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New modification of Adomian decomposition method for solving a system of nonlinear fractional partial differential equations

Research Article

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Abstract: In this paper, we introduce a new modification of Adomian decomposition method (ADM) for solving a system of nonlinear fractional partial differential equations (NFPDEs). This modification has been constructed for a general system of NFPDEs, and it is easy to implement numerically. Therefore, this modification is more practical and helpful for solving abroad systems of NFPDEs.The approximate solution for systems of NFPDEs is easily obtained by the means of Caputo fractional partial derivative based on the properties of fractional calculus. Moreover, the convergence and error analysis of the proposed modification are shown. The approximate and numerical solutions for well-known examples are presented in forms of tables and graphs to make a comparison with the results that previously obtained by some other well-known methods. Numerical results are carried out to confirm the accuracy and efficiency of the proposed modification.

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Keywords: System of nonlinear fractional partial differential equations • Adomian decomposition method • Existence theorem • Error analysis • Approximate solutions

1. Introduction

During last few decades, fractional order partial differential equations have been proposed and investigated in many research fields, such as fluid mechanics, mechanics of materials, biology, plasma physics, finance and chemistry, see [1–6]. The systems of fractional partial differential equations have been increasingly used to represent physical and control systems (see for instant, [7–11] and some references cited therein). Since some of the fractional order partial differential equations do not have exact analytic solutions, approximating or numerical techniques are generally applied. There are many different analytical and numerical methods such as ADM [12], the fractional complex transformation (Elsayed M.E. Zayed ([13]), homotopy perturbation method (S. Momani [14]), a homotopy perturbation technique (S. T. Mohyud-Din [15]), Variational iteration method (Z. Odibat [16]), homotopy perturbation transform method (Brajesh Kumar Singh and Pramod Kumar [17]), generalized differential transform method (A. Ebadian et al. [18]), decomposition method (Zaid Odibat [16]), Modified least squares homotopy perturbation method (H. Thabet and S. Kendre [19]) and so on.

The ADM due to Adomian [20] has been successfully used in solving a wide variety of deterministic as well as stochastic problems in differential equations. The ADM provides the solution in a rapidly convergent series with easily computable components. The main advantage of ADM is that it can be used directly to solve,all types of differential

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equations with homogeneous and inhomogeneous initial and boundary conditions. These include linear and nonlinear ordinary, and partial differential equations, see [21–25].

More recently, the powerful modifications of ADM were proposed by Wazwaz [26], Odibat [16] and Ramana [27]. The established modifications in [16, 26–28] demonstrated a rapid convergence of the series solution if compared with the standard Adomian method, and therefore presented a major progress. The modified decomposition method has been shown to be computationally efficient in several examples that are important to researchers in applied science.

The purpose of this paper is to introduce a new reliable modification of ADM to find the approximate analytical solution for the following general system of nonlinear fractional partial differential system with initial values:

$$\begin{cases} \mathscr{D}_t^{q_i} u_i(\bar{x},t) = f_i(\bar{x},t) + L_i \bar{u}(\bar{x},t) + N_i \bar{u}(\bar{x},t), \ m_i - 1 < q_i \leq m_i \in \mathbb{N}, \\ \dfrac{\partial^{k_i} u_i(\bar{x},0)}{\partial t^{k_i}} = f_{ik_i}(\bar{x}), \ \ k_i = 0,1,2,\ldots,m_i-1, \ i = 1,2,\ldots,m, \end{cases} \tag{1}$$

for $\bar{u}(\bar{x},t) = (u_1(\bar{x},t), u_2(\bar{x},t),\ldots,u_n(\bar{x},t))$, $\bar{x} = (x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ where L_i, N_i are linear and nonlinear operators respectively of $\bar{u} = \bar{u}(\bar{x},t)$ and its derivatives which might include other fractional derivatives of orders less than q_i, and $f_i(\bar{x},t)$ are known analytic functions and $\mathscr{D}_t^{q_i}$ are the Caputo partial derivatives of fractional order q_i. In case of $f_i(\bar{x},t) = 0$, the system (1) becomes in the homogeneous form.

Our aim is to generalize the achievements established by others in away to include the standard Adomian method [21–23, 25] and the modifications of ADM are presented in [16, 26–28].

The rest of the paper is organized as follows: In Section 2, we will present basic definitions, theorems and preliminary results that are needed in the sequel. In Section 3, we introduce a new modification of ADM for solving a system of NFPDEs. Applications and numerical implementations are found out in Section 4.

2. Preliminaries

There are various definitions and properties of fractional integrals and derivatives. In this section, we give some modified definitions, theorems and properties of the fractional calculus theory, which can be found in [14, 29–33].

Definition 2.1.
A real function $u(x,t)$, $x \in \mathbb{R}, t > 0$, is said to be in the space C_μ, $\mu \in \mathbb{R}$ if there exists a real number $p(>\mu)$, such that $u(x,t) = t^p u_1(x,t)$, where $u_1(x,t) \in C(\mathbb{R} \times [0,\infty))$, and it is said to be in the space C_μ^m if and only if $\frac{\partial^m u(x,t)}{\partial t^m} \in C_\mu$, $m \in \mathbb{N}$.

Definition 2.2.
Let $q \in \mathbb{R} \setminus \mathbb{N}$ and $q \geq 0$. The Riemann-Liouville fractional partial integral denoted by $\mathscr{I}_t^q$ of order q for a function $u(x,t)$ is defined as:

$$\begin{cases} \mathscr{I}_t^q u(x,t) = \dfrac{1}{\Gamma(q)} \displaystyle\int_0^t (t-\tau)^{q-1} u(x,\tau)d\tau, \ \ q,t > 0, \\ \mathscr{I}_t^0 u(x,t) = u(x,t), \quad q = 0, \ \ t > 0, \end{cases} \tag{2}$$

where Γ is the well-known Gamma function.

