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Exact solution of fractionalized MHD viscoelastic fluid

Research Article

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Abstract: This paper presents an exact solution for the MHD flow of an incompressible Maxwell fluid due to constantly moving plate. The fractional calculus approach is used in the governing equations. Exact analytic solutions are obtained for the velocity field and shear stress by means of Fourier sine and Laplace transforms. The general solutions present as sum between Newtonian and non-Newtonian contributions and written in terms of the generalized G functions, satisfy all imposed initial and boundary conditions. Finally, the influence of the material parameters on the fluid motion, as well as a comparison between models, is shown by graphical illustrations.

MSC: 76A05 • 76A10

Keywords: Exact solutions • MHD Maxwell fluid • Fractional derivative • Unsteady flow • Velocity field • Shear stress • Laplace • Fourier transforms

1. Introduction

It is an established fact that the flow characteristics of non-Newtonian fluids are quite different when compared with the viscous fluids. Therefore, the well known Navie-Stokes equations are inappropriate for the non-Newtonian fluids. It has generally been recognized that in technological applications non-Newtonian fluids are more appropriate than Newtonian fluids. This is perhaps due to their demands in biorheology, geophysics, chemical and petroleum industries [1]. Because of the difficulty to suggest a single model which exhibits all properties of non-Newtonian fluids, they cannot be described as simply as Newtonian fluids [2]. Hence there are numerous models of non-Newtonian fluids proposed in the literature. The computation of flow of these fluids is at best a complicated affair. Therefore, in order to get some insight into their flow behavior, it is best to focus on a model with a minimum number of parameters in the constitutive equations [3]. There is a subclass of the rate type fluids namely the Maxwell fluids in which the relaxation phenomena can be taken into account. The upper-convected Maxwell model is more realistic viscoelastic fluid models for industrial significance. Specifically the Maxwell fluid model has been used for the viscoelastic flows where the dimensionless relaxation time is small. However in some more concentrated polymeric fluids the Maxwell model is also useful for large dimensionless relaxation time. Some recent investigations dealing with the flows of Maxwell fluids are given in the references [4–10].

In recent years, fractional calculus has encountered much success in the description of complex dynamics and in the field of biorheology, in part, because many tissue-like materials (polymers, gels, emulsions, composites and suspensions) exhibit power-law responses to an applied stress or strain [11, 12]. An example of such power-law behavior

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in elastic tissue was observed recently for viscoelastic measurements of the aorta, both in vivo and in vitro [13, 14], and the analysis of these data was most conveniently performed using fractional order viscoelastic models. The starting point is usually the classical differential equation, which is modified by replacing the classical, time-derivatives of an integer-order, by the so-called the Riemann-Liouville or Caputo differential integral operators. This generalization allows one to define precisely non-integer order integrals or derivatives. Bagley [15], Friedrich [16], Huang Junqi [17], He Guangyu [18], Xu [19, 20] and Tan [21–24] have sequentially introduced the fractional calculus approach into various rheology problems. Fractional derivative constitutive equations have been found to be quite flexible, in describing linear-viscoelastic behavior of polymers from glass-transition to the main or relaxation in the glassy state. Recently, fractional calculus has encountered much success in the description of viscoelasticity [25, 26]. Some recent work on this subject are mention in [27–35].

MHD plays an important role in agriculture, petroleum industries, geophysics and in astrophysics. It has important applications in the study of geological formations, in exploration and thermal recovery of oil, and in the assessment of aquifers, geothermal reservoirs and underground nuclear waste storage sites. MHD flow has application in metrology, solar physics and in motion of earthÂŠs core. In the field of power generation, MHD is receiving considerable attention due to the possibilities it offers for much higher thermal efficiencies in power plants. In the present study, we are going to investigate MHD flow of an upper-convected Maxwell fluid with fractional derivative above an infinite plate. The flow is induced by an infinite plate that slides in its plane with constant velocity. Exact solutions are derived by Fourier sine and Laplace transforms. The expression for velocity field and shear stress are written as sum between Newtonian and non-Newtonian contributions in term of generalized $G_{a,b,c}(\bullet, t)$ functions satisfy all imposed initial and boundary conditions. The solutions for ordinary MHD Maxwell and hydrodynamic fluids, can also be obtained as special cases from present general solutions. Finally the influence of pertinent parameters on fluid motion as well comparison among three different models, in the presence and absence of magnetic effect is discuses through graphical illustrations.

2. Governing equations of fractionalized MHD Maxwell fluid

The equations governing the MHD flow of an incompressible fluid include the continuity equation and the momentum equation. In the absence of body forces, they are

$$\nabla \cdot \mathbf{V} = 0, \quad \nabla \cdot \mathbf{T} = \rho \frac{\partial \mathbf{V}}{\partial t} + \rho\big(\mathbf{V} \cdot \nabla\big)\mathbf{V} + \sigma B_0^2 \mathbf{V}, \tag{1}$$

where the body forces are absent, ρ is the fluid density, $\mathbf{V}$ is the velocity field, σ is the electrical conductivity of the fluid, B_0 is the applied magnetic field, t is the time and ∇ represents the gradient operator. Note that the fluid is electrically conducting in the presence of a uniform magnetic field and the induced magnetic field is neglected for small magnetic Reynolds number assumption. No electric field is applied. The Cauchy stress $\mathbf{T}$ in an incompressible Maxwell fluid is given by [4–10].

