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The Cesàro χ^2 of Tensor products in Orlicz sequence spaces

Research Article

Vandana[a], **Deepmala**[b], **N. Subramanian**[c], **Lakshmi Narayan Mishra**[d, *]

[a] *Department of Management Studies, Indian Institute of Technology, Madras, Chennai -600 036, Tamil Nadu, India*
[b] *Mathematics Discipline, PDPM-Indian Institute of Information Technology, Design and Manufacturing, Jabalpur - 482005 (MP), India*
[c] *Department of Mathematics, SASTRA University, Thanjavur-613 401, India*
[d] *Department of Mathematics, School of Advanced Sciences, Vellore Institute of Technology (VIT) University, Vellore 632 014, Tamil Nadu, India*

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Abstract: Let X be a Banach lattice and χ_f^2 be an double gai Orlicz sequence space associated to an Orlicz function with the Δ_2- condition. In this paper we define the Cesàro χ^2 sequence space $\text{Ces}_p^q\left(\chi_f^2\right)$ generated by a Orlicz sequence space and exhibit some general properties of the spaces.

MSC: 46B42 • 46B28

Keywords: Analytic sequence • Double sequences • χ^2 space • Cesàro χ^2 • Musielak-Orlicz function • $p-$ metric space • Banach metric lattice • Positive tensor product

1. Introduction

Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich [1]. Later on it was investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy et al. ([6] -[17]), Turkmenoglu [18], Raj ([19]-[25]), Mishra et al. ([29]-[33]) and many others. Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) .$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

* Corresponding author.
E-mail address(es): vdrai1988@gmail.com (Vandana), dmrai23@gmail.com (Deepmala), nsmaths@yahoo.com (N. Subramanian), lakshminarayanmishra04@gmail.com (Lakshmi Narayan Mishra).

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m,n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x,y) = sup_{m,n}\left\{\left|x_{mn} - y_{mn}\right|^{\frac{1}{m+n}} : m,n : 1,2,3,...\right\}, \quad (1)$$

forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in\,\Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m,n \in \mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . \\ . \\ . \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $((m+n)!|x_{mn}|)^{\frac{1}{m+n}} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2.

2. Definitions and Preliminaries

Definition 2.1 (26).
An Orlicz function is a function $M : [0,\infty) \to [0,\infty)$ which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function M is replaced by $M(x+y) \le M(x) + M(y)$, then this function is called modulus function. An Orlicz function M is said to satisfy Δ_2- condition for all values u, if there exists $K > 0$ such that $M(2u) \le KM(u), u \ge 0$.

Lemma 2.1.
Let M be an Orlicz function which satisfies Δ_2- condition and let $0 < \delta < 1$. Then for each $t \ge \delta$, we have $M(t) < K\delta^{-1}M(2)$ for some constant $K > 0$.

Definition 2.2 (27).
Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \le m$. A real valued function $d_p(x_1,\dots,x_n) = \|(d_1(x_1,0),\dots,d_n(x_n,0))\|_p$ on X satisfying the following four conditions:
(i) $\|(d_1(x_1,0),\dots,d_n(x_n,0))\|_p = 0$ if and and only if $d_1(x_1,0),\dots,d_n(x_n,0)$ are linearly dependent,
(ii) $\|(d_1(x_1,0),\dots,d_n(x_n,0))\|_p$ is invariant under permutation,
(iii) $\|(\alpha d_1(x_1,0),\dots,\alpha d_n(x_n,0))\|_p = |\alpha|\,\|(d_1(x_1,0),\dots,d_n(x_n,0))\|_p, \alpha \in \mathbb{R}$
(iv) $d_p\left((x_1,y_1),(x_2,y_2)\cdots(x_n,y_n)\right) = \left(d_X(x_1,x_2,\cdots x_n)^p + d_Y(y_1,y_2,\cdots y_n)^p\right)^{1/p}$ $for 1 \le p < \infty$; (or)
(v) $d\left((x_1,y_1),(x_2,y_2),\cdots(x_n,y_n)\right) := \sup\left\{d_X(x_1,x_2,\cdots x_n), d_Y(y_1,y_2,\cdots y_n)\right\}$,
for $x_1,x_2,\cdots x_n \in X, y_1,y_2,\cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1,0),\dots,d_n(x_n,0))\|_E = sup\left(|det(d_{mn}(x_{mn},0))|\right) = sup\left(\begin{vmatrix} d_{11}(x_{11},0) & d_{12}(x_{12},0) & \dots & d_{1n}(x_{1n},0) \\ d_{21}(x_{21},0) & d_{22}(x_{22},0) & \dots & d_{2n}(x_{1n},0) \\ . \\ . \\ . \\ d_{n1}(x_{n1},0) & d_{n2}(x_{n2},0) & \dots & d_{nn}(x_{nn},0) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

