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On the problem of guaranteed package guidance with some terminal quality criterions

Research Article

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Abstract: We consider the class of problems of the control theory related to effective methods for constructing controls. One of approaches to solving control problems under lack of information was suggested by Yu.S. Osipov and called the method of program packages. The initial state of the considered system is unknown, but belongs to a finite set. The control problem contains a group of terminal quality criterions, which depends on initial states. We use a modification of the method of non-anticipatory strategies (quasi-strategies) for the solving such control problem and constructing program packages. We note that in all previous works related to the method of program packages, the guidance problems with one quality criterion were considered. In the present paper the guaranteed guidance problem with some terminal quality criterions under incomplete information about the initial state is considered for a linear autonomous control system. The criterion for the solvability of that problem based on the method of program packages is established. An illustrative example is given.

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1. Introduction

Consider a controlled system of the form

$$\dot{x}(t) = Ax(t) + Bu(t) + c(t), \tag{1}$$

where $t \in T = [t_0, \vartheta]$ is a time variable, $t_0 < \vartheta < +\infty$; $x(t) \in \mathbb{R}^n$ is the system's state at the time t; $u(t) \in \mathbb{R}^m$ is the value of control vector at the time t; A and B are matrices of corresponding dimensions, and the function $c(\cdot)\colon T \mapsto \mathbb{R}^n$ is a piecewise continuous function.

We assume that the controller a priori knows that the initial state x_0 of the system belongs to some finite set $X_0 \subset \mathbb{R}^n$ (the set of *admissible initial states*), but this state itself is not known. Under an *open-loop control* we understand any Lesbeque measurable function $u(\cdot)\colon T \to U$. Here, $U \subset \mathbb{R}^m$ is a convex compact set describing the instantaneous resource of control. The set of all open-loop controls is denoted by $\mathscr{U}$. The systems's *motion* corresponding to an

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admissible initial state $x_0 \in X_0$, and an open-loop control $u(\cdot) \in \mathscr{U}$ is called a Carathéodory solution of system of differential equations (1) defined on the interval T and satisfying the initial condition $x(t_0) = x_0$; this motion in denoted by $x(\cdot; t_0, x_0, u(\cdot))$. Control problems for dynamical systems are one of the intensively developing sections of optimization theory [1–3].

If the initial state x_0 is known the standard open-loop control problem is to find a control $u^{\text{opt}}_{x_0}(\cdot)$ minimizing the quality criterion:

$$u^{\text{opt}}_{x_0}(\cdot) = \operatorname{argmin}\left\{f\left(x(\vartheta; t_0, x_0, u(\cdot))\right) : u(\cdot) \in \mathscr{U}\right\}.$$

Let us introduce the following notation:

$$J_{x_0} = f\left(x\left(\vartheta; t_0, x_0, u^{\text{opt}}_{x_0}(\cdot)\right)\right), \quad M_{x_0} = \left\{x \in \mathbb{R}^n : f(x) \leqslant J_{x_0}\right\}, \quad x_0 \in X_0.$$

We assume below that $f(\cdot): \mathbb{R}^n \to \mathbb{R}$ is a proper convex function.

$$\operatorname{dom} f = \left\{x \in X : f(x) < +\infty\right\}.$$

A function f is called proper if $\operatorname{dom} f \neq 0$ and $f(x) > -\infty$ for all $x \in X$. Let zero belong to the interior of the set $\operatorname{dom} f$. The latest condition, as known, involves the continuity of function f at zero.

The closed-loop guidance problem on a collection of target sets consist in forming an open-loop control guaranteeing that the system's state reaches the set $M_{\bar{x}_0}$, where $\bar{x}_0$ is the real (but unknown) initial state. In the motion process, the controller forms the open-loop control by the feedback principle observing the current signal $y(t) = Q(t)\,x(t)$ on the system's state $x(t)$ at this moment. In accordance with the standard formalism of the theory of guaranteed control [4, 5], the controller corrects the values of an open-loop control $u(\cdot)$ at times $t_0 = \tau_0 < \tau_1 < \ldots < \tau_m = \vartheta$ specified in advance. At any moment τ_j ($j \in [0, m-1]$), the values of the open-loop control are determined for $t \in [\tau_j, \tau_{j+1})$ depending on the history $t \mapsto y(t)$ of the observation on $[t_0, \tau_j]$ and the history $t \to u(t)$ of the control on $[t_0, \tau_j)$. Thus, the problem of guaranteed closed-loop guidance to a group of target sets consist in choosing (by an arbitrary $\varepsilon > 0$ specified in advance) a control forming method such that a motion $x(\cdot)$ of system (1) reaches a closed ε-neighborhood of the target set M_{x_0} for any initial state $x_0 \in X_0$ at the terminal time ϑ.