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}, \quad \mathbf{S} + \Lambda(\dot{\mathbf{S}} - \mathbf{LS} - \mathbf{SL}^T) = \mu \mathbf{A}, \tag{2}$$

where $-p\mathbf{I}$ denotes the indeterminate spherical stress due to the constraint of incompressibility, $\mathbf{S}$ is the extra-stress tensor, $\mathbf{L}$ is the velocity gradient, $\mathbf{A} = \mathbf{L} + \mathbf{L}^T$ is the first Rivlin Ericksen tensor, μ is the dynamic viscosity of the fluid, Λ is relaxation time, the superscript T indicates the transpose operation and the superposed dot indicates the material time derivative. The model characterized by the constitutive Eq. (2) contains as special case the Newtonian fluid model for $\Lambda \to 0$.

For the problem under consideration we assume a velocity field $\mathbf{V}$ and an extra-stress tensor $\mathbf{S}$ of the form

$$\mathbf{V} = \mathbf{V}(y, t) = u(y, t)\mathbf{i}, \quad \mathbf{S} = \mathbf{S}(y, t), \tag{3}$$

where $\mathbf{i}$ is the unit vector along the x-coordinate direction. For these flows the constraint of incompressibility is automatically satisfied. If the fluid is at rest up to the moment $t = 0$, then

$$\mathbf{V}(y, 0) = \mathbf{0}, \qquad \mathbf{S}(y, 0) = \mathbf{0}, \tag{4}$$

and Eqs. (2)-(4) imply $S_{yy} = S_{yz} = S_{zz} = S_{xz} = 0$, and

$$\Big(1 + \Lambda \frac{\partial}{\partial t}\Big)\tau(y, t) = \mu \frac{\partial u(y, t)}{\partial y}, \tag{5}$$

where $\tau(y, t) = S_{xy}(y, t)$ are the non-zero shear stresses. In the absence of body forces, the balance of linear momentum reduces to

$$\frac{\partial \tau(y, t)}{\partial y} - \frac{\partial p}{\partial x} = \rho \frac{\partial u(y, t)}{\partial t} + \sigma B_0^2 u(y, t), \quad -\frac{\partial p}{\partial y} = \sigma B_0^2 v(y, t), \quad -\frac{\partial p}{\partial z} = \sigma B_0^2 w(y, t). \tag{6}$$

Eliminating τ between Eqs. (5) and (6), we find the governing equation under the form

$$\left(1+\Lambda\frac{\partial}{\partial t}\right)\frac{\partial u(y,t)}{\partial t}=-\frac{1}{\rho}\left(1+\Lambda\frac{\partial}{\partial t}\right)\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u(y,t)}{\partial y^2}-\frac{\sigma B_0^2}{\rho}\left(1+\Lambda\frac{\partial}{\partial t}\right)u(y,t);\quad y,t>0, \tag{7}$$

where $\nu=\mu/\rho$ is the kinematic viscosity of the fluid.

The governing equations corresponding to an incompressible MHD Maxwell fluid with fractional derivatives, performing the same motion in the absence of pressure gradient, are (cf. [34, 35])

$$\left(1+\Lambda^\beta D_t^\beta\right)\frac{\partial u(y,t)}{\partial t}=\nu\frac{\partial^2 u(y,t)}{\partial y^2}-B\left(1+\Lambda^\beta D_t^\beta\right)u(y,t),\ \left(1+\Lambda^\beta D_t^\beta\right)\tau(y,t)=\mu\frac{\partial u(y,t)}{\partial y}, \tag{8}$$

where $B=\sigma B_0^2/\rho$ and β is fractional parameter, and the fractional differential operator so called Caputo fractional operator D_t^β defined by [36, 37]

$$D_t^\beta f(t)=\begin{cases}\dfrac{1}{\Gamma(1-\beta)}\displaystyle\int_0^t\frac{f'(\tau)}{(t-\tau)^\beta}\,d\tau, & 0<\beta<1;\\[2mm] \dfrac{df(t)}{dt}, & \beta=1,\end{cases} \tag{9}$$

and $\Gamma(\bullet)$ is the Gamma function.

3. Formation of initial & boundary value Problem

Consider an incompressible fractionalized MHD Maxwell fluid occupying the space lying over an infinitely extended plate which is situated in the (x,z) plane and perpendicular to the y-axis. Initially, the fluid is at rest and at the moment $t=0^+$ the plate is impulsively brought to the constant velocity U in its own plane. Due to the shear, the fluid above the plate is gradually moved. Its velocity is of the form $(3)_1$ while the governing equations are given by Eq. (8). The appropriate initial and boundary conditions are [38]

$$u(y,0)=\frac{\partial u(y,0)}{\partial t}=0;\quad \tau(y,0)=0,\quad y>0, \tag{10}$$

$$u(0,t)=UH(t);\quad t\geq 0, \tag{11}$$

where $H(t)$ is the Heaviside function. Moreover, the natural conditions

$$u(y,t),\ \frac{\partial u(y,t)}{\partial y}\to 0 \text{ as } y\to\infty \text{ and } t>0, \tag{12}$$

have to be also satisfied. They are consequences of the fact that the fluid is at rest at infinity and there is no shear in the free stream. In the following the system of fractional partial differential Eq. (8), with appropriate initial and boundary conditions, will be solved by means of Laplace and finite Hankel transforms. In order to avoid lengthy calculations of residues and contour integrals, the discrete inverse Laplace transform method will be used [27–35].

