}\ \ \theta=\theta_0,\ \text{ if }\ t=0,\ a\le\xi\le b,\ -1\le\eta\le 1;\\ \text{(ii)}\ \ \theta+k_1\frac{\partial\theta}{\partial\xi}=0,\ \text{at}\ \ \xi=a,\ t>0,\ -1\le\eta\le 1;\\ \text{(ii)}\ \ \theta+k_2\frac{\partial\theta}{\partial\xi}=0,\ \text{at}\ \ \xi=b,\ t>0,\ -1\le\eta\le 1\end{array}\right\} \tag{8}$$

2.2. Associated thermal stresses:

The equation of equilibrium of thermoelastic body is

$$(\lambda+2\mu)\,\mathrm{grad}\,\Delta-2\mu\,\mathrm{rot}\,\tilde{\omega}=\mathrm{grad}\left[(\lambda+2\mu)\,\alpha\,\theta\right] \tag{9}$$

We have the equation of equilibrium in spheroidal coordinates as

$$\left.\begin{array}{l}(\lambda+2\mu)\sinh\xi\sin\eta\,\frac{\partial\Delta}{\partial\xi}-2\mu\frac{\partial\tilde{\omega}}{\partial\eta}=\alpha(3\lambda+2\mu)\frac{\partial\theta}{\partial\xi},\\ (\lambda+2\mu)\sinh\xi\sin\eta\,\frac{\partial\Delta}{\partial\eta}+2\mu\frac{\partial\tilde{\omega}}{\partial\alpha}=\alpha(3\lambda+2\mu)\frac{\partial\theta}{\partial\eta}\end{array}\right\} \tag{10}$$

in which the component of rotation as

$$2\tilde{\omega}=\frac{\sinh\xi\sin\eta}{\sinh^2\xi+\sin^2\eta}\left[\frac{\partial u'_\eta}{\partial\xi}-\frac{\partial u'_\xi}{\partial\eta}\right] \tag{11}$$

with the component of displacement as

$$\left.\begin{array}{l}u'_\xi=u_\xi/h_\xi=(\sinh^2\xi+\sin^2\eta)^{1/2}\,u_\xi,\\ u'_\eta=u_\eta/h_\eta=(\sinh^2\xi+\sin^2\eta)^{1/2}\,u_\eta\end{array}\right\} \tag{12}$$

having (λ,μ) as the Lamé's elastic constant and dilatation as

$$\Delta=\frac{1}{(\sinh^2\xi+\sin^2\eta)\sinh\xi\sin\eta}\left\{\frac{\partial}{\partial\xi}[\sinh\xi\sin\eta\,u'_\xi]+\frac{\partial}{\partial\eta}[\sinh\xi\sin\eta\,u'_\eta]\right\} \tag{13}$$

The solution of Eqs. (10) without the body force in a spheroidal coordinate system can be expressed as

$$(\lambda+2\mu)\nabla^2\Delta=K\theta,\ \nabla^2\Delta=0,\ u'_\eta=0,\ at\ t=0 \tag{14}$$

in which restraint coefficient is $K=\alpha(3\lambda+2\mu)/(\lambda+2\mu)=\alpha(1+\upsilon)/(1-\upsilon)$ with υdenotes as Poisson's ratio.The corresponding stress components (taking the notations as $\alpha=\sinh\xi$, $\bar{\alpha}=\cosh\xi$, $\beta=\cos\eta$, $\bar{\beta}=\sin\eta$ for simplicity as) are given by

$$\begin{aligned}\sigma_{\xi\xi}=h^2&\left\{\bar{\alpha}^2\frac{\partial^2u'_\xi}{\partial\xi^2}-c^2h^2\bar{\beta}^2\left(\xi\frac{\partial u'_\xi}{\partial\xi}-\eta\frac{\partial u'_\xi}{\partial\eta}\right)\right\}\\ &+ch^2\left\{\alpha\bar{\alpha}^2\beta\frac{\partial^2u'_\eta}{\partial\xi^2}-[c^2h^2\alpha^2\bar{\beta}^2+2(1-\upsilon)\bar{\alpha}^2]\beta\frac{\partial u'_\eta}{\partial\xi}+[c^2h^2\bar{\beta}^2-2\upsilon]\alpha\beta^2\frac{\partial u'_\eta}{\partial\eta}\right\}\end{aligned} \tag{15}$$