As follows from Theorem 2.1 presented below, this problem is solvable if and only if the package guidance problem is solvable. So, in the present paper we focus on solvability conditions of the latter problem using the method from [6–8]. It should be noted that the method was used for solving the guaranteed guidance problem for linear systems of ordinary differential equations [9, 10], linear stochastic differential equations [11], delay differential equations [12], systems with distributed parameters [13]. In the papers cited above, it was assumed that the target set was the same for all initial states.

Remark 1.1.
Let

$$J^* = \max_{x_0 \in X_0} J_{x_0} = \max_{x_0 \in X_0} \min_{u(\cdot) \in \mathscr{U}} J\left(x(\vartheta; t_0, x_0, u(\cdot))\right). \tag{2}$$

The symbol x_0^* stands for a vector maximizing, with respect to x_0, the right-hand part of (2). Then, if

$$\bar{x}_0 = x_0^*,$$

the control problem is naturally called the guaranteed maximin guidance problem.

2. Preliminaries.

Let us introduce some results of [13, 14] before we proceed to find solvability conditions of the problem in question. For the convenience of the readers the results are formulated in the reduced form. Consider controlled system (1). Let us introduce the fundamental matrix $F(\cdot,\cdot)$ of the homogeneous system $\dot{x}(t) = Ax(t)$. For any $x_0 \in X_0$, we define

$$g_{x_0}(t) = Q(t)\,F(t, t_0)\,x_0 \quad (t \in T);$$

the function $g_{x_0}(\cdot)$ is called the *homogeneous signal* corresponding to the admissible initial state x_0. The set of all admissible initial states x_0 corresponding to a homogeneous signal $g(\cdot)$ *till a time* $\tau \in [t_0, \vartheta]$ is denoted by $X_0(\tau|g(\cdot))$; thus,

$$X_0\left(\tau|g(\cdot)\right) = \left\{x_0 \in X_0 : g(\cdot)|_{[t_0,\tau]} = g_{x_0}(\cdot)|_{[t_0,\tau]}\right\};$$

hereinafter, $g(\cdot)|_{[t_0,\tau]}$, where $\tau \in [t_0, \vartheta]$, is the restriction of the homogeneous signal $g(\cdot)$ onto the interval $[\tau_0, \tau]$.

A family $(w_{x_0}(\cdot))_{x_0\in X_0}$ of open-loop controls is called an *open-loop control package* if it satisfies the following *condition of nonanticipation*: for any homogeneous signal $g(\cdot)$, time $\tau\in(t_0,\vartheta]$, and admissible initial states $x_0',x_0''\in X_0\big(\tau|g(\cdot)\big)$, the equality $w_{x_0'}(t)=w_{x_0''}(t)$ holds for all $t\in[\tau_0,\tau)$. An open-loop control package $\big(w_{x_0}(\cdot)\big)_{x_0\in X_0}$ is called *guiding* if, for any admissible initial state $x_0\in X_0$, the inclusion $x\big(\vartheta;t_0,x_0,w_{x_0}(\cdot)\big)\in M_{x_0}$ takes place. If there exists a guiding open-loop package, we say that the *problem of package guidance is solvable*.

Let G be the set of all homogeneous signals. For an arbitrary homogeneous signal $g(\cdot)$, we introduce the set $T\big(g(\cdot)\big)=\big\{\tau_j\big(g(\cdot)\big):\,j=1,\dots,k_{g(\cdot)}\big\}$ of all *splitting moments*; the strong definition of them is given in [8], and so $T=\cup_{g(\cdot)\in G}T\big(g(\cdot)\big)$. In view of the finiteness of this set, for any homogeneous signal $g(\cdot)$, there exists an index $k_{g(\cdot)}\geqslant 1$ such that $\tau_{k_{g(\cdot)}}\big(g(\cdot)\big)=\vartheta$. Then, the set T can be written in the form $T=\{\tau_1,\dots,\tau_K\}$, where $\tau_j<\tau_{j+1}$ $(i=j,\dots,K-1)$. We suppose $\tau_0=t_0$. For any $k=1,\dots,K$, we introduce the set

$$\mathscr{X}_0(\tau_k)=\big\{X_0\big(\tau_k|g(\cdot)\big):\,g(\cdot)\in G\big\}.$$

