Int. J. Adv. Appl. Math. and Mech. 6(3) (2019) 35 – 41 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com
International Journal of Advances in Applied Mathematics and Mechanics

# Relaxed elastic line on an oriented surface in the Galilean space

**Research Article**

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**Received 18 January 2019; accepted (in revised version) 28 February 2019**

**Abstract:** In this paper, we consider the classical variational problem in the Galilean space. We develop the Euler-Lagrange equations for a elastic line on an oriented surface in the Galilean 3-dimensional space $G_3$. Using the variation method, we give some characterization for the solution curve (the elastic line) of energy equation described by the total squared curvature function of a curve on an oriented surface in $G_3$. Finally, we provide some characterizations related to elastic curves on some sample surfaces.

**MSC:** 53A35 • 53A55 • 49Q20

**Keywords:** Relaxed elastic line • Variational method • Galilean space



## 1. Introduction and Preliminaries

There has been in recent years a wide variety of applications of variational methods to various fields of mechanics and technology. Calculus of variation is an important theory which has common applications in geometry, analysis, physics, chemistry and engineering. Variation method is used to find the maxima or the minima of expressions involving unknown functions called functionals. Therefore, the main problem in the calculus of variations is to minimize (or maximize) not only functions but also functionals. For instance; a typical variational problem is to find the fixed length $l$ of a curve joining two points on a surface. The Euler-Lagrange equations is used for the solution of the problem.

There has been in recent years a wide variety of applications of variational methods to various geometries. For instance, several geometers were interested in studying of calculus of variations in Euclidean Space [1–4].

In literature, there is a few study on relaxed elastic line on an oriented surface in Galilean space $G_3$. Gökçe and Şahin in [5, 6] have obtained intrinsic equations of the relaxed elastic line on an oriented surface in $G_3$ for different energy functions. In this work, we investigate the same problem by using variation method.

Therefore, firstly, we will derive the Euler-Lagrange equations for a relaxed elastic line on an oriented surface in $G_3$. The curves that satisfy these equations along with the given conditions will be defined as relaxed elastic curves. We also present characterizations related to curves on some sample surfaces.

The main results of this paper are given in section 2.

In the nonhomogeneous coordinates the isometries group $B_6$ has the form

$$\begin{aligned}\overline{x} &= a + x\\ \overline{y} &= b + cx + y\cos\varphi + z\sin\varphi \\ \overline{z} &= d + ex - y\sin\varphi + z\cos\varphi\end{aligned} \tag{1}$$

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where $a, b, c, d, e$ and $\varphi$ are real numbers. The group of motions of $G_3$ is a six-parameter group [7].
The Galilean norm of the vector $\mathbf{v} = (x, y, z)$ defined by

$$\|\mathbf{v}\|_G = \begin{Bmatrix} x\ , & if\ \ x \neq 0 \\ \sqrt{y^2+z^2}, & if\ \ x = 0 \end{Bmatrix}. \tag{2}$$

A vector $\mathbf{v} = (x, y, z)$ is said to be non-isotropic if $x \neq 0$. All unit non-isotropic vectors are of the form $(1, y, z)$. For a curve $\alpha : I \to G_3$ , $I \subset \mathbb{R}$ parametrized by the arc-length parameter $s = x$, given in the coordinate form

$$\alpha(x) = (x, y(x), z(x)), \tag{3}$$

the curvature $\kappa(x)$ and the torsion $\tau(x)$ are defined by

$$\kappa(x) = \|\alpha''(x)\|_G = \sqrt{y''(x)^2 + z''(x)^2}, \tag{4}$$

$$\tau(x) = \frac{\det(\alpha'(x), \alpha''(x), \alpha'''(x))}{\kappa^2(x)} \tag{5}$$

and the associated moving trihedron is given by

$$\begin{aligned} T(x) &= \alpha'(x), \\ N(x) &= \frac{\alpha''(x)}{\kappa(x)}, \\ B(x) &= \frac{1}{\kappa(x)}(0, -z''(x), y''(x)). \end{aligned} \tag{6}$$

The vectors $T(x), N(x)$ and $B(x)$ are called the vectors of the tangent, principal normal and the binormal line, respectively [7, 8]. Therefore, the Frenet-Serret formulas can be written in matrix notation as

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix}' = \begin{bmatrix} 0 & \kappa & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix}. \tag{7}$$

Let $\mathbf{a} = (a_1, a_2, a_3)$, $\mathbf{b} = (b_1, b_2, b_3)$ be two vectors in $G_3$.

Therefore, the Galilean cross product $\times_G$ is defined by

$$\mathbf{a} \times_G \mathbf{b} = \begin{vmatrix} 0 & e_2 & e_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{8}$$

as in [9].

For more facts about the Galilean geometry, we refer the reader to [5–7, 9–12] and references therein.

