Int. J. Adv. Appl. Math. and Mech. 5(4) (2018) 43 – 48 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Fuzzy differential equation in a RL-type circuit

Research Article

Silvio A. B. Salgado *, **Leandro Ferreira**, **Danilo M. Pires**, **Felipe A. Velozo**

Institute of Applied Social Sciences, Federal University of Alfenas, CEP 37130-001, Varginha, MG, Brazil

Received 09 May 2018; accepted (in revised version) 15 August 2018

Abstract: In this work, we study the solution of a fuzzy differential equation from an electric circuit of type RL considering the initial current a fuzzy number. For this, we use the method of the fuzzy laplace transform method with the notion of strongly generalized differentiability.

MSC: 94Dxx • 03E72

Keywords: Fuzzy differential equation • RL-type circuit • Fuzzy Laplace transform

1. Introduction

The modeling of real phenomena by means of classical differential equations is almost always incomplete, since the coefficient values of the equations or initial conditions are not usually precisely known. One way to deal with these problems is to use fuzzy theory in differential equations. Fuzzy differential equations arise in several areas of knowledge, such as engineering, physics, demography, economics, among others, with the purpose of describing the dynamics of systems in which the initial condition and/or one of the parameters of the model is uncertain. The study of this class of equations has grown in recent years and represents a promising area for research. The term fuzzy differential equation was first used by [6]. Initial fuzzy value problems (FIVP) were rigorously studied by [7], [8] and [9], using an extension of the Hukuhara derivative.

Several problems of physics can be modeled by fuzzy differential equations. It is the case of the study of the dynamics of electric circuits of type RL whose initial current is uncertain. In [10], the authors present an electric circuit approach using fuzzy differential equations with two distinct notions of differentiability. The electric current is a physical quantity that needs to be measured and there are several factors that influence its measurement process, such as imprecision of the instrument used, influence of the measuring instrument on the electrical circuit, accidental errors, among others. In order to know the dynamics of the current in a circuit of the RL type, it is more precise to consider that the initial current is an information that contains uncertainty and, with this, can be modeled by a fuzzy set.

In this work, we consider a circuit of the type RL, whose initial current is modeled by a triangular fuzzy number. In order to find the solution of the (FIVP), the fuzzy Laplace transform method is used, based on the notion of Hukuhara differentiability.

* Corresponding author.
E-mail address(es): silvio.salgado@unifal-mg.edu.br (Silvio A. B. Salgado).

2. Preliminaries

The definitions of fuzzy number, the Zadeh extension principle and arithmetic for fuzzy numbers were extracted from [2].

Definition 2.1.
Let U be a topological space. A fuzzy subset A of U is characterized by a membership function $\mu_A : U \longrightarrow [0,1]$, where $\mu_A(x)$ denotes the degree that the x element belongs to the set A.

If A is a classical subset of U, its membership function is given by the characteristic function $\chi_A(x)$. We denote both the membership function and the fuzzy set by A.

Definition 2.2.
The α-level of the fuzzy subset is defined by

$$[A]^\alpha = \begin{cases} \{x \in U; A(x) \geq \alpha\} & \text{for } 0 < \alpha \leq 1 \\ \\ \overline{\{x \in U; A(x) > 0\}} & \text{for } \alpha = 0 \end{cases} \tag{1}$$

where $\overline{X}$ denotes the closure of a set X.

Definition 2.3.
A fuzzy subset A is a fuzzy number when the universe set in which μ_A is defined is the set of real numbers $\mathbb{R}$ and it meets:
i) all $\alpha-$levels of A are closed, non-empty intervals of $\mathbb{R}$;
ii) $\{x \in U; A(x) > 0\}$ is a bounded set of $\mathbb{R}$.

It is denoted $\mathbb{R}_{\mathscr{F}}$ the collection of all fuzzy numbers. Thus, the α-levels of the number fuzzy A will be denoted by $[A]^\alpha = \left[a_-^\alpha, a_+^\alpha\right]$.

Remark 2.1.
According to the Zadeh extension principle, given two fuzzy numbers A and B, the addition and multiplication per scalar operations in $\mathbb{R}_{\mathscr{F}}$ are defined, respectively, by

$$(A \oplus B)(z) = \sup_{\phi^{-1}(z)} \min\left\{A(x), B(y)\right\} \quad (k \odot A)(z) = \begin{cases} A\left(\frac{z}{k}\right), & k \neq 0 \\ \chi(0), & k = 0 \end{cases} \tag{2}$$

where $\phi^{-1}(z) = \left\{(x,y); z = x + y\right\}$.