4. Solution of the problem

4.1. Calculation of the velocity field

In order to determine the exact solution, we shall use the Fourier sine transforms [39]. Multiplying both sides of Equation $(8)_1$ by $\sqrt{2/\pi}\sin(y\xi)$, integrating the result with respect to y from 0 to infinity and using $(A1)_1$, and taking into account the boundary conditions (11), we find that

$$\left(1+\Lambda^\beta D_t^\beta\right)\frac{\partial u_S}{\partial t}+\left[\nu\xi^2+B\left(1+\Lambda D_t^\beta\right)\right]u_S=U\xi\sqrt{\frac{2}{\pi}}H(t);\quad \xi,t>0, \tag{13}$$

where the Fourier sine transform $u_S(\xi,t)$ of $u(y,t)$ defined by [39].

$$u_S(\xi,t)=\sqrt{\frac{2}{\pi}}\int_0^\infty u(y,t)\sin(y\xi)\,dy, \tag{14}$$

has to satisfy the initial conditions

$$u_S(\xi,0)=\frac{\partial u_S(\xi,0)}{\partial t}=0;\quad \xi>0, \tag{15}$$

Applying the Laplce transform to Eq. (13), using $(A1)_2$, the Laplce transform formula for sequential fractional derivatives [36, 37] and taking into account the initial conditions (15), we find that

$$\overline{u}_S(\xi,s)=U\xi\sqrt{\frac{2}{\pi}}\frac{\nu}{s\left[s+\Lambda^\beta s^{\beta+1}+\nu\xi^2+B(1+\Lambda^\beta s^\beta)\right]}. \tag{16}$$

In order to obtain the Laplace transform of velocity field $u_S(\xi,t)=\mathscr{L}^{-1}\{\overline{u}(\xi,q)\}$, and to avoid the lengthy and burdensome calculations of residues and contours integrals, we apply the discrete inverse Laplace transform method [27–35]. However, for suitable presentation of the velocity field, we firstly rewrite Eq. (16) in the equivalent form

$$\overline{u}_S(\xi,q)=U\sqrt{\frac{2}{\pi}}\frac{\xi}{\xi^2+B}\left[\frac{1}{q}-\frac{1}{q+B+\nu\xi^2}\right]-\nu U\xi\sqrt{\frac{2}{\pi}}\frac{q}{q+B+\nu\xi^2}\Theta_1(\xi,q), \tag{17}$$

in which

$$\Theta_1(\xi,q)=\frac{\Lambda^\beta q^{\beta-1}+B\Lambda^\beta q^{\beta-2}}{q+\Lambda^\beta q^{\beta+1}+\nu\xi^2+B(1+\Lambda^\beta q^\beta)}.$$

The function $\Theta_1(\xi,q)$ can be written as a double series (see also (A2))

$$\Theta_1(\xi,q)=\sum_{k=0}^{\infty}\frac{(-1)^k(\Lambda^\beta q^{\beta-1}+B\Lambda^\beta q^{\beta-2})\left[\nu\xi^2+B(1+\Lambda^\beta q^\beta)\right]^k}{\left[q+\Lambda^\beta q^{\beta+1}\right]^{k+1}}$$

$$=\sum_{k=0}^{\infty}\sum_{i,j,l\geq 0}^{i+j+l=k}\left(-\frac{1}{\Lambda^\beta}\right)^k\frac{k!}{i!j!l!}(\nu\xi^2)^i\Lambda^{\beta l}B^{j+l}\frac{q^{\beta(1+l)-k-2}+Bq^{\beta(1+l)-k-3}}{(q^\beta+\Lambda^{-\beta})^{k+1}}. \tag{18}$$

Inverting Eq. (17) by means of the Fourier sine formula [39] and using Eq. (A3), we find that

$$\overline{u}(y,q)=\frac{Ue^{-y\sqrt{B/\nu}}}{q}-\frac{2U}{\pi}\int_0^\infty\frac{\xi\sin(y\xi)}{\xi^2+B}\frac{1}{q+B+\nu\xi^2}-\frac{2\nu U}{\pi}\int_0^\infty\frac{q\xi\sin(y\xi)}{q+B+\nu\xi^2}\Theta_1(\xi,q). \tag{19}$$

Introducing (18) into (19), inverting the result by means of the discrete inverse Laplace transform and using (A4)) and (A5), we find the velocity field under form

$$u(y,t)=u_N(y,t)-\frac{2\nu UH(t)}{\pi}\int_0^\infty\xi\sin(y\xi)\theta_1(\xi,t)\,d\xi$$

$$+\frac{2\nu U}{\pi}\int_0^\infty\int_0^t\xi(\nu\xi^2+B)\sin(y\xi)\theta_1(\xi,s)e^{-(\nu\xi^2+B)(t-s)}\,ds\,d\xi, \tag{20}$$

where

$$u_N(y,t)=UH(t)e^{-y\sqrt{B/\nu}}-\frac{2UH(t)}{\sqrt{\pi}}\int_0^{y/2\sqrt{\nu t}}\exp\left(-x^2-\frac{By^2}{4\nu x^2}\right)dx, \tag{21}$$

is the velocity field corresponding to a Newtonian fluid in the magnetic field,

$$\theta_1(\xi,t)=\sum_{k=0}^{\infty}\sum_{i,j,l\geq 0}^{i+j+l=k}\left(-\frac{1}{\Lambda^\beta}\right)^k\frac{k!}{i!j!l!}(\nu\xi^2)^i\Lambda^{\beta l}B^{j+l}\Big\{G_{\beta,\beta(l+1)-k-2,k+1}\left(\frac{-1}{\Lambda^\beta},t\right)$$

$$+BG_{\beta,\beta(l+1)-k-3,k+1}\left(\frac{-1}{\Lambda^\beta},t\right)\Big\}, \tag{22}$$

is the inverse Laplace transform of $F_1(\xi,q)$ and the generalized $G_{a,b,c}(d,t)$ function is defined by [40]

$$G_{a,b,c}(d,t)=\sum_{j=0}^{\infty}\frac{d^j\Gamma(c+j)}{\Gamma(c)\Gamma(j+1)}\frac{t^{(c+j)a-b-1}}{\Gamma[(c+j)a-b]}. \tag{23}$$