Let M and Φ be mutually complementary Orlicz functions. Then, we have
(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality) [See [Kampthan\ et\ al.\ , [28]] \tag{2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u). \tag{4}$$

A sequence $M = (M_{mn})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\, u - M_{mn}(u) : u \geq 0\}, m, n = 1, 2, \cdots$$

is called the complementary function of a Musielak-Orlicz function M. For a given Musielak Orlicz function M, the Musielak-Orlicz sequence space t_f is defined by

$$t_f = \left\{x \in w^2 : I_f(|x_{mn}|)^{1/m+n} \to 0\ as\ m, n \to \infty\right\}.$$

We consider t_f equipped with the Luxemburg metric

$$d(x, y) = sup_{mn}\left\{inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1\right\}.$$

The postivity perspective, it is know that the projective tensor and the injective tensor product of two Banach lattices are, in general not Banach lattices. Throughout w, χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex double sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers.Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Definition 2.3.
Positive tensor products: For Banach lattices X and Y, let $X \otimes Y$ denote the algebraic tensor product of X and Y. For each
$u = \sum_{m=1}^{r}\sum_{n=1}^{s} x_{mn} \otimes y_{mn} \in X \otimes Y$,
define $T_u : X^* \to Y$ by $T_u(x^*) = \sum_{m=1}^{r}\sum_{n=1}^{s} x^*(x_{mn})\, y_{mn}$ for each $x^* \in X^+$. Then injective cone on $X \otimes Y$ is defined to be $C_i = \left\{u \in X \otimes Y : T_u(x^*) \in Y^* \quad \forall x^* \in X^{*+}\right\}$.

Definition 2.4.
Let $X \bar{\otimes}_i Y$ denote the completion of $X \otimes Y$ with respect $d(.,.)$. Then $X \bar{\otimes}_i Y$ with C_i as its positive cone is a Banach lattice called the positive injective tensor product of X and Y.
The positive cone on $x \otimes Y$ is defined to be
$C_p = \left\{\sum_{m=1}^{r}\sum_{n=1}^{s} x_{mn} \otimes y_{mn} : r, s \in \mathbb{N}, x_{mn} \in X^+, y_{mn} \in Y^+\right\}$.

We define the following spaces:
For a Banach metric lattice X, let
$\chi_f^2(X) = \left\{\bar{x} = (x_{ij})_{ij} \in X^{\mathbb{N}\times\mathbb{N}} : \left(x^*\left((i+j)!\,|x_{ij}|\right)^{1/i+j}\right)_{ij} \in \chi^2, \forall x^* \in X^{*+}\right\}$.
The metric defined to be

$$d(x, y) = sup\left\{\left\|\left|\left(x^*\left((i+j)!\,|x_{ij} - y_{ij}|\right)^{1/i+j}\right) : x^* \in B_{X^{*+}}\right|\right\|\right\}, x = (x_{ij})_{ij} \in \chi_f^2(X).$$

Let $\chi_f^2(X) = \left\{\bar{x} \in \chi_f^3(X) : lim_{i,j,n}\left\|\left|\left((i+j)!\,|\bar{x}_{ij}(>n)|\right)^{1/i+j}\right|\right\| \to 0 \quad as \quad i, j \to \quad \infty\right\}$, with the metric
$d(x, y) = sup\left\{\left\|\left|\left((i+j)!\,|(\bar{x}_{ij} - \bar{y}_{ij})(>n)|\right)^{1/i+j}\right|\right\| \forall (\bar{x}_{ij}) \in \chi_f^2(X^{*+})\right\}$.