Definition 2.4.
Let $A, B \in \mathbb{R}_{\mathscr{F}}$ the function $D_\infty : \mathbb{R}_{\mathscr{F}} \times \mathbb{R}_{\mathscr{F}} \to [0, +\infty[$ defined by

$$D_\infty(A,B) = \sup_{0 \leqslant \alpha \leqslant 1} \max\left\{| a_-^\alpha - b_-^\alpha |, | a_+^\alpha - b_+^\alpha |\right\} \tag{3}$$

is called Pompeiu-Hausdorff distance between the fuzzy numbers A and B.

Proposition 2.1 ([5]).
Let $A, B, C, D \in \mathbb{R}_{\mathscr{F}}$ and $k \in \mathbb{R}$, then

(i) $(\mathbb{R}_{\mathscr{F}}, D_\infty)$ *is a metric space;*

(ii) $D_\infty(A \oplus C, B \oplus C) = D_\infty(A,B)$;

(iii) $D_\infty(k \odot A, k \odot B) = | k | D_\infty(A,B)$;

(iv) $D_\infty(A \oplus B, C \oplus D) \leq D_\infty(A,C) + D_\infty(B,D)$.

There are several ways to define the difference between fuzzy numbers. The usual difference between fuzzy numbers, for example, is based on the difference between intervals. In this work we use the $H-$ difference call.

Definition 2.5 ([3]).
Let $A, B \in \mathbb{R}_{\mathscr{F}}$. If it exists $C \in \mathbb{R}_{\mathscr{F}}$ such that $A = B \oplus C$, then C is called the $H-$ difference of A and B, denoted by $A \ominus B$.

Remark 2.2.
It should be noted that $A \ominus B \neq A \oplus (-1) \odot B$.

Definition 2.6 ([3]).
Let $f : (a,b) \longrightarrow \mathbb{R}_{\mathscr{F}}$ and $t_0 \in (a,b)$. We say that f is strongly generalized differentiable at t_0 if there exists an element $f^{'}(t_0) \in \mathbb{R}_{\mathscr{F}}$, such that for all $h > 0$ sufficiently small, one of these conditions holds true

(i)

$$\lim_{h\searrow 0} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\searrow 0} \frac{f(t_0) \ominus f(t_0-h)}{h} = f^{'}(t_0) \tag{4}$$

(ii)

$$\lim_{h\searrow 0} \frac{f(t_0) \ominus f(t_0+h)}{-h} = \lim_{h\searrow 0} \frac{f(t_0-h) \ominus f(t_0)}{-h} = f^{'}(t_0) \tag{5}$$

(iii)

$$\lim_{h\searrow 0} \frac{f(t_0+h) \ominus f(t_0)}{h} = \lim_{h\searrow 0} \frac{f(t_0-h) \ominus f(t_0)}{-h} = f^{'}(t_0) \tag{6}$$

(iv)

$$\lim_{h\searrow 0} \frac{f(t_0) \ominus f(t_0+h)}{-h} = \lim_{h\searrow 0} \frac{f(t_0) \ominus f(t_0-h)}{h} = f^{'}(t_0) \tag{7}$$

in which all limits are calculated in the metric D_∞.

If only the condition (i) is checked, the function f is differentiable in the sense of Hukuhara.

Proposition 2.2 ([4]).
Let $f : \mathbb{R} \longrightarrow \mathbb{R}_{\mathscr{F}}$ and denote $[f(t)]^\alpha = [f_-^\alpha(t), f_+^\alpha(t)]$, for each $\alpha \in [0,1]$. Then

1. *If f is differentiable in the sense (i) then $f_-^\alpha(t)$ and $f_+^\alpha(t)$ are differentiable functions and $\left[f^{'}(t)\right]^\alpha = \left[\left(f_-^\alpha\right)^{'}(t), \left(f_+^\alpha\right)^{'}(t)\right]$;*
2. *If f is differentiable in the sense (ii) then $f_-^\alpha(t)$ and $f_+^\alpha(t)$ are differentiable functions and $\left[f^{'}(t)\right]^\alpha = \left[\left(f_+^\alpha\right)^{'}(t), \left(f_-^\alpha\right)^{'}(t)\right]$.*

Definition 2.7 ([11]).
Let $f : [a,\infty[\longrightarrow \mathbb{R}_{\mathscr{F}}$ with $[f(t)]^\alpha = \left[f_-^\alpha(t), f_+^\alpha(t)\right]$. For any fixed $\alpha \in [0,1]$, assume $f_-^\alpha(t)$ and $f_+^\alpha(t)$ are Riemann integrable on $[a,\infty[$. If

$$\left[\int_a^\infty f(t)dt\right]^\alpha = \left[\int_a^\infty f_-^\alpha(t)dt, \int_a^\alpha f_+^\alpha(t)dt\right] \tag{8}$$

for each $\alpha \in [0,1]$, then f is improper fuzzy Riemann-integrable on $[a,\infty[$ and the improper fuzzy Riemann-integral is a fuzzy number.