4.2. Calculation of the shear stress

Applying the Laplace transform to Eq. (8)$_2$ and using the initial condition (10)$_3$, we find that

$$\overline{\tau}(y,q)=\mu\frac{1}{1+\Lambda^\beta q^\beta}\frac{\partial\overline{u}(y,q)}{\partial y},\tag{24}$$

where $\overline{\tau}(y,q)$ is the Laplace transform of $\tau(y,t)$. In order to get $\partial\overline{u}(y,q)/\partial y$ we apply the inverse Fourier sine transforms to Eq. (16) and find that

$$\overline{u}(\xi,q)=\frac{2U}{\pi}\int_0^\infty\frac{\nu\xi\sin(y\xi)}{q[q+\Lambda^\beta q^{\beta+1}+\nu\xi^2+B(1+\Lambda^\beta q^\beta)]}d\xi,\tag{25}$$

from which

$$\frac{\partial\overline{u}(y,q)}{\partial y}=\frac{2U}{\pi}\int_0^\infty\frac{\nu\xi^2\cos(y\xi)}{q[q+\Lambda^\beta q^{\beta+1}+B(1+\Lambda^\beta q^\beta)]}d\xi.\tag{26}$$

Introducing (26) into (24), we can write $\overline{\tau}(y,t)$ under suitable form

$$\overline{\tau}(y,q)=\frac{2\nu\mu U}{\pi}\int_0^\infty\xi^2\cos(y\xi)\Big[\frac{1}{\nu\xi^2+B}\frac{1}{q}-\frac{1}{\nu\xi^2+B}\frac{1}{q+\nu\xi^2+B}\Big]d\xi$$

$$-\frac{2\nu\mu U}{\pi}\int_0^\infty\xi^2\cos(y\xi)\Theta_2(\xi,q)\Theta_3(\xi,q)d\xi,\tag{27}$$

where

$$\Theta_2(\xi,q)=\frac{q^{\beta-1}}{\Lambda^\beta(q^\beta+\Lambda^{-\beta})}\frac{1}{q+(\nu\xi^2+B)},$$

$$\Theta_3(\xi,q)=\frac{\Lambda^\beta(\nu\xi^2+2B)+2\Lambda^\beta q+\Lambda^{2\beta}q^{\beta+1}+B\Lambda^{2\beta}q^\beta}{q+\Lambda^\beta q^{\beta+1}+\nu\xi^2+B(1+\Lambda^\beta q^\beta)}.$$

Inverting Eq. (27) and using (A4) and (A5), as well as the discrete inverse Laplace transform, we find the shear stress $\tau_1(y,t)$ under simple the form

$$\tau(y,t)=\tau_N(y,t)-\frac{2\nu\mu UH(t)}{\pi}\int_0^\infty\int_0^t\xi^2\cos(y\xi)\theta_2(t-s)\theta_3(\xi,s)ds\,d\xi,\tag{28}$$

where the inverse Laplace transform $\theta_2(t)$ and $\theta_3(\xi,t)$ of $\Theta_2(q)$ and $\Theta_3(\xi,q)$ are given by

$$\tau_N(y,t)=-\mu UB_0\sqrt{\frac{\sigma}{\mu}}e^{-y\sqrt{B/\nu}}-\frac{\mu U}{\sqrt{\pi\nu t}}\exp\Big(-\frac{y^2}{4\nu t}-Bt\Big)$$

$$-\frac{2U}{\sqrt{\pi}}\int_0^{\frac{y}{2\sqrt{\nu t}}}\Big(-\frac{By}{2\nu x^2}\Big)\exp\Big\{-x^2-\Big(\frac{By}{2\nu x}\Big)^2\Big\}dx,\tag{29}$$

$$\theta_2(t)=\frac{1}{\Lambda^\beta}\int_0^t G_{\beta,\beta-1,1}\Big(-\frac{1}{\Lambda^\beta},t-s\Big)e^{-(\nu\xi^2+B)s}ds,\tag{30}$$

respectively

$$\theta_3(\xi,t)=\sum_{k=0}^{\infty}\sum_{i,j,l\geq0}^{i+j+l=k}\Big(-\frac{1}{\Lambda^\beta}\Big)^k\frac{k!}{i!j!l!}(\nu\xi^2)^i\Lambda^{\beta l}B^{j+l}\Big\{(\nu\xi^2+B)G_{\beta,\beta l-k-1,k+1}\Big(\frac{-1}{\Lambda^\beta},t\Big)$$