Elements $X_{0,k}$ of the set $\mathscr{X}_0(\tau_k)$ is called the *clusters of initial states* at the moment τ_k. For any $k=0,\dots,K$, the clusters of initial states at the moment τ_k form a partition of the set X_0 of all admissible initial states, i.e.,

$$X_0=\bigcup_{X_{0,k}\in\mathscr{X}_0(\tau_k)}X_{0,k},\quad X'_{0,k}\bigcap X''_{0,k}=\varnothing\quad\Big(X'_{0,k},X''_{0,k}\in\mathscr{X}_0(\tau_k),\;X'_{0,k}\neq X''_{0,k}\Big).$$

Let $\mathscr{U}_{X_0}$ be the set of all families of vectors $\big(w_{x_0}(\cdot)\big)_{x_0\in X_0}$ in U. Any Lebesque measurable function $t\mapsto\big(w_{x_0}(t)\big)_{x_0\in X_0}:\,T\mapsto\mathscr{U}_{X_0}$ is called an *extended open-loop control*. The family of open-loop controls $\big(w_{x_0}(\cdot)\big)_{x_0\in X_0}$ is identified with the extended open-loop control $t\mapsto\big(w_{x_0}(t)\big)_{x_0\in X_0}$ as in [8, 10–14]. For any $k=0,\dots,K$, we denote by $\mathscr{U}_k$ the set of all families $\big(w_{x_0}(\cdot)\big)_{x_0\in X_0}\in\mathscr{U}_{X_0}$ such that, for any cluster $X_{0,k}\subset\mathscr{X}_0(\tau_k)$ and any initial states $x_0',x_0''\in X_{0,k}$, the equality $w_{x_0'}=w_{x_0''}$ holds. An extended open-loop control $\big(w_{x_0}(\cdot)\big)_{x_0\in X_0}$ is called *admissible* if, for any $k=0,\dots,K$, the inclusion $\big(w_{x_0}(t)\big)_{x_0\in X_0}\in\mathscr{U}_k$ is valid for all $t\in(\tau_{k-1},\tau_k]$ in case $k>1$ and for all $t\in[t_0,\tau_1]$ in case $k=1$.

For $j=1,2,\dots$, we define an *extended control resource* $\mathscr{R}_j$ as the finite-dimentional Hilbert space of all families $l=(l_{x_0})_{x_0\in X_0}$ of vectors in $\mathbb{R}^j$ with the scalar product $\langle\cdot,\cdot\rangle_{\mathscr{R}_j}$ of the form $\langle l',l''\rangle_{\mathscr{R}_j}=\sum_{x_0\in X_0}(l'_{x_0},l''_{x_0})$ $\quad(l'=(l'_{x_0})_{x_0\in X_0}\in\mathscr{R}_j,\;l''=(l''_{x_0})_{x_0\in X_0}\in\mathscr{R}_j)$. Hereinafter, $(\cdot,\cdot)$ is the scalar product in the finite-dimensional Euclidian space $\mathbb{R}^n$. The values of extended open-loop controls are further considered as elements of $\mathscr{R}_n$.

Consider the *extended system* consisting of copies of (1) parameterized by admissible initial states $x_0\in X_0$. A copy parameterized by an admissible initial state x_0 has x_0 as the initial state and is subject to the action of some open-loop control $w_{x_0}(\cdot)$. Thus, the extended system has the form

$$\dot{x}_{x_0}(t)=A\,x_{x_0}(t)+B\,w_{x_0}(t)+c(t),\quad x_{x_0}(t_0)=x_0,\quad(x_0\in X_0).\tag{3}$$