Proposition 2.3 ([11]).
Let $f : [0,\infty[\longrightarrow \mathbb{R}_{\mathscr{F}}$ with $[f(t)]^\alpha = \left[f_-^\alpha(t), f_+^\alpha(t)\right]$. For any fixed $\alpha \in [0,1]$ are Riemann-integrable on $[a,b]$ for every $b \geq a$ and assume there are two positive M_-^α and M_+^α such that $\int_a^b | f_-^\alpha(t) | dt \leq M_-^\alpha$ and $\int_a^b | f_+^\alpha(t) | dt \leq M_+^\alpha$, for all $b \geq a$. Then f is improper fuzzy Riemann-integrable on $[a,\infty[$ and the improper fuzzy Riemann-integral is a fuzzy number. Further more, we have

$$\left[\int_a^\infty f(t)dt\right]^\alpha = \left[\int_a^\infty f_-^\alpha(t)dt, \int_a^\alpha f_+^\alpha(t)dt\right] \tag{9}$$

for all $\alpha \in [0,1]$.

The fuzzy Laplace transform method introduced by [1] can be used to solve the linear fuzzy initial value problems.

Definition 2.8 ([1]).
Let $f : [0,\infty[\longrightarrow \mathbb{R}_{\mathscr{F}}$. If $f(t) \odot e^{-st}$ is improper fuzzy Riemann integrable on $[0,\infty[$ for each $s > 0$ and integer, then

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} f(t) \odot e^{-st} dt \tag{10}$$

is called a fuzzy Laplace transform.

Under the conditions of the definition 2.8, if $[f(t)]^\alpha = [f_-^\alpha(t), f_+^\alpha(t)]$ for $\alpha \in [0,1]$ and $t \geq 0$, then we have that

$$[\mathscr{L}\{f(t)\}]^\alpha = \left[l\{f_-^\alpha(t)\}, l\{f_+^\alpha(t)\}\right] \tag{11}$$

for all $\alpha \in [0,1]$ and $s > 0$, where l denotes the usual Laplace transform, that is, $l\{g(t)\} = \int_0^\infty g(t)e^{-st}dt$. The next propositions establish other properties of the fuzzy Laplace transform operator.

Proposition 2.4 ([1]).
Let $f, g : [0,\infty[\longrightarrow \mathbb{R}_{\mathscr{F}}$ such that its fuzzy Laplace transform exists and c_1, c_2 real constants. Then

$$\mathscr{L}\{(c_1 \odot f(t)) \oplus (c_2 \odot g(t))\} = \left(c_1 \odot \mathscr{L}\{f(t)\}\right) \oplus \left(c_2 \odot \mathscr{L}\{g(t)\}\right). \tag{12}$$

Proposition 2.5 ([1]).
Let $f, f' : [0,\infty[\longrightarrow \mathbb{R}_{\mathscr{F}}$ such that its fuzzy Laplace transform exists. Then

1) $\mathscr{L}\left\{f'(t)\right\} = \left(s \odot \mathscr{L}\{f(t)\}\right) \ominus f(0)$ iff f is differentiable in the sense (i);

2) $\mathscr{L}\left\{f'(t)\right\} = \left((-1) \odot f(0)\right) \ominus \left(s \odot \mathscr{L}\{f(t)\}\right)$ iff f is differentiable in the sense (ii).

3. Fuzzy value initial problem in a RL-type circuit

Consider an electric circuit of type RL, consisting of a voltage source $e(t)$ volts, a resistor of R ohms and an inductor of L henries. It is denoted by $i(t)$ the current, in amperes, passing through the circuit. Thus, by traversing the circuit clockwise, the dynamics of the electric current $i(t)$ is given by

$$L\frac{di}{dt} + Ri = e(t) \tag{13}$$

Considering that the electric current in $t = 0$ is modeled by a triangular fuzzy number, then we have

$$\begin{cases} \dfrac{di}{dt} + \dfrac{R}{L}i = \dfrac{e(t)}{L} \\ i(0) = A \in \mathbb{R}_{\mathscr{F}} \end{cases} \tag{14}$$

where $e, i : [0,\infty[\longrightarrow \mathbb{R}_{\mathscr{F}}$, $i(t)$ is differentiable in the sense (i), that is, differentiable Hukuhara and $\frac{R}{L}$ is a positive real number. It is denoted by $i^\alpha(t) = [i_-^\alpha(t), i_+^\alpha(t)]$ and $e^\alpha(t) = [e_-^\alpha(t), e_+^\alpha(t)]$.