Theorem 2.1.
Let $q_1, q_2 \in \mathbb{R} \setminus \mathbb{N}, q_1, q_2 \geq 0$ and $p > -1$. For a function $u(x,t) \in C_\mu$, $\mu > -1$, the operator $\mathscr{I}_t^q$ satisfies the following properties:

$$\begin{cases} \mathscr{I}_t^{q_1} \mathscr{I}_t^{q_2} u(x,t) = \mathscr{I}_t^{q_1+q_2} u(x,t), \\ \mathscr{I}_t^{q_2} \mathscr{I}_t^{q_1} u(x,t) = \mathscr{I}_t^{q_1} \mathscr{I}_t^{q_2} u(x,t), \\ \mathscr{I}_t^q t^p = \dfrac{\Gamma(p+1)}{\Gamma(p+q+1)} t^{p+q}. \end{cases} \tag{3}$$

Definition 2.3.
For $q \in \mathbb{R}$, $m-1 < q < m \in \mathbb{N}$, the Riemann-Liouville fractional partial derivative of order q for $u(x,t)$ is defined as follows:

$$\mathscr{D}_t^q u(x,t) = \frac{\partial^m}{\partial t^m} \int_0^t \frac{(t-\tau)^{m-q-1}}{\Gamma(m-q)} u(x,\tau)d\tau, \ \ t > 0. \tag{4}$$

Definition 2.4.
Let $q, t \in \mathbb{R}, t > 0$ and $u(x,t) \in C_\mu^m$. Then

$$\begin{cases} \mathscr{D}_t^q u(x,t) = \int_a^t \frac{(t-\tau)^{m-q-1}}{\Gamma(m-q)} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & m-1 < q \leq m \in \mathbb{N}, \\ \mathscr{D}_t^q u(x,t) = \frac{\partial^m u(x,t)}{\partial t^m}, & q = m \in \mathbb{N}, \end{cases} \tag{5}$$

is called the Caputo fractional partial derivative of order q for a function $u(x,t)$.

Theorem 2.2.
Let $t, q \in \mathbb{R},\ t > 0$ and $m-1 < q \leq m \in \mathbb{N}$. Then

$$\begin{cases} \mathscr{I}_t^q \mathscr{D}_t^q u(x,t) = u(x,t) - \sum_{k=0}^{m-1} \frac{t^k}{k!} u_t^{(k)}(x,0), \\ \mathscr{D}_t^q \mathscr{I}_t^q u(x,t) = u(x,t). \end{cases} \tag{6}$$

3. New modification of ADM for solving a system of NFPDEs

The reliable modifications of Adomian decomposition method and its effectiveness had been confirmed through many studies. In this section, we introduce a new modification of ADM to solve a system of NFPDEs.
We assume that the solution functions $u_i(\bar{x},t)$ of the system (1) have the following analytic expansion:

$$u_i(\bar{x},t) = \sum_{k=0}^{\infty} u_{ik}(\bar{x},t),\ i = 1,2,\ldots,m. \tag{7}$$

The modified decomposition method [26] is assumed that the function $u_{i0}(\bar{x},t)$ can be divided into the sum of two parts, namely $\phi_{i1}(\bar{x},t)$ and $\phi_{i2}(\bar{x},t)$ and it can be written as

$$u_{i0}(\bar{x},t) = \phi_{i1}(\bar{x},t) + \phi_{i2}(\bar{x},t),\ i = 1,2,\ldots,m. \tag{8}$$

The variation here is that only one part namely $\phi_{i1}(\bar{x},t)$ is to be assigned to the zeroth component $u_{i0}(\bar{x},t)$, whereas the remaining part $\phi_{i2}(\bar{x},t)$ is combined with the other terms to define $u_{i1}(\bar{x},t)$.
To introduce our modification, first we need to present the following results:

Theorem 3.1.
Let $\bar{u}(\bar{x},t) = \sum_{k=0}^{\infty} \bar{u}_k(\bar{x},t)$, for the parameter λ, we define $\bar{u}_\lambda(\bar{x},t) = \sum_{k=0}^{\infty} \lambda^k \bar{u}_k(\bar{x},t)$, then the nonlinear operators $N_i \bar{u}_\lambda$ satisfy the following property:

$$N_i \bar{u}_\lambda = N_i \sum_{k=0}^{\infty} \lambda^k \bar{u}_k = \sum_{n=0}^{\infty} \Big[\frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [N_i \sum_{k=0}^{n} \lambda^k \bar{u}_k]_{\lambda=0} \Big] \lambda^n,\ i = 1,2,\ldots,m. \tag{9}$$

Proof. According to Maclaurin expansion of $N_i \sum_{k=0}^{\infty} \lambda^k \bar{u}_k$ with respect to λ, we have

$$\begin{aligned} N_i \bar{u}_\lambda = N_i \sum_{k=0}^{\infty} \lambda^k \bar{u}_k &= [N_i \sum_{k=0}^{\infty} \lambda^k \bar{u}_k]_{\lambda=0} + [\frac{\partial}{\partial \lambda} [N_i \sum_{k=0}^{\infty} \lambda^k \bar{u}_k]_{\lambda=0}] \lambda + [\frac{\partial^2}{\partial \lambda^2} [N_i \sum_{k=0}^{\infty} \lambda^k \bar{u}_k]_{\lambda=0}] \lambda^2 + \cdots \\ &= \sum_{n=0}^{\infty} [\frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [N_i \sum_{k=0}^{\infty} \lambda^k \bar{u}_k]_{\lambda=0}] \lambda^n \\ &= \sum_{n=0}^{\infty} \Big[\frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [N_i (\sum_{k=0}^{n} \lambda^k \bar{u}_k + \sum_{k=n+1}^{\infty} \lambda^k \bar{u}_k)]_{\lambda=0} \Big] \lambda^n \\ &= \sum_{n=0}^{\infty} \Big[\frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} [N_i \sum_{k=0}^{n} \lambda^k \bar{u}_k]_{\lambda=0} \Big] \lambda^n,\ i = 1,2,\ldots,m, \end{aligned} \tag{10}$$

which completes the proof. □

Definition 3.1.
Let the polynomials, $A_{in}(u_{i0}, u_{i1}, \ldots, u_{in})$, for $i = 1,2,\ldots m$, to be defined as follows:

$$A_{in}(u_{i0}, u_{i1}, \ldots, u_{in}) = \frac{1}{n!} \frac{\partial^n}{\partial \lambda^n} \Big[N_i \sum_{k=0}^{n} \lambda^k \bar{u}_k \Big]_{\lambda=0},\ i = 1,2,\ldots,m, \tag{11}$$

we call $A_{in}(u_{i0}, u_{i1}, \ldots, u_{in})$ as the generalized Adomian polynomials [34, 35].

Remark 3.1.
Let $A_{in} = A_{in}(u_{i0}, u_{i1}, \dots, u_{in})$, by using Theorem 3.1 and Definition 3.1, the nonlinear operators $N_i \bar{u}_\lambda$ can be expressed in terms of Admomian polynomials as follows:

$$N_i \bar{u}_\lambda = \sum_{n=0}^{\infty} \lambda^n A_{in}, \; i = 1, 2, \dots, m. \tag{12}$$

Remark 3.2.
Let $\lambda \to 1$ in Remark 3.1 and by using Theorem 3.1, we have

$$N_i \sum_{k=0}^{\infty} \bar{u}_k = \sum_{n=0}^{\infty} A_{in}, \; i = 1, 2, \dots, m. \tag{13}$$

3.1. Existence Theorem

The following theorem presents a general solution form obtained by new modification of ADM for a system of nonlinear fractional partial differential equations:

Theorem 3.2.
Let $m_i - 1 < q_i < m_i \in \mathbb{N}$, for $i = 1, 2, \dots m$, and let $f_i(\bar{x}, t), f_i(\bar{x})$ to be as in system (1). Then the system (1) admit at least a solution given by

$$u_i(\bar{x}, t) = \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik}(\bar{x}) + f_{it}^{(-q_i)}(\bar{x}, t) + \sum_{k=2}^{\infty} [L_{it}^{(-q_i)} \bar{u}_{(k-1)} + A_{i(k-1)t}^{(-q_i)}], \tag{14}$$

for $i = 1, 2, \dots m$, where $L_{it}^{(-q_i)} \bar{u}_{(k-1)}$ and $A_{i(k-1)t}^{(-q_i)}$ denote the q_i^{th} fractional partial integral with respect to t for $L_{i(k-1)}$ and $A_{i(k-1)}$ respectively.