3. Notations

For a vector space X, a vector $\bar{x} = (x_{ij})_{ij} \in X^{\mathbb{N}\times\mathbb{N}}$ and $n \in \mathbb{N}$, we write $\bar{x}(\leq n)$ is a two dimensional matrix from first term to nth term and remaining term zero. and $\bar{x}(> n)$ is a two dimensional matrix from first term to nth term zero and start with (n+1)th term.

If X is an ordered set, the usual order on $X^{\mathbb{N}\times\mathbb{N}}$ is defined by $\bar{x} = (x_{ij})_{ij} \geq 0 \Longleftrightarrow (x_{ij}) \geq 0$ for each $i, j \in \mathbb{N}$. for Banach lattice X, X^* denotes its dual space, B_X denotes its closed unit ball, and X^+ denotes its positive cone.

4. Some New Cesàro Orlicz sequence space of Tensor products in Musielak

The main aim of this article is to introduce the following sequence spaces and examine the topological and algebraic properties of the resulting sequence spaces. Let $f = (f_{mn})$ be a sequence Orlicz function, $\left(\bar{X}, \|(d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0))\|_p\right)$ be a $p-$ metric space, and consider $\mu_{mn}(\bar{x}) = \left\|\left(((i+j)!\left|\bar{x}_{ij}(> n)\right|\right)^{1/i+j}\right\|$ and $\eta_{mn}(\bar{x}) = \left\|\left|\bar{x}_{ij}(> n)\right|^{1/i+j}\right\|$.

We define the following sequence spaces as follows:

$\left[\chi^2_f, \|(d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0))\|_p\right] =$
$\lim_{mn\to\infty}\left\{\sum_m\sum_n\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}), (d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0))\right\|_p\right)\right] = 0\right\}$,

and

$\left[\Lambda^2_f, \|(d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0))\|_p\right] =$
$sup\left\{\sum_m\sum_n\left[f_{mn}\left(\left\|\eta_{mn}(\bar{x}), (d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0))\right\|_p\right)\right] < \infty\right\}$.

Let $p \in [1,\infty)$ and q be a double gai sequence of positive real numbers such that

$$Q_{ij} = \sum_{m=0}^{i}\sum_{n=0}^{j} q_{mn}, i, j \in \mathbb{N}$$

$Ces^q_p\left(\chi^2_f\right) = d(x,0) =$
$x \in \chi^2 := lim_{m,n\to\infty}$
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn}\left[f_{mn}\left(\|\mu_{mn}(\bar{x}), (d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0)\|_p\right)\right]^{p_{mn}}\Big)^{1/p_{mn}}$
$= 0$

If $q_{mn} = 1$ for all $m, n \in \mathbb{N}$, then $Ces^q_p\left(\chi^2_f\right)$ reduces to $Ces_p\left(\chi^2_f\right)$ and if $f(x) = x$ then $Ces^q_p\left(\chi^2_f\right)$ reduces to $Ces^q_p\left(\chi^2\right)$.

Let $p \in [1,\infty)$ and q be a double analytic sequence of positive real numbers such that

$$Q_{ij} = \sum_{m=0}^{i}\sum_{n=0}^{j} q_{mn}, i, j \in \mathbb{N}$$

$Ces^q_p\left(\Lambda^2_f\right) = d(x,0) =$
$x \in \Lambda^2 := sup$
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn}\left[f_{mn}\left(\left\|\eta_{mn}(\bar{x}), (d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{p_{mn}}\right)^{1/p_{mn}}$
$< \infty$

If $q_{mn} = 1$ for all $m, n \in \mathbb{N}$, then $Ces^q_p\left(\Lambda^2_f\right)$ reduces to $Ces_p\left(\Lambda^2_f\right)$ and if $f(x) = x$ then $Ces^q_p\left(\Lambda^2_f\right)$ reduces to $Ces^q_p\left(\Lambda^2\right)$.