We take the space $\mathscr{R}_n$ as the phase space of the extended system. The extended system's control is chosen from the class of all admissible extended open-loop controls. For any admissible extended open-loop control $t\to\big(w_{x_0}(t)\big)_{x_0\in X_0}$, we understand the corresponding *motion* of the extended system as the function $t\mapsto\big(x\big(t;t_0,x_0,w_{x_0}(\cdot)\big)\big)_{x_0\in X_0}:\,T\mapsto\mathscr{R}_n$. The *extended target set* is the set $\mathscr{M}$ of all families $(x_{x_0})_{x_0\in X_0}\in\mathscr{R}_n$ such that $x_{x_0}\in M_{x_0}$ for all $x_0\in X_0$. An admissible extended open-loop control $t\to\big(w_{x_0}(t)\big)_{x_0\in X_0}$ is said to be *guiding for the extended system* if the corresponding motion $\big(x\big(\cdot;t_0,x_0,w_{x_0}(\cdot)\big)\big)_{x_0\in X_0}$ of this system takes a value in the extended target set at the time ϑ: $\big(x\big(\vartheta;t_0,x_0,w_{x_0}(\cdot)\big)\big)_{x_0\in X_0}\in\mathscr{M}$. We say that the *extended problem of open-loop guidance is solvable* if there exists an admissible extended open-loop control that is guiding for the extended system.

The theorem below is proved by analogy with [6–8].

Theorem 2.1.
1) An extended open-loop control $t\to\big(w_{x_0}(t)\big)_{x_0\in X_0}$ is an open-loop control package if and only if it is admissible.

2) An admissible extended open-loop control is a guiding open-loop control package if and only if it is guiding for the extended system.

3) The problem of guaranteed closed-loop guidance to a collection of target sets is solvable if and only if the problem of open-loop control package guidance is solvable.

4) The problem of open-loop control package guidance to a collection of target sets is solvable if and only if the problem of open-loop control guidance is solvable.

Let S be some subspace of the space $\mathbb{R}^n$ orthogonal to all $l\in\mathbb{R}^n$ such that $\rho^+(l|M_{x_0})=\infty$ for any $x_0\in X_0$. We denote by $L\subset S$ a convex compact set containing an image of the unit sphere. In this case, there exist constants $r_1,\,r_2>0$ satisfying the inequality $r_2>r_1$ and such that, for any vector $z\in S$ of the unit norm there exists $r\in[r_1,r_2]$ such that the inclusion $r\,z\in L$ holds. Then, the symbol $\mathscr{L}$ means the set of all $(l_{x_0})_{x_0\in X_0}\in\mathscr{R}_n$ such that $l_{x_0}\in L$ for all $x_0\in X_0$.

By analogy to [8], the solvability criterion of the open-loop guidance control problem is established. In our case, the criterion takes the form

$$\sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}}\gamma\big((l_{x_0})_{x_0\in X_0}\big)\leqslant 0.\tag{4}$$

Here,

$$\gamma\big((l_{x_0})_{x_0\in X_0}\big) = \mathcal{V}(l_{x_0};\vartheta,t_0) - \sum_{x_0\in X_0} p^+(l_{x_0}|M_{x_0}), \quad D(\tau) = B^\top F^\top(\vartheta,\tau),$$

$$\mathcal{V}(l_{x_0};\vartheta,t_0) = \sum_{x_0\in X_0}\left(l_{x_0}, F(\vartheta,t_0)\,x_0 + \int_{t_0}^{\vartheta} F(\vartheta,\tau)\,c(\tau)\,d\tau\right) + \sum_{k=1}^{K}\int_{\tau_{k-1}}^{\tau_k} \sum_{X_{0,k}\in\mathscr{X}(\tau_k)} \varrho^-\left(D(\tau)\sum_{x_0\in X_{0,k}} l_{x_0}\Big|U\right) d\tau,$$

$\varrho^-(l|U) = \inf\{(l,x)\colon x\in U\}$ and $\varrho^+(l|M_{x_0}) = \sup\{(l,x)\colon x\in M_{x_0}\}$ are the lower and upper support functions, respectively; $|\cdot|_{\mathbb{R}^n}$ denotes the norm in the same space $\mathbb{R}^n$, and $\top$ means transposition.

3. Solvability of guaranteed open-loop guidance problem

Assume that, for any x_0 there exists a solution of the problem

$$\min_{u(\cdot)\in\mathscr{U}} f\,(x\,(\vartheta;t_0,x_0,u(\cdot))) = J_{x_0}.$$

For any $a>0$, we introduce the set

$$M_{x_0}^a = \{x\in\mathbb{R}^n\colon f(x)\leqslant J_{x_0}-a\}.$$

Lemma 3.1.
Let $a>0$. Then, there exists a number $\mu\in(0,a)$ such that, whatever the initial state $x_0\in X_0$ may be, the inclusion

$$O_\mu(M_{x_0}^a)\subset M_{x_0} \tag{5}$$

is valid. Here, $O_\mu(M)$ means a μ-neighborhood of the set M.