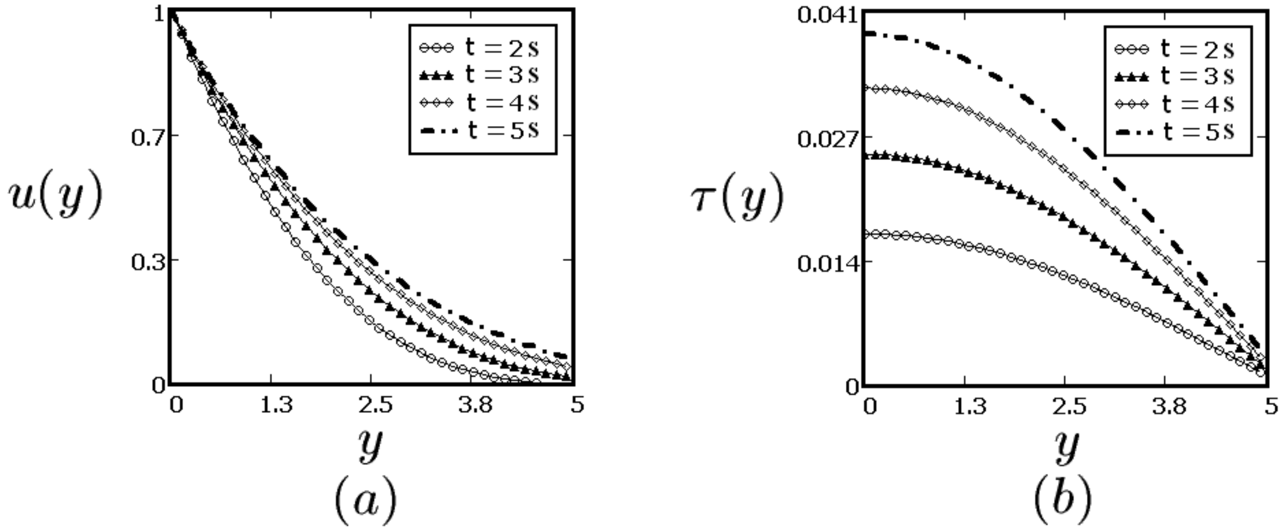
$$+2G_{\beta,\beta l-k,k+1}\Big(\frac{-1}{\Lambda^\beta},t\Big)+\Lambda^\beta G_{\beta,\beta(l+1)-k,k+1}\Big(\frac{-1}{\Lambda^\beta},t\Big)+B\Lambda^\beta G_{\beta,\beta(l+1)-k-1,k+1}\Big(\frac{-1}{\Lambda^\beta},t\Big)\Big\}.\tag{31}$$

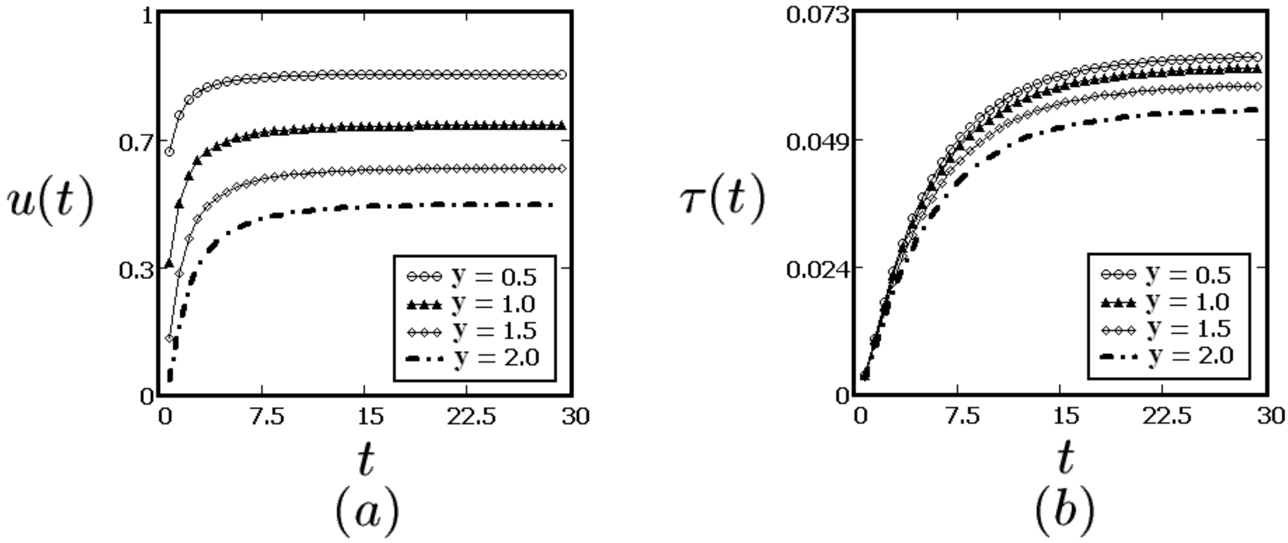
5. Special cases

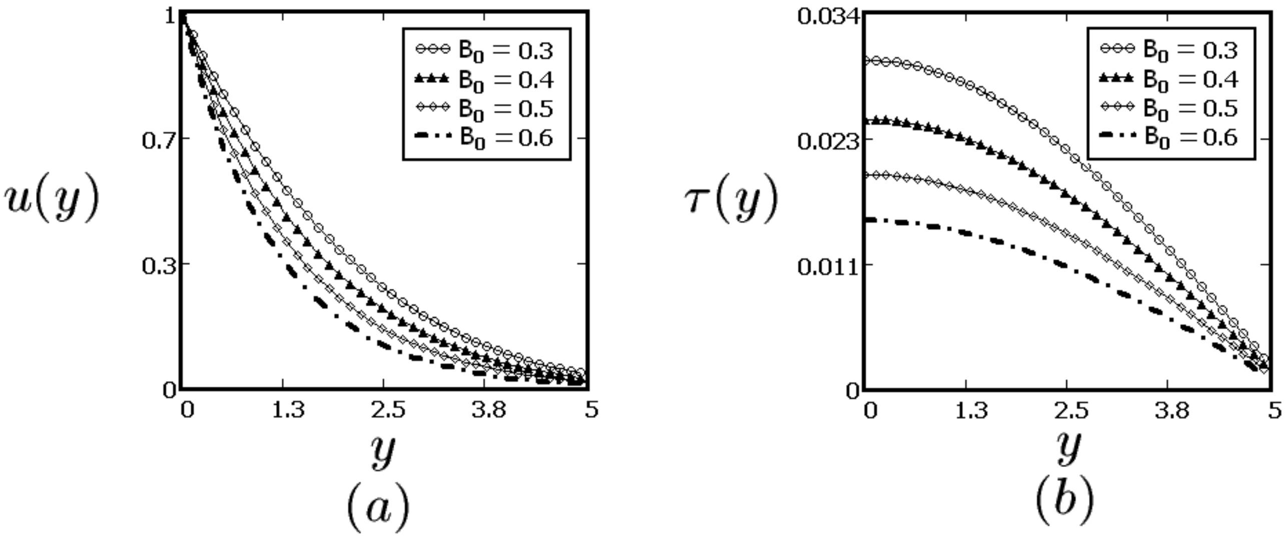
The similar solution for ordinary MHD Maxwell fluids are also obtained making $\beta\to1$ in Eqs. (20), (22), (28), (30) and (31).