$$\begin{aligned}\sigma_{\eta\eta}=h^2&\left\{\bar{\beta}^2\frac{\partial^2u'_\xi}{\partial\eta^2}-c^2h^2\bar{\alpha}^2\left(\alpha\frac{\partial u'_\xi}{\partial\xi}-\beta\frac{\partial u'_\xi}{\partial\eta}\right)\right\}\\ &+ch^2\left\{\alpha\beta\bar{\beta}^2\frac{\partial^2u'_\eta}{\partial\eta^2}-[c^2h^2\bar{\alpha}^2\beta^2+2(1-\upsilon)\bar{\beta}^2]\alpha\frac{\partial u'_\eta}{\partial\eta}+[c^2h^2\alpha^2-2\upsilon]\bar{\alpha}^2\beta\frac{\partial u'_\eta}{\partial\xi}\right\}\end{aligned} \tag{16}$$

$$\begin{aligned}\sigma_{\xi\eta}=h^2\bar{\alpha}\beta&\left\{-\frac{\partial^2u'_\xi}{\partial\xi\partial\eta}+c^2h^2\left(\alpha\frac{\partial u'_\xi}{\partial\eta}+\beta\frac{\partial u'_\xi}{\partial\xi}\right)\right\}\\ &-ch^2\bar{\alpha}\beta\left\{\alpha\beta\frac{\partial^2u'_\eta}{\partial\xi\partial\eta}-[c^2h^2\beta^2+(1-2\upsilon)]\alpha\frac{\partial u'_\eta}{\partial\xi}-[c^2h^2\alpha^2+(1-2\upsilon)]\beta\frac{\partial u'_\eta}{\partial\eta}\right\}\end{aligned} \tag{17}$$

For traction-free surfaces the stress functions

$$\sigma_{\xi\xi}=\sigma_{\xi\eta}=0\ \text{at}\ \xi=a \tag{18}$$

3. Solution to the problem

3.1. The required transformation

To solve fundamental differential Eq.(7), we firstly introduce a new integral transformation of order n and p over the variable ξ and η as

$$\bar{f}(C_{m,n}) = \int_a^b \int_{-1}^1 f(\xi,\eta)\,(\xi^2-\eta^2)\,S_{n,p}(k_1,k_2,\xi,\eta,C_{m,n,p})\,d\xi\,d\eta \tag{19}$$

and its inversion theorem of (19) is

$$f(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty} \frac{\bar{f}(C_{m,n,p})}{\Lambda_{n,p}}\,S_{n,p}(k_1,k_2,\xi,\eta,C_{m,n,p}) \tag{20}$$

in which the kernel $S_{n,p}(k_1,k_2,\xi,\eta,q_{n,p})$ is given in equation

$$\begin{aligned} S_{n,p}(k_1,k_2,\xi,\eta,C_{m,n}) &= \Psi^{(1)}_{m,n}(\xi,C_{m,n})\,[\Psi^{(2)}_{m,n}(k_1,a,C_{m,n})+\Psi^{(2)}_{m,n}(k_2,b,C_{m,n})] \\ &\quad -\Psi^{(2)}_{m,n}(\xi,C_{m,n})[\Psi^{(1)}_{m,n}(k_1,a,C_{m,n})+\Psi^{(1)}_{m,n}(k_2,b,C_{m,n})]\Phi_{m,n}(\eta,C_{m,n}) \end{aligned}$$

and

$$\Lambda_{np} = \int_a^b \int_0^1 (\xi^2-\eta^2)\,S^2_{n,p}(k_1,k_2,\xi,\eta,C_{m,n,p})\,d\xi\,d\eta,$$

in which $C_{m,n}$ is the positive root of the equation of

$$[\Psi^{(1)}_{m,n}(k_1,a,C)\,\Psi^{(2)}_{m,n}(k_2,b,C)-\Psi^{(1)}_{m,n}(k_2,b,C)\Psi^{(2)}_{m,n}(k_1,a,C)]\,\Phi_{m,n}(\eta,C)=0$$