Proof. To prove the above theorem, we perform Riemann-Liouville fractional partial integral given by Definition2.2 with respect to t to both sides of the equations of the system (1), and by using Theorem 2.2, we obtain

$$u_i(\bar{x}, t) = \sum_{j=0}^{m_i-1} \frac{t^{k_i}}{k_i!} u_{it}^{(k_i)}(\bar{x}, 0) + f_{it}^{(-q_i)}(\bar{x}, t) + \mathscr{I}_t^{q_i}[L_i \bar{u}] + \mathscr{I}_t^{q_i}[N_i \bar{u}], \tag{15}$$

for $i = 1, 2, \dots, m$. Substituting the initial condition from system (1) in system (9), we get

$$u_i(\bar{x}, t) = f_{it}^{(-q_i)}(\bar{x}, t) + \sum_{k_i=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik_i}(\bar{x}) + \mathscr{I}_t^{q_i}[L_i \bar{u}] + \mathscr{I}_t^{q_i}[N_i \bar{u}], \; i = 1, 2, \dots, m. \tag{16}$$

According to the decomposition method [12], we decompose the unknown functions $u_{ik}(\bar{x}, t)$ into sums of components defined by the following decomposition series:

$$u_i(\bar{x}, t) = \sum_{k=0}^{\infty} u_{ik}(\bar{x}, t), \; i = 1, 2, \dots, m. \tag{17}$$

By substituting the system (17) into the system (16), we obtain

$$\sum_{k=0}^{\infty} u_{ik}(\bar{x}, t) = \sum_{k=0}^{m-1} \frac{t^k}{k!} f_{ik_i}(\bar{x}) + f_i^{(-q_i)}(\bar{x}, t) + \mathscr{I}_t^{q}[L_i \sum_{k=0}^{\infty} \bar{u}_k] + \mathscr{I}_t^{q}[N_i \sum_{k=0}^{\infty} \bar{u}_k], \tag{18}$$

for $i = 1, 2, \dots, m$. The linear terms $L_i \bar{u}$ satisfy

$$L_i \bar{u}(\bar{x}, t) = L_i \sum_{k=0}^{\infty} \bar{u}_k(\bar{x}, t) = \sum_{n=0}^{\infty} L_i \bar{u}_k(\bar{x}, t), \; i = 1, 2, \dots, m. \tag{19}$$

By using Remark 3.2, the system (18) can be rewritten as:

$$\sum_{k=0}^{\infty} \bar{u}_{ik}(x, t) = \sum_{k=0}^{m_i-1} \frac{t^{k_i}}{k_i!} f_{ik_i}(\bar{x}) + f_{it}^{(-q_i)}(\bar{x}, t) + \mathscr{I}_t^{q_i}[\sum_{k=0}^{\infty} L_i \bar{u}_k] + \mathscr{I}_t^{q_i}[\sum_{n=0}^{\infty} A_{in}], \tag{20}$$

for $i = 1, 2, \dots, m$. Under the assumption of modified ADM [26], we propose a slight variation in $u_{i0}(\bar{x}, t)$ and $u_{i1}(\bar{x}, t)$ as compared to ADM [12]. So we assume that $u_{0i}(\bar{x}, t) = \phi_{i1}(\bar{x}, t)$ for $i = 1, 2, \dots, m$, and the variation here is that only one part $\phi_{i1}(\bar{x},t)=\sum_{k=0}^{m_i-1}\frac{t^{k_i}}{k_i!}f_{ik_i}(\bar{x})$ be assigned to the zeroth component $u_{i0}(\bar{x},t)$, whereas the remaining part $\phi_{i2}(\bar{x},t)=f_{it}^{(-q_i)}(\bar{x},t)$ is combined with the other terms to define $u_{i1}(\bar{x},t)$. According to these suggestions, it formulates the modified recursive algorithm as follows:

$$\begin{cases} u_{i0}(\bar{x},t)=\sum_{k_i=0}^{m_i-1}\frac{t^{k_i}}{k_i!}f_{ik_i}(\bar{x}),\\ u_{i1}(\bar{x},t)=f_{it}^{(-q_i)}(\bar{x},t)+L_{it}^{(-q_i)}\bar{u}_0]+A_{i0t}^{(-q_i)},\\ u_{i2}(\bar{x},t)=L_{it}^{(-q_i)}\bar{u}_1+A_{i1t}^{(-q_i)},\\ \quad\vdots\\ u_{ik}(\bar{x},t)=L_{it}^{(-q_i)}\bar{u}_{(k-1)}+A_{i(k-1)t}^{(-q_i)},\ k=2,3,\dots,\ i=1,2,\dots,m. \end{cases} \tag{21}$$

The decomposition series given by the system (17) can be written as:

$$u_i(\bar{x},t)=\sum_{k=0}^{n}u_{ik}(\bar{x},t)=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t)+\sum_{k=2}^{n}u_{ik}(\bar{x},t),\ i=1,2,\dots,m. \tag{22}$$

Inserting (21) into (22) completes the proof. □

3.2. Convergence and Error analysis

The following theorems shows the convergence and the error analysis for the solution of a system of NFPDEs obtained by the proposed modification of ADM:

Theorem 3.3.
Let B be a Banach space. Then the series $\{u_{in}(\bar{x},t)\}_{n=0}^{\infty}$ obtained by the system (1) converges to $S_i\in B$ for $i=1,2,\dots,m$, if there exists γ_i, $0\le\gamma_i<1$, then $\|u_{in}\|\le\gamma_i\|u_{i(n-1)}\|$, $\forall n\in\mathbb{N}$.