The space $Ces^q_p\left(\chi^2_f\right)$ is a metric space with the metric
$d(x,y) = inf$
$sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn}$
$\left[f_{mn}\left\|\mu_{mn}(\bar{x}) - 0, (d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0)\|_p\right]^{p_{mn}}\right)^{1/p_{mn}} \leq 1$

The space $Ces^q_p\left(\Lambda^2_f\right)$ is a metric space with the metric
$d(x,y) = inf$
$sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn}$
$\left[f_{mn}\left(\left\|\eta_{mn}(\bar{x}) - 0, (d(\bar{x}_1,0), d(\bar{x}_2,0), \cdots, d(\bar{x}_{n-1},0)\|_p\right]\right)^{p_{mn}}\right)^{1/p_{mn}} \leq 1.$

5. Main Results

Proposition 5.1.

Let, the tensor produce of Orlicz sequence space $x,y \in Ces_p^q\left(\chi_f^2\right)$. Then for any $\epsilon > 0$ and $L > 0$, there exists $\delta > 0$ such that $\left(d\left(x+y,0\right),0\right)^{p_{mn}} = d\left(x,0\right)^{p_{mn}} + \epsilon$, whenever $d\left(x,0\right)^{p_{mn}} \le L$ and $d\left(y,0\right)^{p_{mn}} \le \delta$

Proof. For any fix $\epsilon > 0$

$d\left(x+y,0\right)^{p_{mn}} =$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}+\bar{y}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}}$

$\le \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}} +$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{y}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}} \le \left(1-\beta\right)$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}} +$

$\beta\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}} +$

$\beta\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\frac{\mu_{mn}(\bar{y})}{\beta},\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}}$

$\le \left(1-\beta\right)$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}} +$

$\frac{\beta}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}2q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}}$

$+\frac{\beta}{2}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}2q_{mn}\left[f_{mn}\left(\left\|\frac{\mu_{mn}(\bar{y})}{\beta},\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}}$

$\le d\left(x,0\right)^{p_{mn}} + \frac{\epsilon}{2} +$

$\left(\frac{2}{\beta}\right)^{p_{mn}-1}$

$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}}$

$\le d\left(x,0\right)^{p_{mn}} + \frac{\epsilon}{2} + \frac{\epsilon}{2}$

$\le d\left(x,0\right)^{p_{mn}} + \epsilon.$ □

Proposition 5.2.

For every $p = (p_{mn})$, $\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta} = \left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\alpha} = \left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\gamma} = \left[Ces_p^q\left(\eta_f^2\right)\right]^{\beta}$, where $\left[Ces_p^q\left(\eta_f^2\right)\right] = \bigcap_{N\in N-\{1\}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}$

$\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right)N^{m+n/p_{mn}},\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right)\right\|_p\right)\right]\right)^{p_{mn}} < \infty$

Proof. (1): First we show that $\left[Ces_p^q\left(\eta_f^2\right)\right] \subset \left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}$

Let $x \in \left[Ces_p^q\left(\eta_f^2\right)\right]$ and $y \in \left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}$. Then we can find a positive integer N such that $\left(\left|\mu\left(\bar{y}\right)\right|\right) < max$ $1, sup_{m,n\ge1}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}$ $\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p\right)\right]^{p_{mn}}\right)^{1/p_{mn}} < N$, for all m,n.

Hence we may write

$\left|\sum_{m,n}\mu_{mn}\left(\bar{x}\bar{y}\right)\right| \le \sum_{m,n}\left|\mu_{mn}\left(\bar{x}\bar{y}\right)\right| \le \sum_{m,n}\mu_{mn}\left(\bar{x}\right)N^{m+n}.$

Since $x \in Ces_p^q\left(\eta_f^2\right)$. the series on the right side of the above inequality is convergent, whence $x \in Ces_p^q\left(\Lambda_f^2\right)$. Hence $\left[Ces_p^q\left(\eta_f^2\right)\right] \subset \left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}$

Now we show that $\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta} \subset \left[Ces_p^q\left(\eta_f^2\right)\right]$

For this, let $x \in \left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}$ and suppose that $x \notin \left[Ces_p^q\left(\Lambda_f^2\right)\right]$. Then there exists a positive integer $N > 1$ such that $\mu_{mn}\left(\bar{x}\right)N^{m+n} = \infty$.