Applying the fuzzy Laplace transform in both members of the equation given by (14) and, using Proposition (2.5), we have to

$$(s \odot \mathscr{L}\{i(t)\}) \ominus i(0) \oplus \left(\frac{R}{L} \odot \mathscr{L}\{i(t)\}\right) = \left(\frac{1}{L} \odot \mathscr{L}\{e(t)\}\right) \tag{15}$$

Equivalently

$$\begin{cases} l\{i_-^\alpha(t)\} = \dfrac{L}{sL+R} i_-^\alpha(t) + \dfrac{1}{sL+R} l\{e_-^\alpha(t)\} \\[2ex] l\{i_+^\alpha(t)\} = \dfrac{L}{sL+R} i_+^\alpha(t) + \dfrac{1}{sL+R} l\{e_+^\alpha(t)\} \end{cases} \tag{16}$$

Hence solution of system (16) is as follows:

$$\begin{cases} i_-^\alpha(t) = e^{-\frac{R}{L}t} i_-^\alpha(0) + \dfrac{1}{L} e^{-\frac{R}{L}t} * e_-^\alpha(t) \\ i_+^\alpha(t) = e^{-\frac{R}{L}t} i_+^\alpha(0) + \dfrac{1}{L} e^{-\frac{R}{L}t} * e_+^\alpha(t) \end{cases} \tag{17}$$

that is

$$i^\alpha(t) = \left[e^{-\frac{R}{L}t} i_-^\alpha(0) + \frac{1}{L} e^{-\frac{R}{L}t} * e_-^\alpha(t), e^{-\frac{R}{L}t} i_+^\alpha(0) + \frac{1}{L} e^{-\frac{R}{L}t} * e_+^\alpha(t) \right] \tag{18}$$

The expression (18) satisfies the fuzzy differential equation (14), in which the sum $\frac{di}{dt} + \frac{R}{L} i = \frac{e(t)}{L}$ represents the sum of Minkowski, that is, if $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are closed and limited intervals of the line, the sum of Minkowski and defined by $A + B = [a_1 + b_1, a_2 + b_2]$. The solution (18) represents an application with fuzzy values, that is, for each $t \in [0, \infty[$ associates a single fuzzy set, but precisely, a single fuzzy number. The Fig. 1 illustrates the geometric behavior of the problem solution (14), with a scale between 0 and 1 representing the membership degrees, considering A a triangular fuzzy number whose $\alpha-$ levels are given by $[A]^\alpha = [2+\alpha, 4-\alpha]$, $\frac{R}{L} = 0.01$ and $e(t)$ identically null. It can be seen that the darkest region represents the values of $i(t)$ whose membership is maximum at a given instant $t > 0$, that is, the classical solution. It is noticed that in the initial instants, the uncertainty has a greater influence in the behavior of the electric current and that, as t grows, such uncertainty becomes smaller and smaller. Fixed $\alpha = \alpha_0$, for a given instant $t > 0$, there are different intensities of electric current with different membership degrees, that is

$$\left[e^{-\frac{R}{L}t} i_-^{\alpha_0}(0) + \frac{1}{L} e^{-\frac{R}{L}t} * e_-^{\alpha_0}(t), e^{-\frac{R}{L}t} i_+^{\alpha_0}(0) + \frac{1}{L} e^{-\frac{R}{L}t} * e_+^{\alpha_0}(t) \right] \tag{19}$$

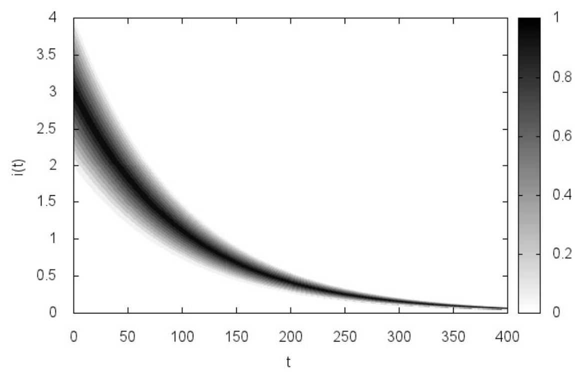

Fig. 1. Geometric behavior of the solution (18), with $e(t)$ identically null, $\frac{R}{L} = 0.01$ and $[A]^\alpha = [2+\alpha, 4-\alpha]$.