Proof. Define that S_{in} are the sequences of partial sums of the series given by the system (21) as:

$$\begin{cases} S_{i0}=u_{i0}(\bar{x},t),\\ S_{i1}=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t),\\ S_{i2}=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t)+u_{i2}(\bar{x},t),\\ \ \vdots\\ S_{in}=u_{i0}(\bar{x},t)+u_{i1}(\bar{x},t)+u_{i2}(\bar{x},t)+\cdots+u_{in}(\bar{x},t),\ i=1,2,\dots,m, \end{cases} \tag{23}$$

and we need to show that $\{S_{in}\}_{n=0}^{\infty}$ are a Cauchy sequences in Banach space B. For this purpose, we consider

$$\|S_{i(n+1)}-S_{in}\|=\|u_{i(n+1)}(\bar{x},t)\|\le\gamma_i\|u_{in}(\bar{x},t)\|\le\gamma_i^2\|u_{i(n-1)}(\bar{x},t)\|\le\cdots\le\gamma_i^{n+1}\|u_{i0}\|,\ i=1,2,\dots,m. \tag{24}$$

For every $n,r\in\mathbb{N}$, $n\ge r$, by using the system (24) and the triangle inequality successively, we have

$$\begin{aligned} \|S_{in}-S_{ir}\| &= \|(S_{in}-S_{i(n-1)})+(S_{i(n-1)}-S_{i(n-2)})+\cdots+(S_{i(r+1)}-S_{ir})\|\\ &\le \|(S_{in}-S_{i(n-1)})\|+\|(S_{i(n-1)}-S_{i(n-2)})\|+\cdots+\|(S_{i(r+1)}-S_{ir})\|\\ &\le \gamma_i^n\|u_{i0}(\bar{x},t)\|+\gamma_i^{n-1}\|u_{i0}(\bar{x},t)\|+\cdots+\gamma_i^{r+1}\|u_{i0}(\bar{x},t)\|\\ &\le \gamma_i^{r+1}(1+\gamma_i+\cdots+\gamma_i^n+\cdots)\|u_{i0}(\bar{x},t)\|=\frac{\gamma_i^{r+1}}{1-\gamma_i}\|u_{i0}(\bar{x},t)\|, \end{aligned} \tag{25}$$

for $i=1,2,\dots,m$. Since $u_{i0}(\bar{x},t)$ is bounded, we have

$$\lim_{n,r\to\infty}\|S_{in}-S_{ir}\|=0,\ i=1,2,\dots,m. \tag{26}$$

Therefore, the sequences $\{S_{in}\}_{n=0}^{\infty}$, $i=1,2,\dots,m$ are Cauchy sequences in the Banach space B, so the series solution defined in the system (22) converges. This completes the proof. □

Theorem 3.4.
The maximum absolute truncation error of the series solution (14) *of the nonlinear time-space fractional partial differential system* (2) *is estimated to be*

$$\sup_{(\bar{x},t)\in\Omega} \left|u_i(\bar{x},t) - \sum_{k=0}^{r} u_{ik}(\bar{x},t)\right| \le \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ \Omega \subset \mathbb{R}^{n+1},\ i = 1,2,\dots,m. \tag{27}$$

Proof. From Theorem 3.3, we have

$$\|S_{in} - S_{ir}\| \le \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ i = 1,2,\dots,m. \tag{28}$$

But we assume that $S_{in} = \sum_{k=0}^{n} u_{ik}(\bar{x},t)$ for $i = 1,2,\dots,m$, and since $n \to \infty$, we obtain $S_{in} \to u_i(\bar{x},t)$, so the system (28) can be rewritten as:

$$\|u_i(\bar{x},t) - S_{ir}\| = \|u_i(\bar{x},t) - \sum_{k=0}^{r} u_{ik}(\bar{x},t)\| \le \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ i = 1,2,\dots,m. \tag{29}$$

So, the maximum absolute truncation error in the region Ω is

$$\sup_{(\bar{x},t)\in\Omega} \left|u_i(\bar{x},t) - \sum_{k=0}^{r} u_{ik}(\bar{x},t)\right| \le \frac{\gamma_i^{r+1}}{1-\gamma_i} \sup_{(\bar{x},t)\in\Omega} |u_{i0}(\bar{x},t)|,\ i = 1,2,\dots,m, \tag{30}$$

and this completes the proof. □

4. Applications and numerical implementations

In this section, we apply the new modification of ADM for solving systems of NFPDEs. These examples are chosen because their closed form solutions are available or they have been solved previously by some other well-known methods.

Example 4.1.
Consider the following system of nonlinear homogeneous dispersive long wave equations of time fractional order with initial values:

$$\begin{cases} \mathscr{D}_t^q u(x,t) + v_x(x,t) + \frac{1}{2}u_x^2(x,t) = 0,\ u(x,0) = \alpha\big[1 + \tanh(\frac{1}{2}[\beta + \alpha x])\big], \\ \mathscr{D}_t^q v(x,t) + (u(x,t)v(x,t) + u(x,t) + u_{xx}(x,t))_x = 0,\ v(x,0) = -1 + \frac{1}{2}\alpha^2 \operatorname{sech}^2(\frac{1}{2}[\beta + \alpha x]). \end{cases} \tag{31}$$

For $q = 1$, the exact solitary wave solution for the system (31) given by [36, 37] is as follows:

$$\begin{aligned} u(x,t) &= \alpha\big[1 + \tanh(\frac{1}{2}[\beta + \alpha x - \alpha^2 t])\big], \\ v(x,t) &= -1 + \frac{1}{2}\alpha^2 \operatorname{sech}^2(\frac{1}{2}[\beta + \alpha x - \alpha^2 t]), \end{aligned} \tag{32}$$

where α, β are arbitrary constants.
By comparing the system (31) with system (1), we observe that, $f_1(x,t) = f_2(x,t) = 0$, $N_1(u,v) = -\frac{1}{2}(u^2)_x$ and $N_2(u,v) = -(uv)_x$, and the system (31) can be rewritten as:

$$\mathscr{D}_t^q u(x,t) = -v_x(x,t) + N_1(u(x,t),v(x,t)),\ \mathscr{D}_t^q v(x,t) = -u_x(x,t) - u_{xxx}(x,t) + N_2(u(x,t),v(x,t)). \tag{33}$$

To obtain the approximate solution for the system (31), we operate the Riemann-Liouville fractional integral given by Definition 2.2 with respect to t to both sides of the equations of the system (31), and by using Theorem 2.2, we obtain

$$\begin{aligned} u(x,t) &= u(x,0) - \mathscr{I}_t^q[v_x(x,t)] + \mathscr{I}_t^q[N_1(u(x,t),v(x,t))], \\ v(x,t) &= v(x,0) - \mathscr{I}_t^q[u(x,t) + u_{xx}(x,t)] + \mathscr{I}_t^q[N_2(u(x,t),v(x,t))]. \end{aligned} \tag{34}$$

Assume that the solution is given as:

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t),\ v(x,t) = \sum_{k=0}^{\infty} v_k(x,t). \tag{35}$$