If we define $\mu_{mn}\left(\bar{y}\right) = N^{m+n}Sgn\mu_{mn}\left(\bar{x}\right)$ $m,n = 1,2,\cdots$, then $\mu_{mn}\left(\bar{y}\right) \in \left[Ces_p^q\left(\Lambda_f^2\right)\right]$.

But, since

$\left|\sum_{m,n}\mu_{mn}\left(\bar{x}\bar{y}\right)\right| = \sum_{mn}\left|\mu_{mn}\left(\bar{x}\bar{y}\right)\right| = \sum_{m,n}\left|\mu_{mn}\left(\bar{x}\right)\right|N^{m+n} = \infty$, we get $x \notin \left[Ces_p^q\left(\Lambda_f^2\right)\right]^\beta$, which contradicts to the assumption $x \in \left[Ces_p^q\left(\Lambda_f^2\right)\right]^\beta$. Therefore $x \in \left[Ces_p^q\left(\eta_f^2\right)\right]$. Therefore $\left[Ces_p^q\left(\Lambda_f^2\right)\right]^\beta = \left[Ces_p^q\left(\eta_f^2\right)\right]$.

(ii)and (iii) can be shown in a similar way of (i). Therefore we omit it. □

Proposition 5.3.
Let $p = \left(p_{mn}\right)$ be a tensor products of Cesàro space of double analytic Orlicz sequence of strictly positive real numbers p_{mn}. Then
(i) $Ces_p^q\left(\Lambda_f^2\right)$ is a paranormed space with

$$g\left(x\right) = sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p^{1/M}\right]\right) \tag{5}$$

if and only if $h = inf\, p_{mn} > 0$, where $M = max\left(1,H\right)$ and $H = supp_{mn}$.

(ii) $Ces_p^q\left(\Lambda_f^2\right)$ is a complete paranormed linear metric space if the condition p in (5.1) is satisfied.

Proof. **(i): Sufficiency:** Let $h > 0$. It is trivial that $g\left(\theta\right) = 0$ and $g\left(-x\right) = g\left(x\right)$.
The inequality $g\left(x+y\right) \le g\left(x\right) + g\left(y\right)$ follows from the inequality (5.1), since $p_{mn}/M \le 1$ for all positive integers m,n. We also may write $g\left(\lambda x\right) \le max\left(\left|\lambda\right|,\left|\lambda\right|^{h/M}\right)g\left(x\right)$, since $\left|\lambda\right|^{p_{mn}} \le max\left(\left|\lambda\right|^h,\left|\lambda\right|^M\right)$ for all positive integers m,n and for any $\lambda \in \mathbb{C}$, the set of complex numbers. Using this inequality, it can be proved that $\lambda x \to \theta$, when x is fixed and $\lambda \to 0$, or $\lambda \to 0$ and $x \to \theta$, or λ is fixed and $x \to \theta$.

Necessity: Let $Ces_p^q\left(\Lambda_f^2\right)$ be a paranormed space with the paranorm
$g\left(x\right) = sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}$
$\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p\right)\right]^{1/M}\right)$ and suppose that $h = 0$. Since $\left|\lambda\right|^{p_{mn}/M} \le \left|\lambda\right|^{h/M} = 1$ for all positive integers m,n and $\lambda \in \mathbb{C}$ such that $0 < \left|\lambda\right| \le 1$, we have
$sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}$
$\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p\right)\right]^{1/M}\right) = 1$.
Hence it follows that
$g\left(\lambda x\right) = sup\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}$
$\left[f_{mn}\left(\left\|\mu_{mn}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p\right)\right]^{1/M}\right) = 1$ for $x = \left(\alpha\right) \in Ces_p^q\left(\Lambda_f^2\right)$ as $\lambda \to 0$. But this contradicts the assumption $Ces_p^q\left(\Lambda_f^2\right)$ is a paranormed space with $g\left(x\right)$.