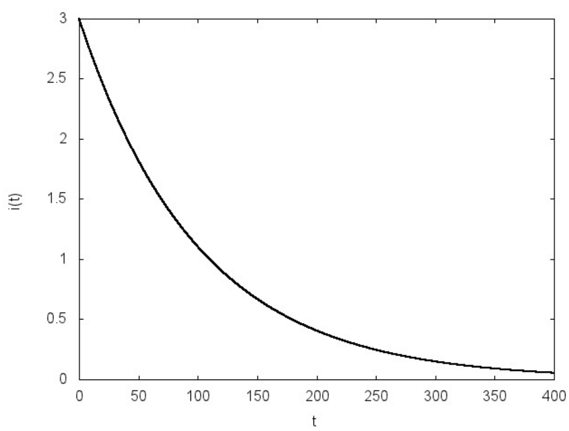

Fig. 2. Geometric behavior of the solution (18), with $e(t)$ identically null, $\frac{R}{L} = 0.01$ and $[A]^\alpha = [2+\alpha, 4-\alpha]$.

It is easy see that if the initial condition is real number, that is, $i_-^\alpha(t) = i_+^\alpha(t) = i_0$ and $e_-^\alpha(t) = e_+^\alpha(t) = e(t)$ for all $\alpha \in [0,1]$, then the solution (18) becomes the classical solution of RL type circuit

$$i(t) = i_0 e^{-\frac{R}{L}t} + \frac{e^{-\frac{R}{L}}t}{L} \int_0^t e^{\frac{R}{L}\xi} e(\xi) d\xi \tag{20}$$

Moreover, this solution is preferred in the sense that, for each $t > 0$, it belongs to fuzzy solution (18) with membership degree equal to 1. The deterministic solution (20) is depicted in Fig. 2, for $i_0 = 3$, $e(t) = 0$ and $\frac{R}{L} = 0.01$.

[0] *The symbol $*$ denotes the usual convolution product, that is, given $f_1(t)$ and $f_2(t)$, then $(f_1 * f_2)(t) = \int_0^t f_1(\xi) f_2(t-\xi) d\xi$.*

4. Conclusions

It is common to use differential equations to describe the temporal evolution of certain phenomena such as in demography, stock pricing, and so on, even those with oscillatory behavior. In this work we studied the differential equation $\frac{di}{dt}+\frac{R}{L}i=\frac{e(t)}{L}$ which models the electric current in an electric circuit of type RL. Assuming that the phenomena present uncertainties in its variable state, we treated $i(t)$ by means of the fuzzy set theory. We investigated the dynamics of the electric current $i(t)$ in a circuit of the type RL, considering that $i(0)$ is modeled by a triangular fuzzy number. The (FIVP) was solved using the fuzzy Laplace transform method with the notion of Hukuhara differentiability. The fuzzy solution found contains the solution of the classical model.

References

[1] T. Allahviranloo, M. B. Ahmadi, Fuzzy Laplace Transform, Soft Comput, Springer, 14(2010) 235–243.

[2] L.C. Barros, R.C. Bassanezi, W.A.Lodwick, A First Course in Fuzzy Logic, Fuzzy Dynamical Systems, and Biomathematics, Springer-Verlag Berlin Heidelberg, 2017.

[3] B. Bede, S.G. Gal, Almost periodic fuzzy number value functions, Fuzzy Sets and Systems, Elsevier, 147 (2004) 385–403.

[4] Y. Chalco-Cano, H. Roman Flores, On new solutions of fuzzy differential equations, Chaos Solitons Fractals, 38 (2006) 112–119.

[5] P. Dimond, P. Kloedem, Metric topology of fuzzy numbers and fuzzy analysis, In: Handbook fuzzy sets, (7)(2000) 583–641.

[6] A. Kandel, W. Byatt, Fuzzy differential equations. In: Proceedings of the International Conference on Cybernetics and Society New York, 1978.

[7] O. Kaleva, Fuzzy differential equations, Fuzzy Sets and Systems, (24)(1987) 301–317.

[8] O. Kaleva, The cauchy problem for fuzzy differential equations, Fuzzy Sets and Systems, (35)(1990) 389–396.

[9] S. Seikkala, On the fuzzy initial value problem, Fuzzy Sets and Systems, (24)(1987) 319–330.

[10] H.S.Yazdi, M. Pakdaman, S. Effati, Fuzzy circuit analysis, International journal of applied engineering research, (3)(8)(2008) 1061–1071.

[11] H. S. Wu, The improper fuzzy Riemann integral and its numerical integration, Information Sciences, (111)(1998) 109–137.