By using the system (35) in the system (34), we obtain

$$\begin{cases} \sum_{k=0}^{\infty} u_k(x,t) = \alpha\big[1+\tanh(\frac{1}{2}[\beta+\alpha x])\big] - \mathscr{I}_t^q[\sum_{k=0}^{\infty} v_{kx}(x,t)] + \mathscr{I}_t^q[N_1(\sum_{k=0}^{\infty} u_k(x,t), \sum_{k=0}^{\infty} v_k(x,t))], \\ \sum_{k=0}^{\infty} v_k(x,t) = -1+\frac{1}{2}\alpha^2\text{sech}^2(\frac{1}{2}[\beta+\alpha x]) - \mathscr{I}_t^q[\sum_{k=0}^{\infty}(u_{kx}(x,t)+u_{kxx}(x,t))] + \mathscr{I}_t^q[N_2(\sum_{k=0}^{\infty} u_k(x,t), \sum_{k=0}^{\infty} v_k(x,t))]. \end{cases} \tag{36}$$

By using Remark 3.2 in the system (36), we get

$$\begin{cases} \sum_{k=0}^{\infty} u_k(x,t) = \alpha\big[1+\tanh(\frac{1}{2}[\beta+\alpha x])\big] - \mathscr{I}_t^q[\sum_{k=0}^{\infty} v_{kx}(x,t)] + \mathscr{I}_t^q[\sum_{k=0}^{\infty} A_{1k}], \\ \sum_{k=0}^{\infty} v_k(x,t) = -1+\frac{1}{2}\alpha^2\text{sech}^2(\frac{1}{2}[\beta+\alpha x]) - \mathscr{I}_t^q[\sum_{k=0}^{\infty}(u_{kx}(x,t)+u_{kxx}(x,t))] + \mathscr{I}_t^q[\sum_{k=0}^{\infty} A_{2k}]. \end{cases} \tag{37}$$

By using Theorem 3.3, the components of decomposition series in the system (35) can be obtained as

$$\begin{aligned} &u_0(x,t) = \alpha\big[1+\tanh(\frac{1}{2}[\beta+\alpha x])\big],\ v_0(x,t) = -1+\frac{1}{2}\alpha^2\text{sech}^2(\frac{1}{2}[\beta+\alpha x]), \\ &u_1(x,t) = -\mathscr{I}_t^q[v_{0x}] + \mathscr{I}_t^q[A_{10}],\ v_1(x,t) = -\mathscr{I}_t^q[u_{0x}+u_{0xx}] + \mathscr{I}_t^q[A_{20}], \\ &u_2(x,t) = -\mathscr{I}_t^q[v_{1x}] + \mathscr{I}_t^q[A_{11}],\ v_2(x,t) = -\mathscr{I}_t^q[u_{1x}+u_{1xx}] + \mathscr{I}_t^q[A_{21}], \\ &\quad\vdots \\ &u_k(x,t) = -\mathscr{I}_t^q[v_{(k-1)x}] + \mathscr{I}_t^q[A_{1(k-1)}], v_k(x,t) = -\mathscr{I}_t^q[u_{(k-1)x}+u_{(k-1)xx}] + \mathscr{I}_t^q[A_{2(k-1)}], \end{aligned} \tag{38}$$

where $A_{1(k-1)}, A_{2(k-1)}, k = 1,2,\ldots$ are generalized Adomian polynomials which can be obtain by using the system (11). Consequently, after a few calculations using Mathematica software, we obtain

$$\begin{aligned} &u_0(x,t) = \alpha\big[1+\tanh(\frac{1}{2}[\beta+\alpha x])\big],\ v_0(x,t) = -1+\frac{1}{2}\alpha^2\text{sech}^2(\frac{1}{2}[\beta+\alpha x]), \\ &u_1(x,t) = -\frac{\alpha^3\text{sech}^2(\frac{1}{2}[\beta+\alpha x])}{2\Gamma(q+1)}t^q, v_1(x,t) = \frac{\alpha^4\sinh(\beta+\alpha x)\text{sech}^4(\frac{1}{2}[\beta+\alpha x])}{4\Gamma(q+1)}t^q, \\ &u_2(x,t) = -\frac{4\alpha^5 t^{2q}\sinh^4\big(\frac{1}{2}(\beta+\alpha x)\big)\text{csch}^3(\beta+\alpha x)}{\Gamma(2q+1)},\ v_2(x,t) = \frac{\alpha^6 t^{2q}(\cosh(\beta+\alpha x)-2)\text{sech}^4\big(\frac{1}{2}(\beta+\alpha x)\big)}{4\Gamma(2q+1)}, \\ &u_3(x,t) = \frac{\alpha^7 t^{3q}\text{sech}^4(\frac{1}{2}[\beta+\alpha x])}{4\Gamma(q+1)^2\Gamma(3q+1)}\Big[\Gamma(q+1)^2[2-\cosh(\beta+\alpha x)] + \big[\Gamma(2q+1)-2\Gamma(q+1)^2\big]\tanh(\frac{1}{2}[\beta+\alpha x])\Big], \\ &v_3(x,t) = \frac{\alpha^8 t^{3q}\text{sech}^6(\frac{1}{2}[\beta+\alpha x])}{16\Gamma(q+1)^2\Gamma(3q+1)}\Big[\Gamma(2q+1)(3-2\cosh(\beta+\alpha x)) + \Gamma(q+1)^2\big[-10\sinh(\beta+\alpha x) \\ &\qquad + \sinh(2[\beta+\alpha x]) + 8\cosh(\beta+\alpha x) - 12\big]\Big], \\ &\qquad\vdots \end{aligned}$$

and so on. Hence the third-order term approximate solution for the system (23) is given by

$$\begin{aligned} u(x,t) &= \alpha\big[1+\tanh(\frac{1}{2}[\beta+\alpha x])\big] - \frac{\alpha^3\text{sech}^2(\frac{1}{2}[\beta+\alpha x])}{2\Gamma(q+1)}t^q - 4\alpha^5\sinh^4(\frac{1}{2}[\beta+\alpha x])\frac{\text{csch}^3(\beta+\alpha x)t^{2q}}{\Gamma(2q+1)} \\ &\quad + \frac{\alpha^7 t^{3q}\text{sech}^4(\frac{1}{2}[\beta+\alpha x])}{4\Gamma(q+1)^2\Gamma(3q+1)}\Big[\Gamma(q+1)^2[2-\cosh(\beta+\alpha x)] + \big[\Gamma(2q+1)-2\Gamma(q+1)^2\big]\tanh(\frac{1}{2}[\beta+\alpha x])\Big], \\ v(x,t) &= -1+\frac{1}{2}\alpha^2\text{sech}^2(\frac{1}{2}[\beta+\alpha x]) + \frac{\alpha^4\sinh(\beta+\alpha x)\text{sech}^4(\frac{1}{2}[\beta+\alpha x])}{4\Gamma(q+1)}t^q \\ &\quad + \frac{\alpha^6 t^{2q}(\cosh(\beta+\alpha x)-2)\text{sech}^4\big(\frac{1}{2}(\beta+\alpha x)\big)}{4\Gamma(2q+1)} + \frac{\alpha^8 t^{3q}\text{sech}^6(\frac{1}{2}[\beta+\alpha x])}{16\Gamma(q+1)^2\Gamma(3q+1)}\Big[\Gamma(2q+1)(3-2\cosh(\beta+\alpha x)) \\ &\quad + \Gamma(q+1)^2\big[-10\sinh(\beta+\alpha x) + \sinh(2[\beta+\alpha x]) + 8\cosh(\beta+\alpha x) - 12\big]\Big]. \end{aligned}$$