Proof. By the finiteness of the set X_0, it is sufficient to establish that inclusion (5) is valid for one x_0. Let us assume the contrary; i.e, relation (5) does not hold for some $x_0\in X_0$. We take an arbitrary sequence of numbers $\mu_i\to 0$ as $i\to\infty$. Then, for any natural number i, we find a vector $x_i\in O_{\mu_i}(M_{x_0}^a)$ such that $x_i\notin M_{x_0}$ and $f(x_i)>J_{x_0}$. Therefore,

$$\liminf_{i\to\infty} f(x_i)\geqslant J_{x_0}. \tag{6}$$

Note that each vector x_i can be represented as the sum of two vectors $x_i=y_i+z_i$, where $y_i\in M_{x_0}^a$; i.e., the inequalities $f(y_i)\leqslant J_{x_0}-a$ and $\|z_i\|\leqslant\mu_i$ are valid. Then, by the convexity of the function f, we obtain

$$f(x_i)\leqslant f(y_i)+f(z_i)\leqslant J_{x_0}-a+f(z_i).$$

Because of the continuity of the function f at zero, we have $f(z_i)\to f(0)=0$ as $i\to\infty$. Consequently,

$$\limsup_{i\to\infty} f(x_i)\leqslant J_{x_0}-a.$$

Thus, we have a contradiction with (6), which proves the lemma. □

Further, we need
Condition 1. For any $\gamma_*>0$, one can specify $\mu\in(0,\gamma_*)$ such that, for any $x_0\in X_0$, the inclusion $M_{x_0}\in O_\mu(M_{x_0}^{\gamma_*})$ holds.

Remark 3.1.
It is clear that Condition 1 does not always hold. For example, for the quality criterion $f(x)=c|x|_{\mathbb{R}^n}$ it is valid for $c>1$, but it is not for $c<1$. Condition 1 is, in some sense, "inverse" to the statement of Lemma 1.

Theorem 3.1.
Let the function f satisfy Condition 1. Then, the equality

$$\max_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \gamma\big((l_{x_0})_{x_0\in X_0}\big)=0 \tag{7}$$

represents necessary and sufficient conditions for the solvability of the guaranteed open-loop control problem on a collection of target sets.

Proof. We start from the necessity. Let us assume the contrary: the problem is solvable and

$$\gamma \equiv \sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \gamma(l_{x_0}) = -\gamma^* < 0,$$

where $\gamma^* > 0$. Then, for any $x_0 \in X_0$, we have $\gamma(l_{x_0}) \leqslant -\gamma^*$; i.e.,

$$\sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \left(\mathcal{V}(l_{x_0};\vartheta,t_0) - \sum_{x_0\in X_0} \rho^+(l_{x_0}|M_{x_0})\right) \leqslant -\gamma^*. \tag{8}$$

By the definition of the set $\mathscr{L}$, one can derive the existence of the number $\mu^* > 0$ from the equality

$$\sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} |l_{x_0}|_{\mathbb{R}^n} = \mu^*.$$

Then, following Condition 1, we find number $\mu \in \left(0, \frac{\gamma^*}{\mu^* K}\right)$ such that

$$\rho^+\left(l_{x_0}|O_\mu(M_{x_0}^{\gamma^*})\right) = \rho^+\left(l_{x_0}|M_{x_0}^{\gamma^*}\right) + \mu|l_{x_0}|_{\mathbb{R}^n} \geqslant \rho^+(l_{x_0}|M_{x_0}), \tag{9}$$

where K is the number of elements of the set X_0. In its turn, inequality (8) implies the inequality

$$\sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \left(\mathcal{V}(l_{x_0};\vartheta,t_0) - \sum_{x_0\in X_0} \rho^+\left(l_{x_0}|O_\mu(M_{x_0}^{\gamma^*})\right)\right) \leqslant -\gamma^*. \tag{10}$$

Note that

$$\sum_{x_0\in X_0} \sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \mu|l_{x_0}|_{\mathbb{R}^n} \leqslant \mu\mu^* K < \gamma^*.$$