(ii) The proof is clear. □

Proposition 5.4.
$Ces_p^q\left(\Lambda_f^2\right)$ is a complete paranormed space with the natural paranorm if and only if $Ces_p^q\left(\Lambda_f^2\right) = Ces^q\left(\Lambda_f^2\right)$.

Proposition 5.5.
For every $p = \left(p_{mn}\right)$, then $Ces_p^q\left(\eta_f^2\right) \subset \left[Ces_p^q\left(\chi_f^2\right)\right]^\beta \underset{\neq}{\subset} Ces_p^q\left(\Lambda_f^2\right)$

Proof. **Case (i)** First we show that $Ces_p^q\left(\eta_f^2\right) \subset \left[Ces_p^q\left(\chi_f^2\right)\right]^\beta$. We know that $\left[Ces_p^q\left(\chi_f^2\right)\right] \subset Ces_p^q\left(\Lambda_f^2\right)$.
$\left[Ces_p^q\left(\Lambda_f^2\right)\right]^\beta \subset \left[Ces_p^q\left(\chi_f^2\right)\right]^\beta$. But $\left[Ces_p^q\left(\Lambda_f^2\right)\right]^\beta = Ces_p^q\left(\eta_f^2\right)$. Therefore

$$Ces_p^q\left(\eta_f^2\right) \subset \left[Ces_p^q\left(\chi_f^2\right)\right]^\beta \tag{6}$$

Case (2): Now we show that $\left[Ces_p^q\left(\chi_f^2\right)\right]^\beta \underset{\neq}{\subset} Ces_p^q\left(\Lambda_f^2\right)$.
Let $y = \mu_{mn}\left(\bar{y}\right)$ be an arbitrary point $\left[Ces_p^q\left(\chi_f^2\right)\right]^\beta$. If y is not $Ces_p^q\left(\Lambda_f^2\right)$, then for each natural number d, we can find an index $m_d n_d$ such that
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{m_dn_d}\left[f_{mn}\left(\left\|\mu_{m_dn_d}\left(\bar{x}\right),\left(d\left(\bar{x}_1,0\right),d\left(\bar{x}_2,0\right),\cdots,d\left(\bar{x}_{n-1},0\right)\right\|_p\right)\right]^{1/M}\right) > d, \left(1,2,3,\cdots\right)$. Define $x =$

$\{x_{mn}\}$ by
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{m_d n_d}$
$\left[f_{mn}\left(\|\mu_{m_d n_d}(\bar{x}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0)\|_p)\right]^{1/M}\right) > d$, for $(mn)=(m_d n_d)$ for some $d\in\mathbb{N}$; and
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{m_d n_d}$
$\left[f_{mn}\left(\|\mu_{m_d n_d}(\bar{x}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0)\|_p)\right]^{1/M}\right)=0$, *otherwise*. Then x is $Ces_p^q\left(\chi_f^2\right)$, but for infinitely mn,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}>1, \tag{7}$$

Consdier the sequence $z=\{z_{mn}\}$, where $\mu_{mn}(\bar{z})=\mu_{mn}(\bar{x})-s$ with
$s=\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}$;
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}=$
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}$
The z is a point of $Ces_p^q\left(\chi_f^2\right)$. Also
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}=0$. Hence z is in $Ces_p^q\left(\chi_f^2\right)$. But, by the equation (7),
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}$ does not converge.
$\mu_{mn}(\bar{x}\bar{y})$ diverges. Thus, the sequence $\mu_{mn}(\bar{y})$ would not be $\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta}$. This contradiction proves that

$$\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta}\subset Ces_p^q\left(\Lambda_f^2\right) \tag{8}$$