3.2. Transient heat conduction analysis

Applying the transform defined by Eq.(19), and taking into account last two equations of(8), we obtain

$$-\lambda^2_{m,n,p}\,\ddot{\bar{\theta}} + \frac{\ddot{\bar{\Theta}}\,(n,p,t)}{\lambda} = \frac{1}{\kappa}\frac{\partial\ddot{\bar{\theta}}}{\partial t} \tag{21}$$

in which

$$\lambda^2_{m,n,p} = \frac{2\kappa}{c^2}C^2_{m,n,p}$$

The solution of linear Eq. (21) is

$$\ddot{\bar{\theta}} = \theta_0\exp(-\lambda^2_{m,n,p}t) + \kappa\int_o^t \dddot{\bar{\Theta}}\,(n,p,t-u)\exp(-\lambda^2_{m,n,p}u)\,du \tag{22}$$

On applying inversion theorems defined in (20), one obtains the expression for temperature as

$$\begin{aligned}\theta(\xi,\eta,t) &= \textstyle\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}[\theta_0\exp(-\lambda^2_{m,n,p}t)+\kappa\int_o^t \dddot{\bar{\Theta}}\,(n,p,t-u) \\ &\quad\times\exp(-\lambda^2_{m,n,p}u)du]S_{n,p}(k_1,k_2,\xi,\eta,C_{m,n,p})/\Lambda_{n,p}\end{aligned} \tag{23}$$

3.3. Thermal displacements and thermal stresses

Now, we assume the components of displacements which satisfies the Eq. (14) as

$$u'_\xi = \frac{K}{(\lambda+2\mu)}\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\left\{\left(\theta_0\exp(-\lambda^2_{m,n,p}t)\;\;\xi + [B_{np}\sin\xi + C_{np}\cos\eta]\right)/\Lambda_{n,p}\;\right\} \tag{24}$$

with B_{np} and C_{np} are the arbitrary constants, which can be determined finally by using condition (18) as

$$B_{np} = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\exp(-\lambda^2_{m,n,p}t)\,\theta_0\,c^2h^2\,\bar{\beta}^2(a\,\alpha+\beta\eta)/Z_{np}$$

$$C_{np} = \sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\exp(-\lambda^2_{m,n,p}t)\,\theta_0\,\beta\,\bar{\alpha}^2\,\csc\eta\,\sin a/Z_{np}$$

in which,

$$Z_{np} = c^4 h^2 (a\,\alpha + \beta\eta) \cos a\, \bar{\beta}^2 + \alpha\, \bar{\alpha}^2 \sin a$$

and

$$u'_\eta = \sum_{n=0}^{\infty} \sum_{p=1}^{\infty} \left\{ (D_{np} \sinh\xi + \lambda\xi) + K\ (\theta_0 \exp(-\lambda^2_{m,n,p} t))/(\lambda + 2\mu)\Lambda_{n,p} \right\} \tag{25}$$

in which D_{np} is the arbitrary constant to be determined by using condition (14) in Eq. (25), one obtains

$$D_{np} = -\frac{1}{\sinh\xi}\left(\lambda\xi + \frac{K\theta_0}{(\lambda + 2\mu)\Lambda_{n,p}}\right)$$