Table 1. Numerical values of the approximate and exact solutions when $\alpha = \beta = 0.5$ and $q = 0.5, 1$ for Example 4.1.

x	t	$q = 0.5$		$q = 1$		$\alpha = 0.5$	$\beta = 0.5$	Absolute Error	
		$u(x,t)$	$v(x,t)$	$u(x,t)$	$v(x,t)$	$u_{EX}(x,t)$	v_{EX}	$\lvert u_{EX}(x,t) - u(x,t)\rvert$	$\lvert v_{EX}(x,t) - v(x,t)\rvert$
0.25	0.20	0.621909	-0.882706	0.639916	-0.884788	0.639916	-0.884788	3.06513×10^{-8}	3.40335×10^{-8}
	0.40	0.609298	-0.881558	0.628316	-0.883233	0.628316	-0.883233	4.86441×10^{-7}	1.94750×10^{-6}
	0.60	0.599481	-0.880867	0.616564	-0.881795	0.616567	-0.881794	2.44103×10^{-6}	1.02872×10^{-6}
0.50	0.20	0.650758	-0.886618	0.668188	-0.889144	0.668188	-0.889144	3.27352×10^{-8}	5.06759×10^{-9}
	0.40	0.638477	-0.885130	0.657010	-0.887326	0.657010	-0.887326	5.21894×10^{-7}	8.93617×10^{-8}
	0.60	0.628869	-0.884155	0.645654	-0.885608	0.645656	-0.885608	2.63121×10^{-6}	1.55116×10^{-6}
0.75	0.20	0.678547	-0.891172	0.695297	-0.894070	0.695297	-0.894070	3.32246×10^{-8}	1.03626×10^{-9}
	0.40	0.666676	-0.889385	0.684601	-0.892039	0.684602	-0.892039	5.31703×10^{-7}	9.12339×10^{-9}
	0.60	0.106929	-0.952285	0.673704	-0.890087	0.673707	-0.890087	2.69101×10^{-6}	7.75541×10^{-9}

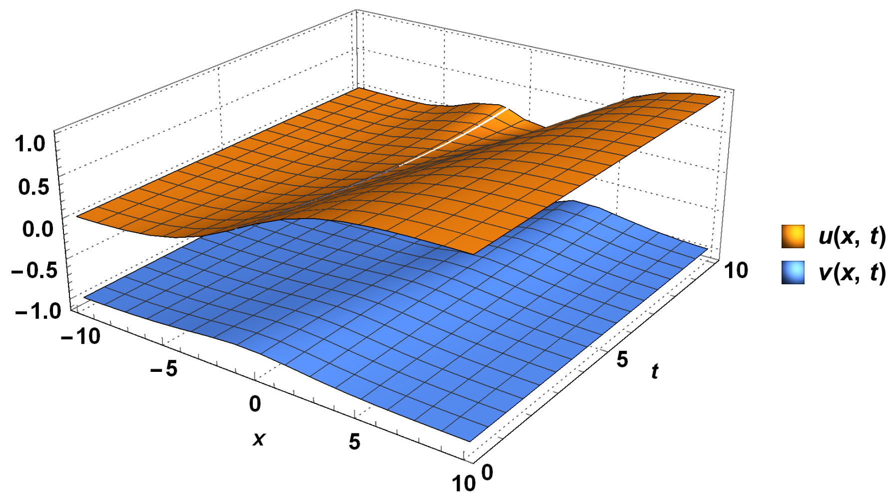

Fig. 1. The graph of the approximate solution when $\alpha = \beta = 0.5$ and $q = 1$ for Example 4.1.

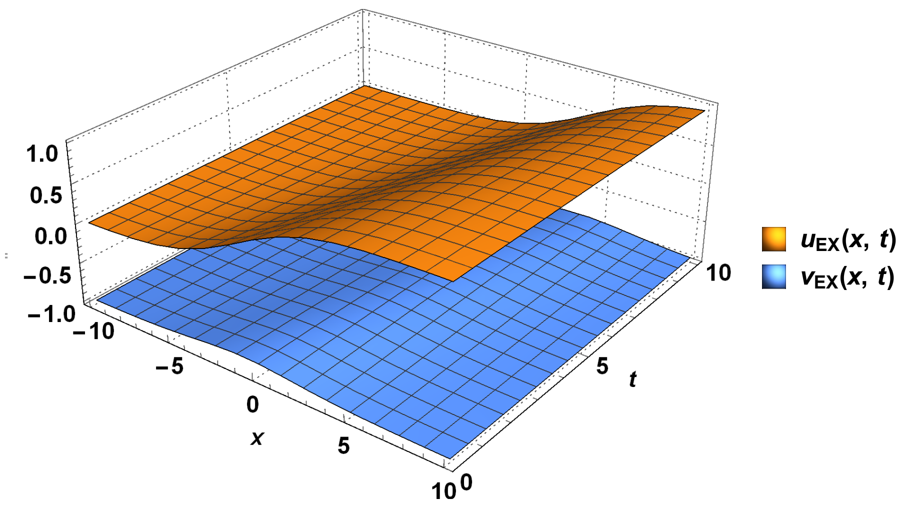

Fig. 2. The graph of the exact solution when $\alpha = \beta = 0.5$ for Example 4.1.