Taking into account this inequality and (9) we obtain from (10)

$$\sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \left(\mathcal{V}(l_{x_0};\vartheta,t_0) - \sum_{x_0\in X_0} \rho^+\left(l_{x_0}|M_{x_0}^{\gamma^*}\right)\right) - \sum_{x_0\in X_0} \sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \mu|l_{x_0}|_{\mathbb{R}^n} \leqslant$$

$$\leqslant \sup_{(l_{x_0})_{x_0\in X_0}\in\mathscr{L}} \left(\mathcal{V}(l_{x_0};\vartheta,t_0) - \sum_{x_0\in X_0} \rho^+\left(l_{x_0}|M_{x_0}^{\gamma^*}\right)\right) \leqslant -\gamma^* + \mu\mu^* K < 0.$$

However, the latter inequality contradicts the solvability condition of open-loop package guidance problem (4) and the definition of the set M_{x_0}. The necessity is proved. The sufficiency follows from condition (4). The theorem is proved. □

Theorem 3.2.
Let $(l^*_{x_0})_{x_0\in X_0} \in \mathscr{L}$ be a vector maximazing expression (7) and let the vector $D(\tau)\sum_{x_0\in X_{0,k}} l^*_{x_0}$ be non-zero for all $\tau \in T$. Let the zero extended open-loop control is not guiding for the extended system and, for any $k = 1,\dots,K$ and any cluster $X_{0,k} \in \mathscr{X}(\tau_k)$, the following equality holds:

$$\left(D(\tau)\sum_{x_0\in X_{0,k}} l^*_{x_0}, w_{X_{0,k}}(\tau)\right) = \varrho^-\left(D(\tau)\sum_{x_0\in X_{0,k}} l^*_{x_0}|U\right) \quad (\tau \in [\tau_{k-1},\tau_k)). \tag{11}$$

Then, the extended open-loop control $t \to \left(w_{x_0}(t)\right)_{x_0\in X_0}$ is guiding for system (3).

Proof. The correctness of Theorem 3.2 is established by analogy to [14]. □

Corollary 3.1.
The open-loop control package corresponding to the extended open-loop control and defined according to Theorem 3.2 is guiding.

4. Example.

Consider the linear controlled system of ordinary differential equations

$$\begin{aligned} \dot{x}_1 &= -2\,x_1 - x_2 + t, \\ \dot{x}_2 &= x_2 + u. \end{aligned} \tag{12}$$

Here, $x_1(t)$ and $x_2(t)$ are the coordinates of the phase vector $x(t) = (x_1(t), x_2(t))^\top$. The values of the control $u(t)$ are bounded by the interval $[-p, p]$, where $p = 0.1151$. Thus, we have the following parameters of system (1): $n = 2$, $m = 1$, and $U = [-p, p]$. The matrices A, B, and the vector $c(t)$ are of the form

$$A = \begin{pmatrix} -2 & -1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad c(t) = \begin{pmatrix} t \\ 0 \end{pmatrix}.$$

Let $[t_0, \vartheta] = [0, 2]$ and let the set of admissible initial states consist of two various elements, $X_0 = \{x', x''\}$, where

$$x' = (x_1', x_2')^\top = (-1, 0.5)^\top, \quad x'' = (x_1'', x_2'')^\top = (-2, 1)^\top.$$

We assume that, in the process of the motion, the information on the position of the system on the interval $[0,1]$ is unavailable, whereas the system's state is completely observed on the half-open interval $(1,2]$; i.e., we have the signal

$$y(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{cases} (0,0)^\top, & t \in [0,1], \\ (t-1)\,(x_1(t), x_2(t))^\top, & t \in (1,2], \end{cases}$$

which corresponds to the continuous observation matrix

$$Q(t) = \begin{cases} 0, & t \in [0,1], \\ (t-1)\,I_2, & t \in (1,2], \end{cases}$$

where $I_2 \in \mathbb{R}^{2\times 2}$ is the unit matrix. Let the quality criterion be $f(x) = 2|x_1|$. The control aim consists in forming, by available values of the signal, an open-loop control of the system and such that it provides at the time $t = 2$ the fulfillment of conditions

$$J_{x'} = 0.5, \quad J_{x''} = 3.$$

Then, the target sets are cylindrical sets

$$M_{x'} = \left\{(x_1, x_2)^\top \in \mathbb{R}^2 \colon |x_1| \leqslant m_1, x_2 \in \mathbb{R}\right\},$$