If we now choose $f=id$, where id is the identity and $\frac{1}{Q_{1j}}\left(q_{1n}\mu_{1n}(\bar{y})\right)=\frac{1}{Q_{1j}}\left(q_{1n}\mu_{1n}(\bar{x})\right)$ and $\frac{1}{Q_{1j}}\left(q_{1n}\mu_{mn}(\bar{y})\right)=\frac{1}{Q_{1j}}\left(q_{1n}\mu_{mn}(\bar{x})\right)=0\ (m,i>1)$ for all n,j, then obviously $x\in Ces_p^q\left(\chi_f^2\right)$ and $y\in Ces_p^q\left(\Lambda_f^2\right)$, but

$$\sum\sum\mu_{mn}(\bar{x}\bar{y})=\infty.\ \textit{Hence}\ \mu_{mn}(\bar{y})\notin\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta} \tag{9}$$

From (8) and (9), we are granted $\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta}\overset{\subset}{\neq}Ces_p^q\left(\Lambda_f^2\right)$. □

Proposition 5.6.
In tensor product of Orlicz sequence space of $Ces_p^q\left(\chi_f^2\right)$ weak convergence does not imply strongly convergence.

Proof. Assume that weak convergence implies strong convergence $Ces_p^q\left(\chi_f^2\right)$. Then we would have $\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta\beta}=Ces_p^q\left(\chi_f^2\right)$ [see Wilansky]. But $\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta\beta}\overset{\subset}{\neq}\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}=Ces_p^q\left(\eta_f^2\right)$. Thus $\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta\beta}\neq Ces_p^q\left(\chi_f^2\right)$. Hence tensor product of Orlicz sequence space is weak convergence does not imply strong convergence in $Ces_p^q\left(\chi_f^2\right)$.

Proposition 5.7.
Let tensor product of Orlicz sequence space of f satisfies the Δ_2- condition. Then $Ces_p^q\left(\chi^2\right)\subset Ces_p^q\left(\chi_f^2\right)$.

Proof: Let

$$x\in Ces_p^q\left(\chi^2\right) \tag{10}$$

Then $\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}$
$\le\epsilon$ for sufficiently large m,n and every $\epsilon>0$.
$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j}q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}\le f(\epsilon)$ (because the tensor product of Orlicz sequence space of f is non-decreasing)

$\Rightarrow \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M} \le Kf(\epsilon)<\epsilon$ (by the tensor product of Orlicz sequence space of Δ_2- condition, for some $K>0$ and by defining $f(\epsilon)<\frac{\epsilon}{K}$)

$$lim_{m,n\to\infty}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}=0 \quad (11)$$

Hence

$$x\in Ces_p^q\left(\chi_f^2\right). \quad (12)$$

From (10) and (12) we get $Ces_p^q\left(\chi^2\right)\subset Ces_p^q\left(\chi_f^2\right)$. □

Proposition 5.8.
$\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta} \overset{\subset}{\neq} Ces_p^q\left(\chi_f^2\right)$

Proof. Let $\mu_{mn}(\bar{x})\in\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}$

$$\sum\sum\mu_{mn}(\bar{x}\bar{y})<\infty\forall\mu_{mn}(\bar{y})\in\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta} \quad (13)$$

Assume that $\mu_{mn}(\bar{x})\notin Ces_p^q\left(\chi_f^2\right)$. Then there exist a sequence positive integers

$$\mu_{m_r n_r}(\bar{x})>\frac{1}{\left((m_r+n_r!\,2)^{(m_r+n_r)}\right)},(r=1,2,3,\cdots)$$

Take

$$\mu_{m_r n_r}(\bar{y})=(2\,(m_r+n_r)!)^{m_r+n_r}\text{ for }r=1,2,3,\cdots$$
$$\mu_{m_r n_r}(\bar{y})=0\text{ otherwise.}$$