Now, using equations (24) and (25) in equations (15)-(17), one obtains the expressions for stresses as

$$\begin{aligned}\sigma_{\xi\xi} = &-\textstyle\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\{(\exp(-\lambda^2_{m,n,p} t) c^2 h^4 K \theta_0 \bar{\beta}^2 (c^4 h^4 (a\alpha + \beta\eta)\xi \\ &\times(\cos a - \cos\xi)\bar{\beta}^2 + \alpha\bar{\alpha}^4 \sin a((\alpha\xi + \beta\eta \csc^2\eta) \sin a - (a\,\alpha + \beta\eta) \sin\xi) \\ &+ c^2 h^2 \bar{\alpha}^2 \bar{\beta}^2 (-\alpha \sin a((a\,\alpha + \beta\eta)\xi \cos\xi - \beta\eta \sin\xi) + \cos a((\beta^2\eta \cot\eta \\ &+ (a\,\alpha + \beta\eta)(2\,\alpha\xi + \beta\eta \csc^2\eta)) \sin a - (a\,\alpha + \beta\eta)^2 \sin\xi)))) \\ &/(\Lambda_{n,p}(\lambda + 2\mu) Z^2_{np}\}\end{aligned} \tag{26}$$

$$\begin{aligned}\sigma_{\eta\eta} = &-\textstyle\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\{(\exp(-\lambda^2_{m,n,p} t) h^2 K\theta_0 \bar{\alpha}^2 (c^4 h^2 \alpha^2 (\alpha^2 + \beta^2 \csc^2\eta)\bar{\alpha}^6 \sin^3 a \\ &+ c^4 h^4 \bar{\beta}^6 \cos a\, (c^4 h^4 \alpha (a\,\alpha + \beta\eta)^3 \cos^2 a - \cos a\, (c^4 h^4 \alpha (a\,\alpha + \beta\eta) \cos\xi \\ &-\tfrac{1}{2}\beta \csc^2\eta(-(4\,a^2\alpha^2 + 8\,a\,\alpha\,\beta\eta + \beta^2(1 + 4\eta^2)) \cot\eta + \beta(-4\,(a\,\alpha + \beta\eta) \\ &+\beta \cos 3\eta \csc\eta)) \sin a) - 2\alpha\beta^2 \sin a\, \sin\xi) + c^2 h^2 \cos a\bar{\alpha}^2 \bar{\beta}^4 \sin a(c^4 h^4 \\ &\times(a\,\alpha + \beta\eta) \cos a\,(\beta^3 \cot\eta + (a\,\alpha + \beta\eta)\,(3\alpha^2 + \beta^2 \csc^2\eta)) - \alpha(2c^4 h^4 \alpha \\ &\times(a\,\alpha + \beta\eta) \cos\xi + \beta(2(\beta + 2(a\,\alpha + \beta\eta) \cot\eta) \times \csc^2\eta\, \sin a)) - c^4 h^4 \beta \\ &\times(a\,\alpha + \beta\eta) \sin\xi))) + \alpha\,\bar{\alpha}^4\,\bar{\beta}^2 \sin a\,(-c^4 h^4 \alpha^2 (a\,\alpha + \beta\eta) \cos\xi\, \sin a \\ &+ c^4 h^4 \cos a(3\alpha^2\,(a\,\alpha + \beta\eta) + \beta^3 \cot\eta) \sin a + \beta(-2\,\alpha \cot\eta \csc^2\eta\, \sin^2 a \\ &+ c^4 h^4 \beta((a\,\alpha + \beta\eta) \csc^2\eta\, \sin 2a + \alpha \sin a \sin\xi)))))/\,(\Lambda_{n,p}(\lambda + 2\mu)\, Z^3_{np}\}\end{aligned} \tag{27}$$

$$\begin{aligned}\sigma_{\xi\eta} = &\textstyle\sum_{n=0}^{\infty}\sum_{p=1}^{\infty}\{(\exp(-\lambda^2_{m,n,p} t)\, c^2 h^4 K \beta^2 \theta_0 \alpha^2 (c^4 h^4 (a\,\alpha + \beta\eta)^2 \cos a \\ &\times(\cos a - \cos\xi)\bar{\beta}^4 - \alpha^2 \bar{\alpha}^4 \cot^2\eta\, \sin^2 a\ - \alpha\,\bar{\alpha}^2 \bar{\beta}^2 \sin a \\ &\times((-1 + a\,c^2 h^2 \alpha + c^2 h^2 \beta\eta) \cos\xi + c^2 h^2 (\cos a(-a\alpha - \beta\eta + \beta \cot\eta \\ &+ (a\,\alpha + \beta\eta) \cot^2\eta) + a \sin\xi))))/\,(\Lambda_{n,p}(\lambda + 2\mu)\, Z^2_{np}\}\end{aligned} \tag{28}$$