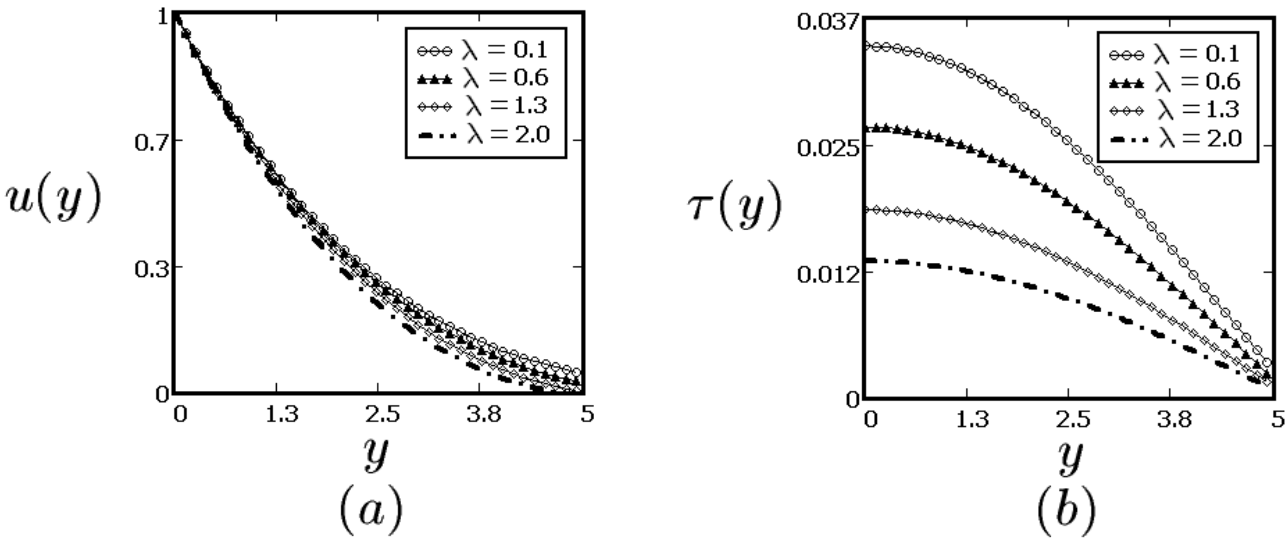
Finally, making B_0 or $B\to0$ in Eqs. (20), (22), (28), (30) and (31), the solutions corresponding to the hydrodynamic flow of ordinary as well as fractionalized Maxwell fluid are obtained.