Example 4.2.
Consider the following system of inhomogeneous nonlinear time-fractional partial differential equations with initial

values:

$$\begin{cases} D_t^{q_1} u(x,t) + u(x,t) + v(x,t)u_x(x,t) = 1, \; u(x,0) = e^x, \\ D_t^{q_2} v(x,t) - v(x,t) - u(x,t)v_x(x,t) = 1, \; v(x,0) = e^{-x}. \end{cases} \tag{39}$$

For $q_1 = q_2 = 1$, the exact solution for the system (39) given by [12] is as follows:

$$u(x,t) = e^{x-t}, \; v(x,t) = e^{-x+t}. \tag{40}$$

By comparing the system (39) with system (1), for $i = 1,2$, we define

$$\begin{aligned} &N_1(u(x,t), v(x,t)) = -v(x,t)u_x(x,t), \\ &N_2(u(x,t), v(x,t)) = u(x,t)v_x(x,t), \\ &f_1(x,t) = f_2(x,t) = 1. \end{aligned} \tag{41}$$

So the system (39) can be rewritten as:

$$\begin{aligned} &D_t^{q_1} u(x,t) = 1 - u(x,t) + N_1(u(x,t), v(x,t)), \\ &D_t^{q_2} v(x,t) = 1 + v(x,t) + N_2(u(x,t), v(x,t)). \end{aligned} \tag{42}$$

To find out the approximate solution for the system (31), we operate the Riemann-Liouville fractional integral given by Definition2.2 with respect to t to both sides of the equations of the system (31), and by using Theorem 2.2, we obtain

$$\begin{aligned} &u(x,t) = u(x,0) + \mathscr{I}_t^{q_1}[1 - u(x,t)] + \mathscr{I}_t^{q_2}[N_1(u(x,t), v(x,t))], \\ &v(x,t) = v(x,0) + \mathscr{I}_t^{q_2}[1 + v(x,t)] + \mathscr{I}_t^{q_2}[N_2(u(x,t), v(x,t))]. \end{aligned} \tag{43}$$

Next, we assume that the solution is given by

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t), \; v(x,t) = \sum_{k=0}^{\infty} v_k(x,t). \tag{44}$$

By using the system (44) in the system (44), we obtain

$$\begin{cases} \sum_{k=0}^{\infty} u_k(x,t) = e^x + \mathscr{I}_t^{q_1}\big[1 - \sum_{k=0}^{\infty} u_k(x,t) + N_1(\sum_{k=0}^{\infty} u_k(x,t), \sum_{k=0}^{\infty} v_k(x,t))\big], \\ \sum_{k=0}^{\infty} v_k(x,t) = g_2(x) + \mathscr{I}_t^{q_2}\big[1 + \sum_{k=0}^{\infty} v_k(x,t) + N_2(\sum_{k=0}^{\infty} u_k(x,t), \sum_{k=0}^{\infty} v_k(x,t))\big]. \end{cases} \tag{45}$$

By using Remark 3.2 in the system (45), we get

$$\begin{cases} \sum_{k=0}^{\infty} u_k(x,t) = e^x + \mathscr{I}_t^{q_1}\big[1 - \sum_{k=0}^{\infty} u_k(x,t) + \sum_{n=0}^{\infty} A_{1n}\big], \\ \sum_{k=0}^{\infty} v_k(x,t) = e^{-x} + \mathscr{I}_t^{q_2}\big[1 + \sum_{k=0}^{\infty} v_k(x,t) + \sum_{n=0}^{\infty} A_{2n}\big]. \end{cases} \tag{46}$$

By using Theorem 3.3, the components of decomposition series in the system (35) can be obtained as:

$$\begin{cases} u_0(x,t) = e^x, \; v_0(x,t) = e^{-x}, \\ u_1(x,t) = \mathscr{I}_t^{q_1}\big[1 - u_0(x,t) + A_{10}\big], \; v_1(x,t) = \mathscr{I}_t^{q}\big[1 + v_0(x,t) + A_{20}\big], \\ u_2(x,t) = \mathscr{I}_t^{q_2}\big[-u_1(x,t) + A_{11}\big], \; v_2(x,t) = \mathscr{I}_t^{q}\big[v_1(x,t) + A_{21}\big], \\ \quad\vdots \\ u_k(x,t) = \mathscr{I}_t^{q_1}\big[-u_{k-1}(x,t) + A_{1(k-1)}\big], \; v_k(x,t) = \mathscr{I}_t^{q_2}\big[v_{(k-1)}(x,t) + A_{2(k-1)}\big], \; k = 1,2,\ldots \end{cases} \tag{47}$$

where $A_{1(k-1)}, A_{2(k-1)}, k = 1,2,\ldots$ are generalized Adomian polynomials which can be obtain by using the system (20). Consequently, by explicit calculations using Mathematica software, we obtain

$$\begin{aligned} &u_0(x,t) = e^x, \; v_0(x,t) = e^{-x}, \\ &u_1(x,t) = -\frac{e^x t^{q_1}}{q_1\Gamma(q_1)}, \; v_1(x,t) = \frac{e^{-x} t^{q_2}}{q_2\Gamma(q_2)}, \\ &u_2(x,t) = \frac{[e^x+1]t^{2q_1}}{\Gamma(2q_1+1)} - \frac{t^{q_1+q_2}}{\Gamma(q_1+q_2+1)}, \\ &v_2(x,t) = \frac{t^{2q_1}}{\Gamma(2q_1+1)} - \frac{[e^{-x}-1]t^{q_1+q_2}}{\Gamma(q_1+q_2+1)}, \\ &u_3(x,t) = \Big[\frac{[e^x-1]t^{2q_2}}{\Gamma(q_1+2q_2+1)} + \frac{\Gamma(q_1+q_2+1)t^{q_1+q_2}}{\Gamma(q_1+1)\Gamma(q+1)\Gamma(2q_1+q_2+1)} - \frac{[e^x-1]t^{q_1+q_2}}{\Gamma(2q_1+q_2+1)} - \frac{[e^x+2]t^{2q_1}}{\Gamma(3q_1+1)}\Big]t^{q_1}, \\ &v_3(x,t) = \Big[-\frac{[1+e^{-x}]t^{2q_1}}{\Gamma(2q_1+q_2+1)} + \frac{\Gamma(q_1+q_2+1)t^{q_1+q_2}}{\Gamma(q_1+1)\Gamma(q_2+1)\Gamma(q_1+2q_2+1)} + \frac{[1+e^{-x}]t^{q_1+q_2}}{\Gamma(q_1+2q_2+1)} + \frac{e^{-x}[1-2e^x]t^{2q_2}}{\Gamma(3q_2+1)}\Big]t^{q_2}, \\ &\quad\vdots \end{aligned}$$

and so on.
Hence the third-order term approximate solution for the system (39) is given by