4. Numerical results, discussion and remarks

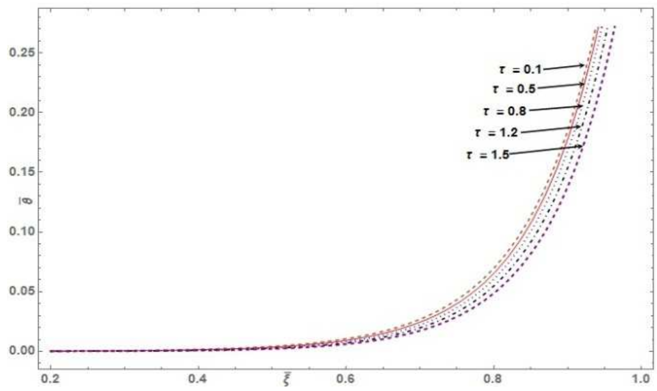

Fig. 2. Temperature distribution versus ξ at η=1 for different values of time.

For the sake of simplicity of calculation, we introduce the following dimensionless values

$$\left.\begin{aligned}&\bar{\xi} = \xi/b,\ e = c/b,\ \bar{h}^2 = h^2 b^2,\ \tau = \kappa\, t/b^2, \bar{\phi}(\xi,\eta,t) = \phi(\xi,\eta,t)/E\alpha_t\theta_k b^2 \\ &\overline{T}(\xi,\eta,\ t) = T(\xi,\eta,\ t)/\theta_k,\ (\overline{T}_i, \overline{T}_o) = (T_i, T_o)/T_k\ (k = i, o),\end{aligned}\right\} \tag{29}$$

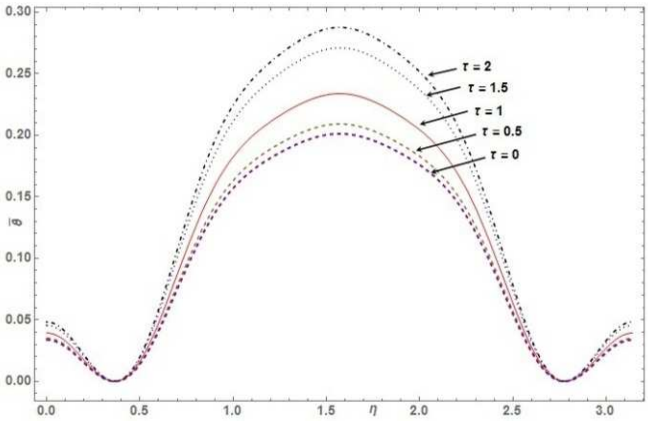

Fig. 3. Temperature distribution versus η at ξ=0.4 for different values of time.

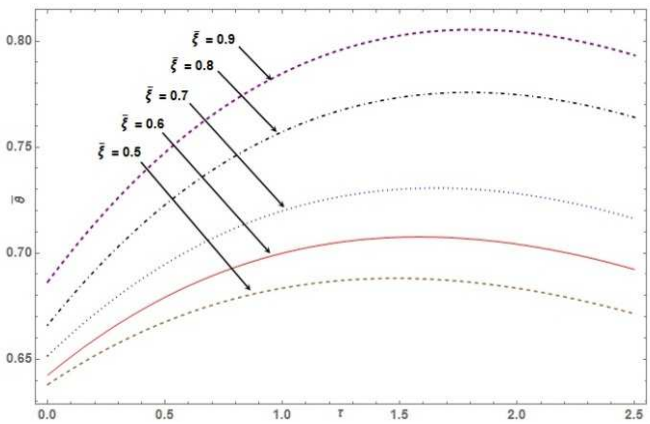

Fig. 4. Temperature distribution versus τ at η=1 for different values of ξ.