$$M_{x''} = \left\{(x_1, x_2)^\top \in \mathbb{R}^2 \colon |x_1| \leqslant m_2, x_2 \in \mathbb{R}\right\},$$

where $m_1 = 0.25$ and $m_2 = 1.5$. Let us check the validness of solvability criterion (4) of the guaranteed closed-loop control guidance. The eigenvalues of the matrix A are $\lambda_1 = -2$ and $\lambda_2 = 1$. Since system (12) is autonomous, the fundamental matrix $F(\cdot,\cdot)$ depends only on the difference between arguments and defined by the formula

$$F(t,s) = F(t-s) = \begin{pmatrix} e^{-2(t-s)} & -\frac{1}{3}\,e^{t-s} \\ 0 & e^{t-s} \end{pmatrix}.$$

The Cauchy formula for an autonomous control system is written in the form

$$x(t) = F(t)\,x(0) + \int_0^t F(t-s)\,c(s)\,ds + \int_0^t F(t-s)\,B\,u(s)\,ds.$$

The homogeneous signals corresponding to the admissible initial states x' and x'' are of the form

$$g_{x'}(t) = Q(t)\,F(t)\,x' = \begin{cases} (0,0)^\top, & t \in [0,1], \\ (t-1)\begin{pmatrix} e^{-2t}x_1' - \frac{1}{3}\,e^t x_2' \\ e^t x_2' \end{pmatrix}, & t \in (1,2], \end{cases}$$

$$g_{x''}(t) = Q(t)\,F(t)\,x'' = \begin{cases} (0,0)^\top, & t \in [0,1], \\ (t-1)\begin{pmatrix} e^{-2t}x_1'' - \frac{1}{3}\,e^t x_2'' \\ e^t x_2'' \end{pmatrix}, & t \in (1,2], \end{cases}$$

Since the initial states are different $x' \neq x''$, we have $\tau_1 = 1$ as the first splitting moment of each homogeneous signal; the second splitting moment τ_2 is the terminal time $\vartheta = 2$; the number K of the splitting moments of the homogeneous signals is 2; the cluster position $\mathscr{X}_0(1)$ at the time $t = 1$ contains the single set X_0; and the cluster position $\mathscr{X}_0(2)$ at the time $t = 2$ contains two sets, $\{x'\}$ and $\{x''\}$.

To form a guiding open-loop control package using Theorem 3.2, we need

$$D(s) = B^\top F^\top(\vartheta - s) = \left(-\frac{1}{3}\,e^{\vartheta - s}, e^{\vartheta - s}\right), \quad s \in [0,2].$$

Let $\mathscr{L}$ be the set of all families $\left(\bar{l}_1', \bar{l}_1''\right) \in \mathscr{R}_2$ $(\bar{l}_1' = (l_1', 0)^\top, \bar{l}_1'' = (l_1', 0)^\top)$ such that one of these two pairs of relations holds:

$$|l_1'| = 1,\ |l_1''| \leqslant 1 \quad \text{or} \quad |l_1'| \leqslant 1,\ |l_1''| = 1.$$

Taking into account the expressions

$$\rho^+\left((l,0)^\top | M_{x'}\right) = m_1\,|l|, \quad \rho^+\left((l,0)^\top | M_{x''}\right) = m_2\,|l|, \quad \rho^-(l|U) = -p\,|l|,$$

for the values of the function $\gamma(\cdot)$, for arbitrary real l' and l'' we obtain from formula (4) the equality

$$\gamma\left((l',0)^\top, (l'',0)^\top\right) = l'z' + l''z'' - \frac{1}{3}\,p\int_0^1 e^{2-s}|l' + l''|\,ds - \frac{1}{3}\,p\int_1^2 e^{2-s}\left(|l'| + |l''|\right)ds - m_1\,|l'| - m_2\,|l''| =$$

$$= l'z' + l''z'' - \frac{1}{3}\,p\,(e^2 - e)\,|l' + l''| - \frac{1}{3}\,p\,(e-1)\left(|l'| + |l''|\right) - m_1|l'| - m_2|l''|,$$

where $z' = e^{-4}x_1' - \frac{1}{3}\,e^2 x_2' + \frac{1}{4}(3 + e^{-4})$ and $z'' = e^{-4}x_1'' - \frac{1}{3}\,e^2 x_2'' + \frac{1}{4}(3 + e^{-4})$.