6. Numerical results and discussion





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λ = 1.3
λ = 2.0
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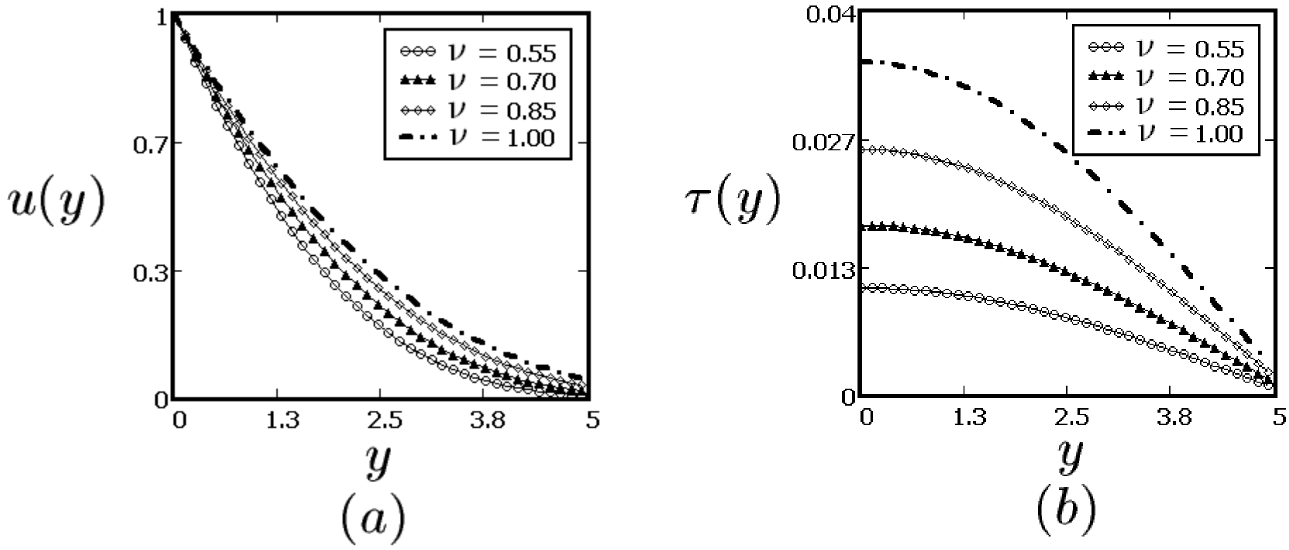
τ(y)
y
(a)
(b)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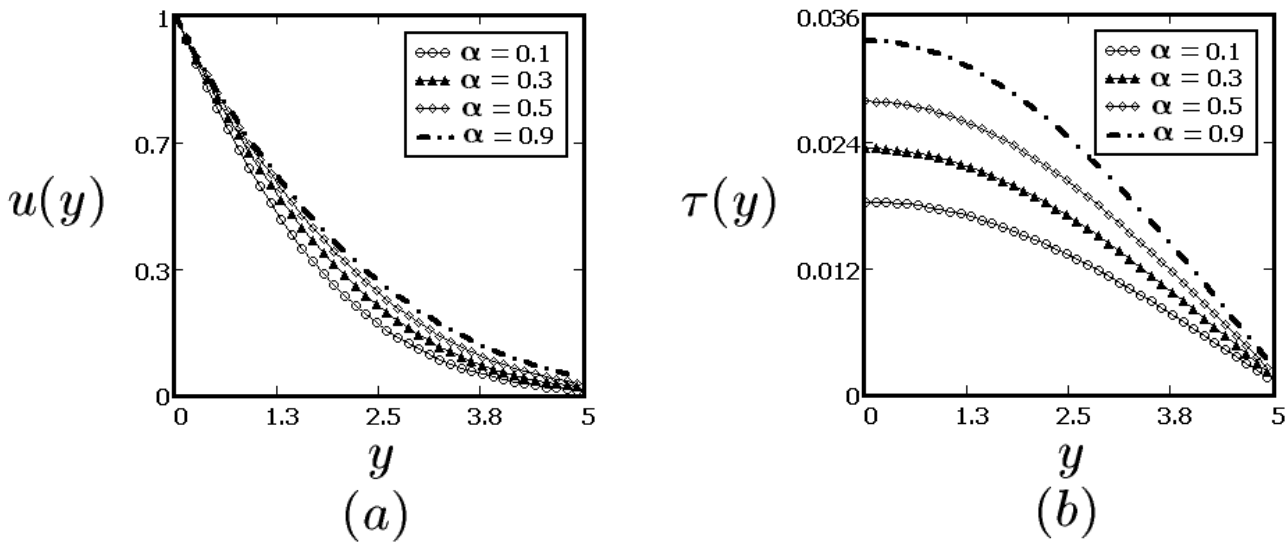
y
(a)
(b)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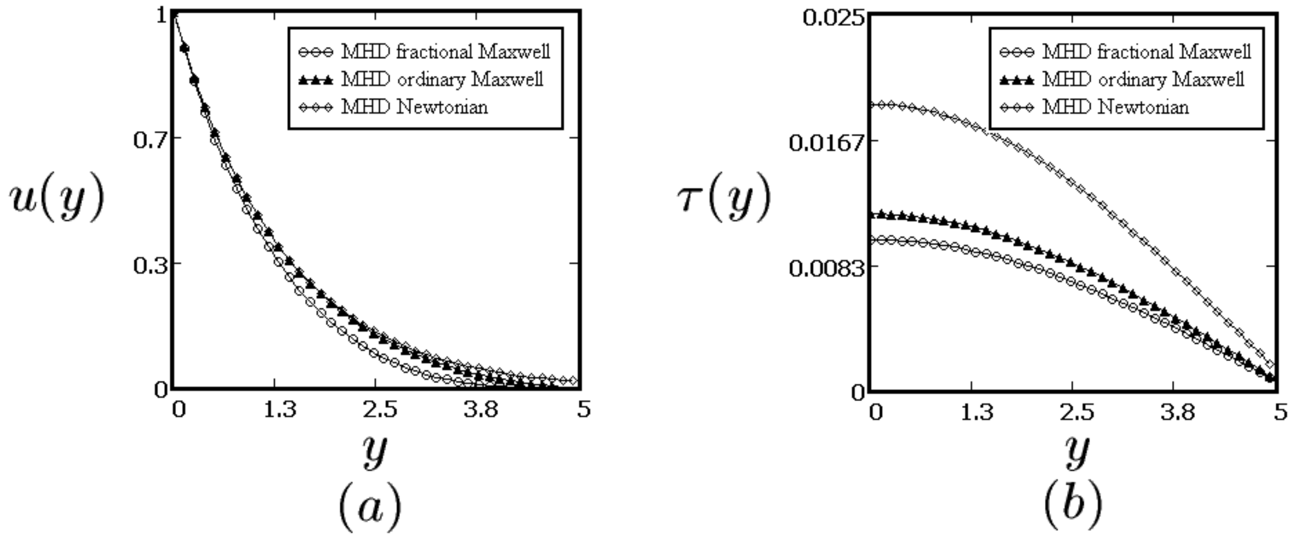
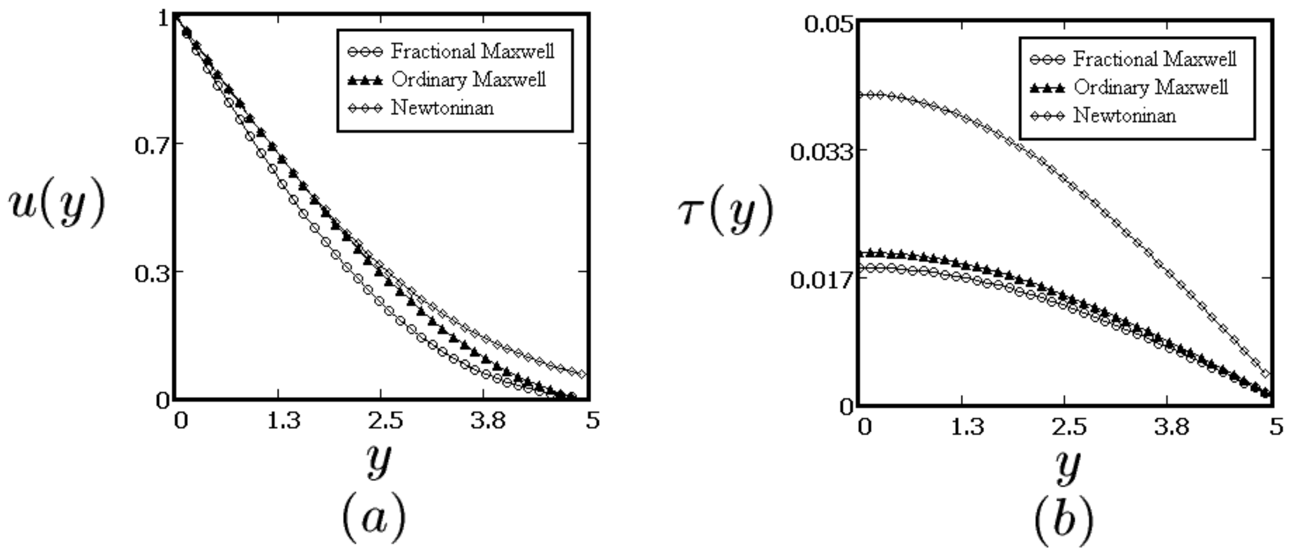
τ(y)
y
(a)
(b)