Then $\mu_{mn}(\bar{y})\in\left[Ces_p^q\left(\Lambda_f^2\right)\right]$. But
$\sum\sum\mu_{mn}(\bar{x}\bar{y})>1+1+1+\cdots$.
We know that the infinite series 1+1+1+⋯ diverges. Hence $\sum\sum\mu_{mn}(\bar{x}\bar{y})$ diverges. Which is contradicts. Hence $\mu_{mn}(\bar{x})\in Ces_p^q\left(\chi_f^2\right)$. Therefore

$$\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}\subset Ces_p^q\left(\chi_f^2\right) \quad (14)$$

If we now choose $p=(p_{mn})$ is a constant $f=id$, where id is the identity and $\frac{1}{Q_{1j}}\left(q_{1n}\mu_{1n}(\bar{y})\right)=\frac{1}{Q_{1j}}\left(q_{1n}\mu_{1n}(\bar{x})\right)$ and $\frac{1}{Q_{1j}}\left(q_{1n}\mu_{mn}(\bar{y})\right)=\frac{1}{Q_{1j}}\left(q_{1n}\mu_{mn}(\bar{x})\right)=0\ (m,i>1)$ for all n,j, then obviously $\mu_{mn}(\bar{x})\in Ces_p^q\left(\chi_f^2\right)$ and $\mu_{mn}(\bar{y})\in Ces_p^q\left(\Lambda_f^2\right)$, but

$$\sum\sum\mu_{mn}(\bar{x}\bar{y})=\infty.\,Hence\,\mu_{mn}(\bar{y})\notin\left[Ces_p^q\left(\chi_f^2\right)\right]^{\beta} \quad (15)$$

From (14) and (15) we are granted $\left[Ces_p^q\left(\Lambda_f^2\right)\right]^{\beta}\overset{\subset}{\neq}Ces_p^q\left(\chi_f^2\right)$. □

Proposition 5.9.
Let $\left(Ces_p^q\left(\chi_f^2\right)\right)^*$ denote the dual space of $Ces_p^q\left(\chi_f^2\right)$. Then we have $\left(Ces_p^q\left(\chi_f^2\right)\right)^*=Ces_p^q\left(\Lambda_f^2\right)$.

Proof. We recall that

$$x = \Im_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\frac{1}{(m+n)!} & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}.$$

with $\frac{1}{(m+n)!}$ in the $(m,n)^{th}$ position and zero other wise, with

$x = \Im_{mn}$,

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{1}{Q_{ij}}\sum_{m=1}^{i}\sum_{n=1}^{j} q_{mn}\left[f_{mn}\left(\left\|\mu_{mn}(\bar{x}\bar{z}),(d(\bar{x}_1,0),d(\bar{x}_2,0),\cdots,d(\bar{x}_{n-1},0))\right\|_p\right)\right]^{1/M}\right) =$$

$$\begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...\left(\frac{(m+n)!}{(m+n)!}\right) & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \\ . \\ . \\ . \\ 0 & 0 & ...1^{1/m+n} & 0 & ... & 0 \\ 0 & 0 & ...0 & 0 & ... & 0 \end{pmatrix}.$$

which is a $Ces_p^q\left(\chi_f^2\right)$ sequence . Hence $\Im_{mn} \in Ces_p^q\left(\chi_f^2\right)$. Let us take $f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\mu_{mn}(\bar{x}\bar{y})$ with $\mu_{mn}(\bar{x}) \in Ces_p^q\left(\chi_f^2\right)$. and $f \in \left(Ces_p^q\left(\chi_f^2\right)\right)^*$. Take $x = \mu_{mn}(\bar{x}) = \Im_{mn} \in Ces_p^q\left(\chi_f^2\right)$. Then

$$\mu_{mn}(\bar{y}) \le \|f\|\, d(\Im_{mn},0) < \infty \text{ for each } m,n$$

Thus $\mu_{mn}(\bar{y})$ is a bounded sequence and hence an Cesàro double analytic sequence of modulus. In other words $\mu_{mn}(\bar{y}) \in Ces_p^q\left(\Lambda_f^2\right)$. Therefore $\left(Ces_p^q\left(\chi_f^2\right)\right)^* = Ces_p^q\left(\Lambda_f^2\right)$. □

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this research paper.

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