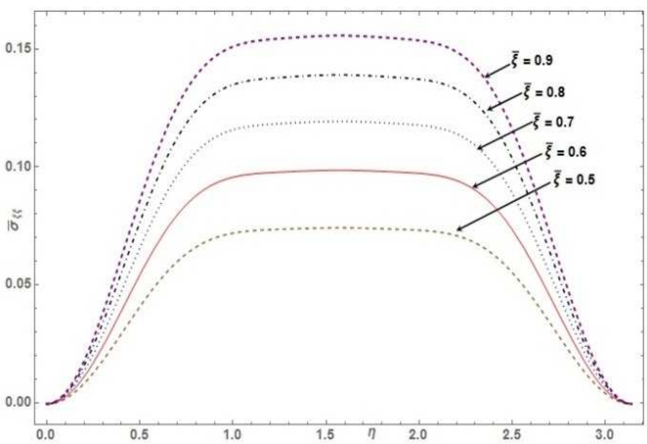

Fig. 5. Radial stress versusη at fixed τ for different values of ξ.

Substituting Eq. (29) in Eqs. (23), (26), (27), (28), one obtains the expressions for the temperature distribution and associated stresses for numerical discussion. The numerical computations have been carried out for Aluminum metal with parameter a=0.65 cm, b=3.22 cm, Modulus of Elasticity E= 6.9 × 10^{-6}N/cm^2, Shear modulus G= 2.7 × 10^6N/cm^2, Poisson ratio $\upsilon = 0.35$, Thermal expansion coefficient $\alpha_{\ t}$= 25.5× 10^6cm/cm-^{0}C, Thermal diffusivity κ= 0.86 cm^2/sec, Thermal conductivity λ= 0.48 calsec^{-1}/cm^{0}C with $p = 0, 1, 2, 3, 4...22$; n = 0, 2, 4, 6, and $C_{m,n,p} = 0.2, 0.4, 0.6, 0.8, ...15.2$ [5] are the positive and real roots of the transcendental equation. The foregoing analysis is performed by setting the radiation coefficients constants, $k_i = 0.86$ $(i = 1, 2)$ so as to obtain considerable mathematical simplicities. In order to examine the influence of uniform heating on the disc, we performed the numerical calculation of time

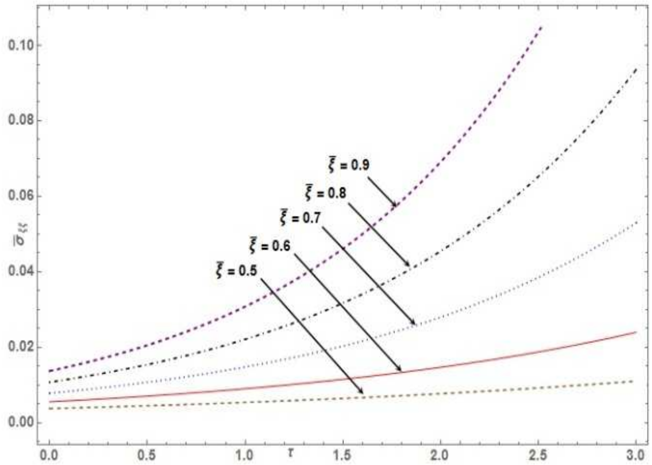

Fig. 6. Radial stress versus τ at η=1 for different values of ξ.