We find

$$\gamma = \max_{\bar{l}', \bar{l}'' \in \mathscr{L}} \gamma\left((l',0)', (l'',0)'\right) = 0,$$

which is reached for $l' = -1$ and $l'' = 0$. Consequently, solvability criterion (4) is valid. To form an open-loop control package using Theorem 3.2, we compute

$$\sum_{x_0 \in X_{0,0}} l_{x_0}^* = \begin{pmatrix}-1\\0\end{pmatrix} + \begin{pmatrix}0\\0\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

and

$$D(s) \sum_{x_0 \in X_{0,0}} l_{x_0}^* = -\frac{1}{3}\,e^{\vartheta - s}, \quad s \in [0,2].$$

Then, using formula (11), we set $u_{X_{0,0}}^*(s) = -p$ on the interval $[0,1]$. Similarly, we find $u_{x'}(s) = -p$ on the half-open interval $(1,2]$, but for $u_{x''}(s)$ the conditions of Theorem 3.2 do not hold because of $\sum_{x_0 \in X_{0,1}} l_{x_0}^* = 0$. In this case, we take $u_{x''}(s) = -p$. As a result, we obtain the open-loop control package

$$u_{x'}(t) = -0.1151, \quad u_{x''}(t) = -0.1151, \quad t \in [0,2],$$

which guides the system to the positions

$$x_1\left(2|x', u_{x'}(\cdot)\right) = z' - \frac{1}{3}\int_0^1 e^{t-s}u_{x',x''}(s)\,ds - \frac{1}{3}\int_1^2 e^{t-s}u_{x'}(s)\,ds = -0.25,$$

$$x_1\left(2|x'', u_{x''}(\cdot)\right) = z'' - \frac{1}{3}\int_0^1 e^{t-s}u_{x',x''}(s)\,ds - \frac{1}{3}\int_1^2 e^{t-s}u_{x''}(s)\,ds = -1.5.$$

The figures show the dependence of the first coordinate on time if the initial state is x' (Fig. 1) and x'' (Fig. 2). The part of the trajectory $(t \in [0,1])$ when we have no signal on the state of the system is given by the dashed line, and the plot when the system is observable $(t \in [1,2])$ is given by the solid line. The targets sets $M_{x'}$ and $M_{x''}$ are represented by shaded rectangles in Fig. 1 and Fig. 2, respectively.

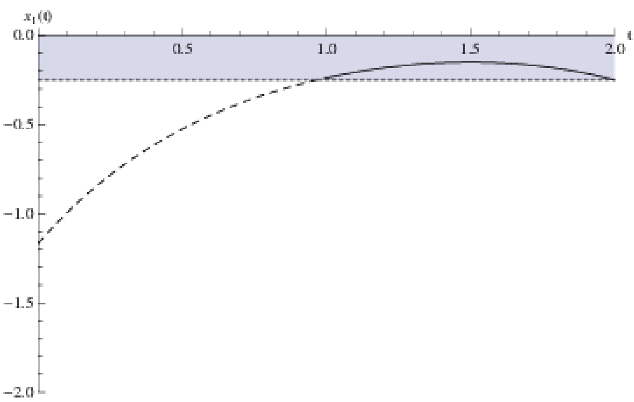

Fig. 1. The trajectory x_1 of the system from x'.

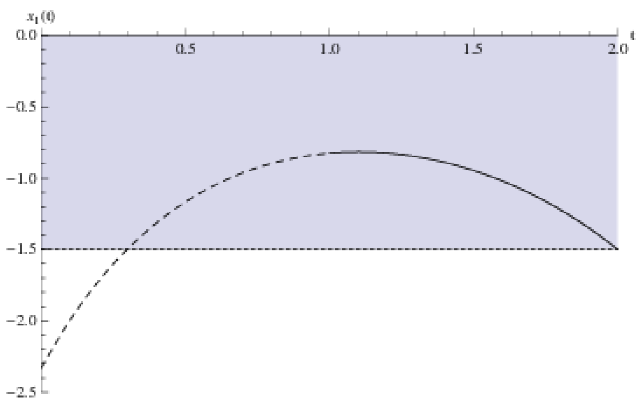

Fig. 2. The trajectory x_1 of the system from x''.

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