In this paper the unsteady MHD flow of an incompressible fractional Maxwell fluid over an infinite plate is obtained by means of the Fourier sine and laplace transforms. The motion of the fluid is due to the plate that at time $t = 0^+$ is suddenly moved with a constant velocity U in its plane. Exact solutions are obtained for the velocity $u(y,t)$ and the shear stress $\tau(y,t)$ under integral and series form in terms of the generalized $G_{a,b,c}(d,t)$ functions. These solutions, presented as a sum of the Newtonian solutions and the corresponding non-Newtonian contributions, satisfy all imposed initial and boundary conditions. The solutions for ordinary MHD Maxwell and hydrodynamic fluids, can also be obtained as special cases from present general solutions.

In order to reveal some relevant physical aspects of the obtained results, the diagrams of the velocity field $u(y,t)$ and the shear stress $\tau(y,t)$ have been drawn against y for different values of t and the material constants Λ, ν and fractional parameter β. The influence of the applied magnetic field on the fluid motion is also underlined by graphical illustrations. Figs. 1 and 2 are presented the diagrams of the velocity and the shear stress at four different times t and vertical distance y. As expected, both the velocity and the shear stress are increasing functions with respect to t and decreasing ones with respect to y. For large values of y they are going to zero. Figs. 3 show the effect of magnetic field B_0 on $u(y,t)$ and $\tau(y,t)$. The presence of the magnetic field, as it results from Fig. 3, leads to a slowness of the motion. The influence of relaxation time Λ on fluid motion is shown in Figs. 4. The effect of relaxation time on fluid motion is quite opposite to that of magnetic field. The impact of kinematic viscosity ν on fluid flows is shown in Figs. 5. As it was to be expected, both the velocity and the shear stress are increasing functions with regard to ν. The important parameter for for the underlying fluid is the fractional parameter β, whose influence is shown in Figs. 6. The velocity field $u(y,t)$ and shear stress $\tau(y,t)$ are increasing functions of fractional parameter β.

Finally, for comparison, the profiles of the velocity and the shear stress corresponding to the three models (fractional Maxwell, ordinary Maxwell, Newtonian) with and without magnetic field are depicted in Fig. 7 and 8 for the same values of t and of the common material parameter. It is clearly seen from these figures that, for these values of the material parameters, the Newtonian fluid is the swiftest and the fractional Maxwell fluid corresponding to $\beta = 0.5$

is the slowest independent of the effect of magnetic field. The shear stress corresponding to Newtonian fluid is largest. Of course, these results are entirely agreed with those resulting from Figs. 4 and 6. If $\Lambda \to 0$, for instance, the velocity of the fractional Maxwell fluid increases to that of the Newtonian fluid. The choice of the reasonable values of the fractional parameter β corresponding to the optional dynamical model, results by corresponding with the experimental data. The units of the material constants in all figures are SI units.

Appendix

$$D_t^p[H(t)] = \frac{t^{-p}}{\Gamma(1-p)}; \quad \mathscr{L}\left\{\frac{t^{-p}}{\Gamma(1-p)}\right\} = q^{p-1}; \; p \in (0,1), \tag{A1}$$

$$\frac{1}{x+a} = \sum_{k=0}^{\infty}(-1)^k \frac{x^k}{a^{k+1}}; \quad (a+b+c)^k = \sum_{i,j,\ell\geq 0}^{i+j+\ell=k} \frac{a^i b^j c^\ell k!}{i!j!\ell!}, \tag{A2}$$

$$\int_0^\infty \frac{\xi \sin(y\xi)}{\xi^2+a^2} d\xi = \frac{\pi}{2} e^{-ay}; \quad \text{Re(a)} \geq 0, \tag{A3}$$

$$\mathscr{L}^{-1}\left\{\overline{u}_1(q)\overline{u}_2(q)\right\} = (u_1 * u_2)(t) = \int_0^t u_1(t-s)u_2(s)ds = \int_0^t u_1(s)u_2(t-s)ds \;\text{ if} \tag{A4}$$

$$u_1(t) = \mathscr{L}^{-1}\left\{\overline{u}_1(q)\right\} \text{ and } u_2(t) = \mathscr{L}^{-1}\left\{\overline{u}_2(q)\right\},$$

$$\mathscr{L}^{-1}\left\{\frac{q^b}{(q^a-d)^c}\right\} = G_{a,b,c}(d,t); \quad \text{Re}\,(ac-b) > 0, \; \text{Re}(q) > 0, \; \left|\frac{d}{q^a}\right| < 1, \tag{A5}$$

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