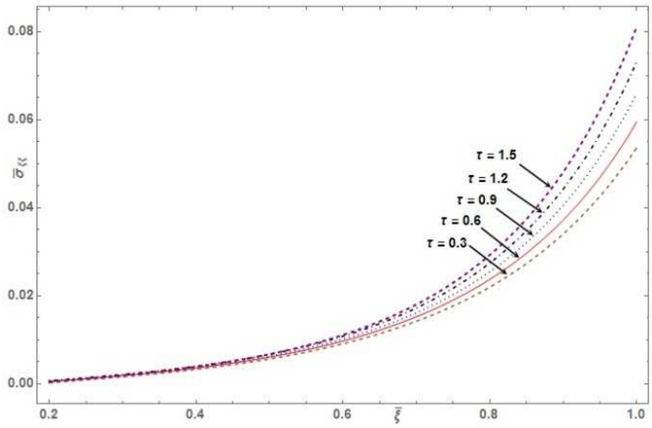

Fig. 7. Radial stress versus ξ for different values of time.

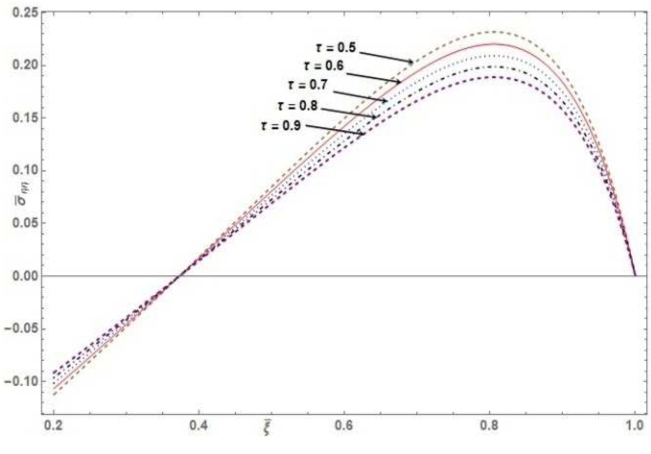

Fig. 8. Tangential stress versus ξ at fixed η for different values of time.

$\tau = 0.001, 0.05, 0.12, 0.30, 0.70, ...\infty$ and numerical calculations are depicted in the following figures with the help of MATHEMATICA software. For the sake of brevity, discussion of these effects is omitted here and graphical illustration is investigated for a thermoelastic response for an elliptical disc considering interior heat generation. From Fig. 2, it can be seen that the temperature change on the heated surface increases when the radius of plate increases, this may be due to the shape of the spheroid. Fig. 3 shows the time variation of temperature distribution along the angular direction of the disc. Initially, the temperature attains minimum, it increases with time and the maximum value of temperature magnitude occurs at mid-core of the disc due to the accumulation of heat energy and again at-

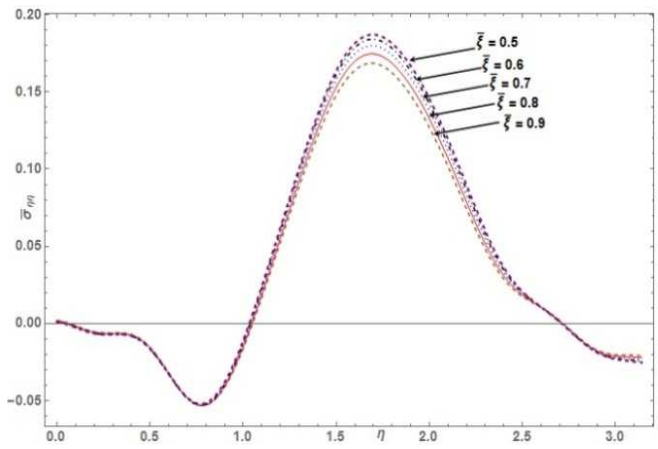

Fig. 9. Tangential stress versus η at a fixed time for different values of ξ.