$$
\begin{aligned}
u(x,t) = &-\frac{t^{q_1+q_2}}{\Gamma(q_1+q_2+1)} + \Big[\frac{[e^x-1]\,t^{2q_2}}{\Gamma(q_1+2q_2+1)} - \frac{[e^x-1]\,t^{q_1+q_2}}{\Gamma(2q_1+q_2+1)} - \frac{[e^x+2]\,t^{2q_1}}{\Gamma(3q_1+1)}\Big]t^{q_1} \\
&+\frac{\Gamma(q_1+q_2+1)\,t^{q_1+q_2}}{\Gamma(q_1+1)\Gamma(q_2+1)\Gamma(2q_1+q_2+1)} + \frac{[e^x+1]\,t^{2q_1}}{\Gamma(2q_1+1)} - \frac{e^x t^{q_1}}{q_1\Gamma(q_1)} + e^x, \\
v(x,t) = &\frac{t^{q_1+q_2}}{\Gamma\left(q_1+q_2+1\right)} - \frac{[1-e^{-x}]\,t^{2q_2}}{\Gamma\left(2q_2+1\right)} + \frac{[1+e^{-x}]\,t^{q_1+q_2}}{\Gamma\left(q_1+2q_2+1\right)} - t^{q_2}\Big[\frac{[1+e^{-x}]\,t^{2q_1}}{\Gamma\left(2q_1+q_2+1\right)} \\
&-\frac{[e^{-x}-2]\,t^{2q_2}}{\Gamma\left(3q_2+1\right)} - \frac{t^{q_1+q_2}\Gamma\left(q_1+q_2+1\right)}{\Gamma\left(q_1+1\right)\Gamma\left(q_2+1\right)\Gamma\left(q_1+2q_2+1\right)}\Big] + \frac{e^{-x}t^{q_2}}{q_2\Gamma\left(q_2\right)} + e^{-x}.
\end{aligned}
$$

Table 2. Numerical values for the approximate and exact solutions when $q_1 = q_2 = 0.5$ and $q_1 = q_2 = 1$ for Example 4.2.

x	t	$q_1 = q_2 = 0.5$		$q_1 = q_2 = 1$				Absolute Error	
		$u(x,t)$	$v(x,t)$	$u(x,t)$	$v(x,t)$	$u_{EX}(x,t)$	v_{EX}	$\lvert u_{EX}(x,t)-u(x,t)\rvert$	$\lvert v_{EX}(x,t)-v(x,t)\rvert$
0.25	0.20	0.757584	1.331070	1.051190	0.951175	1.051270	0.951229	0.00008229	0.000054068
	0.40	0.498626	1.656020	0.859441	1.160930	0.860708	1.161830	0.00126696	0.000901875
	0.60	0.229151	1.944980	0.698510	1.414300	0.704688	1.419070	0.00617826	0.004765330
0.50	0.20	0.986646	1.025820	1.349750	0.740776	1.349860	0.740818	0.00010566	0.000042108
	0.40	0.679531	1.259110	1.103540	0.904135	1.105170	0.904837	0.00162681	0.000702381
	0.60	0.366403	1.458550	0.896904	1.101460	0.904837	1.105170	0.00793305	0.003711240
0.75	0.20	1.280770	0.788092	1.733120	0.576917	1.733250	0.576950	0.00013567	0.000032794
	0.40	0.911817	0.950005	1.416980	0.704141	1.419070	0.704688	0.00208887	0.000547015
	0.60	0.542637	1.079710	1.151650	0.857818	1.161830	0.860708	0.01018620	0.002890320

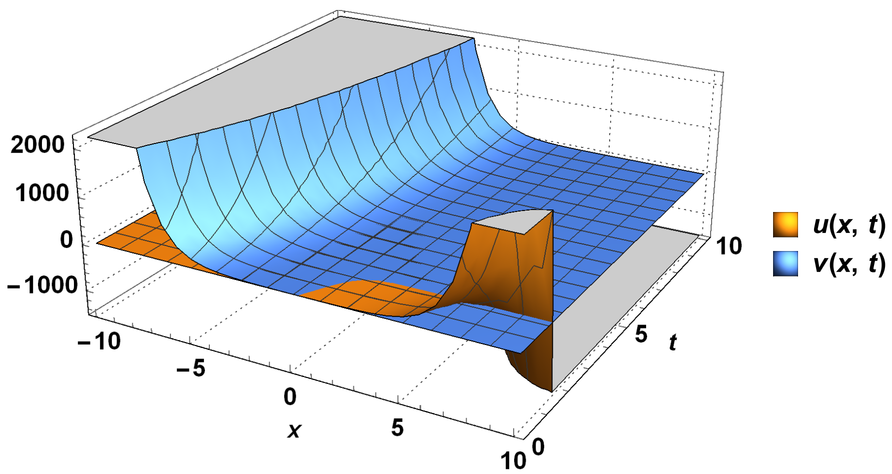

Fig. 3. The graph of the approximate solution for Example 4.2 when $q_1 = q_2 = 1$.

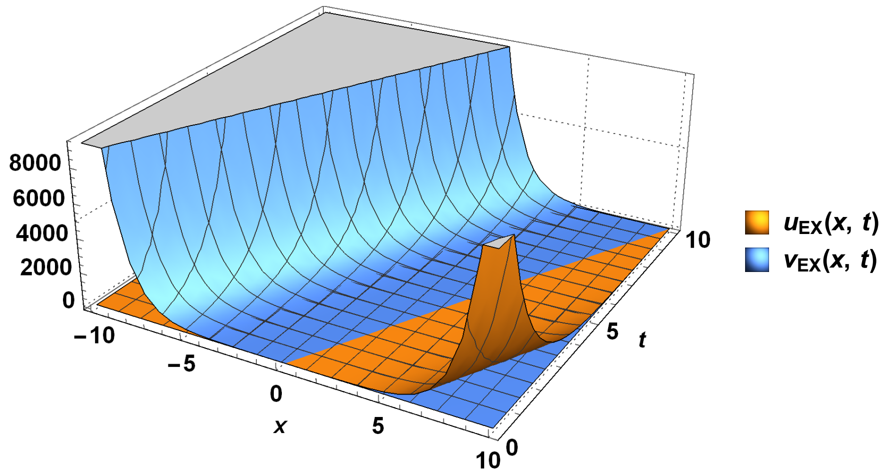

Fig. 4. The graph of the exact solution for Example 4.2.

5. Discussion and Conclusions

Table 1 shows the numerical comparison between the approximate solution and the exact solution for Example 4.1 among different values of x, t when $q = 0.5, q = 1$ and $\alpha = \beta = 0.5$. In Fig. 1, we plot the graph of the approximate solution for Example 4.1 when $\alpha = \beta = 0.5$ and $q = 1$. Fig. 2 presents the graph of the approximate solution for Example 4.1 when $\alpha = \beta = 0.5$. In Table 2, we evaluated the numerical values for the approximate and exact solutions for Example 4.2 among different values of x, t when $q_1 = q_2 = 0.5, q_1 = q_2 = 1$ to make a comparison between the approximate solution and the exact solution for Example 4.2. Fig. 3 presents the graph of the approximate solution for Example 4.2 when $q_1 = q_2 = 1$. In Fig. 4, we plot the graph of the exact solution for Example 4.2.

In this paper, a new modification of ADM for solving a fully general system of NFPDEs was introduced. We have seen that the approximate analytical and numerical solutions present in this paper were in very good conformity with the exact solutions that previously obtained by some other well-known methods to confirm the effectiveness and accuracy of this modification. We used Mathematica software to obtain the approximate solutions and plotting the graphs.

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