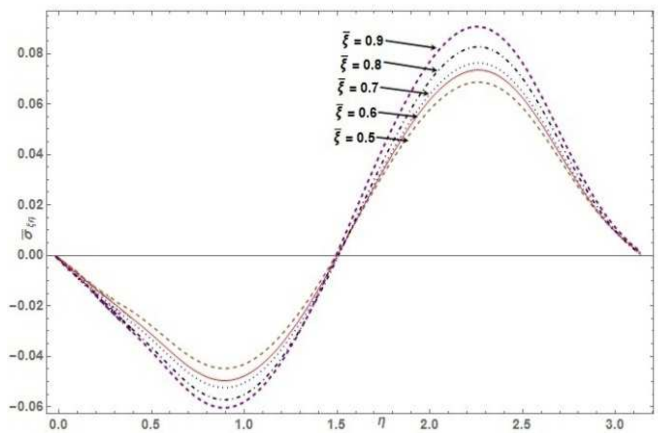

Fig. 10. Shear stress versus η at fixed time for different values of ξ.

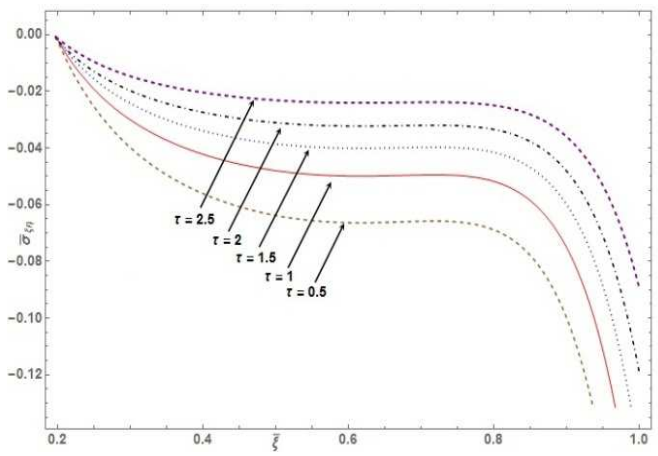

Fig. 11. Shear stress versus ξ at fixed η for different values of time.

taining minimum value on the other side due to compressive force occur on both the ends following a bell-shaped curve. Fig. 4 indicates the temperature distribution along time parameters; temperature attains maximum expansion as time increases. Fig. 5 shows that radial stress attains zero due to compressive stress occurring on the inner part of the plate satisfying the condition (18), whereas maximum tensile stress occurs towards outer surface due to the accumulation of thermal energy dissipated by sectional and internal heat supply. Fig. 6 shows thermal radial stresses along time, but having the same trend almost with Fig. 7, except with the magnitude. From Fig. 8, the tangential stress is observed from negative to a positive direction, the large compressive stress occurs at the inner edge while the tensile stress occurs at the outer edge which drops along the radial direction. Fig. 9 and Fig. 10 explain the tangential and shear stress along the angular direction. In Fig. 9, expansion occurs on the outer edge due to sectional heat supply followed by the compressive stress occurring on the inner core of the ellipse, and the absolute value tends to be zero as its proceeds for shear stress profile.From Fig. 11, it is clear that at the early stage, zero stress occurs in the interior to the plate satisfying the condition (18), and then attains minimum towards the outer curve along the radial direction.

5. Conclusion

The proposed analytical solution of transient thermal stress problem in a generalised prolate spheroidal region subjected to the radiation type boundary conditions on the inner and outer curved surfaces was dealt in a spheroidal coordinates system with the presence of a source of internal heat. The approach integral transformation is used to obtain an exact solution of heat conduction equation and the results can be deduced by specializing the coefficients and parameters involved therein. Some results which are derived using computational tools are accurate enough for the practical purpose was depicted graphically. The following results were obtained during our research.

1. The advantage of this method is its generality and its mathematical power to handle different types of mechanical and thermal boundary conditions.

2. The maximum tensile stress shifting from central core to outer part in the spheroidal region may be due to heat, stress, concentration or available internal heat sources under considered temperature field.

3. The maximum tensile stress occurring in the circular core on the major axis compared to spheroidal central part indicates the distribution of weak heating. It might be due to insufficient penetration of heat through the spheroidal inner surface.

4. Any particular case of interest can be deduced by assigning suitable values to the parameter and source functions involved in the solutions

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