Let $S$ be a surface in $G_3$ and $\alpha$ be a curve on $S$. At a point $\alpha(x)$ of $\alpha$, let $T$ denote the unit tangent vector of $\alpha$ at $\alpha(x)$, $n$ the unit normal to $S$ and $n \times_G T = Q$ the tangential-normal. Then $\{T, Q, n\}$ is an orthonormal basis at $\alpha(x)$ in $S$. This frame is called Galilean Darboux frame or tangent-normal frame. The Frenet-Serret formulas the for the Galilean Darboux frame can be written in matrix notation as

$$\begin{bmatrix} T \\ Q \\ n \end{bmatrix}' = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ 0 & 0 & \tau_g \\ 0 & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ Q \\ n \end{bmatrix} \tag{9}$$

where $\kappa_g$,$\kappa_n$, $\tau_g$ are geodesic curvature, normal curvature and geodesic torsion, respectively [6].

Let $\psi(u_1, u_2) = (\mathbf{X}(u_1, u_2), \mathbf{Y}(u_1, u_2), \mathbf{Z}(u_1, u_2))$ be parametrization of an oriented surface $S$ in $G_3$. Here $\mathbf{X}, \mathbf{Y}$ and $\mathbf{Z}$ are the coordinate functions of $\psi$ . In this case, a unit normal field of surface at a point $p$ is defined by

$$n = \frac{\psi_1 \times_G \psi_2}{\|\psi_1 \times_G \psi_2\|} = \frac{1}{W}(0, \mathbf{X}_1\mathbf{Z}_2 - \mathbf{X}_2\mathbf{Z}_1, \mathbf{X}_2\mathbf{Y}_1 - \mathbf{X}_1\mathbf{Y}_2), \tag{10}$$

where $\psi_i = \frac{\partial \psi}{\partial u_i}$, $W = \|\psi_{u_1} \times_G \psi_{u_2}\|$, $\mathbf{X}_i = \frac{\partial \mathbf{X}}{\partial u_i}$, $\mathbf{Y}_i = \frac{\partial \mathbf{Y}}{\partial u_i}$ and $\mathbf{Z}_i = \frac{\partial \mathbf{Z}}{\partial u_i}$, $i = 1, 2$. In a tangent plane of the surface at the point $p$, there is a unique isotropic direction defined by the condition $\mathbf{X}_1 du_1 + \mathbf{X}_2 du_2 = 0$. A side tangential vector $Q = \frac{1}{W}(\mathbf{X}_2\psi_1 - \mathbf{X}_1\psi_2)$ is a unit isotropic vector in a tangent plane. The curve $\alpha$ on the oriented surface $S$ is specified by $\psi(x) = \psi(u_1(x), u_2(x))$. Then the unit tangent vector along $\alpha$ is

$$T = \frac{d\psi}{dx} = \psi_1 \dot{u}_1 + \psi_2 \dot{u}_2, \tag{11}$$

where $\dot{u}_1 = \frac{du_1}{dx}$ and $\dot{u}_2 = \frac{du_2}{dx}$. The first fundamental form of a surface is introduced in the following way

$$dx^2 = (\mathbf{X}_1 du_1 + \mathbf{X}_2 du_2)^2 + \varepsilon \left\{ (\mathbf{Y}_1 du_1 + \mathbf{Y}_2 du_2)^2 + (\mathbf{Z}_1 du_1 + \mathbf{Z}_2 du_2)^2 \right\}, \tag{12}$$

where

$$\varepsilon = \left\{ \begin{array}{ll} 0, & dx \text{ is non-isotropic,} \\ 1, & dx \text{ is isotropic} \end{array} \right\}. \tag{13}$$

Since $T.T = 1$, we have a constraining relation between any pair of functions $(u_1(x), u_2(x))$ that define the curve $\alpha$ on the surface $S$:

$$g\left(u_1, u_2, \dot{u}_1, \dot{u}_2\right) = 1, \tag{14}$$

where

$$g = \sum_{i,j=1}^{2} g_{ij} \dot{u}_i \dot{u}_j, \tag{15}$$

and $g_1 = \mathbf{X}_1$, $g_2 = \mathbf{X}_2$ and $g_{ij} = g_i.g_j$, $i, j = 1,2$, where $g_{ij}$, $i, j = 1,2$, are the coefficients of the first fundamental form. The normal curvature can be expressed in terms of the coordinates $(u_1(x), u_2(x))$ along the curve $\alpha$ as

$$\kappa_n = \sum_{i,j=1}^{2} L_{ij} u_i' u_j', \tag{16}$$

where $L_{11}, L_{12}$ and $L_{22}$ are the coefficients of the second fundamental form $II$ and the normal components of $\psi_{11}, \psi_{12}$ and $\psi_{22}$, respectively. It holds [11]

$$L_{ij} = \frac{\mathbf{X}_2 \psi_{ij} - \mathbf{X}_{ij} \psi_2}{\mathbf{X}_2} \cdot n \text{ or } L_{ij} = \frac{\mathbf{X}_1 \psi_{ij} - \mathbf{X}_{ij} \psi_1}{\mathbf{X}_1} \cdot n. \tag{17}$$

Similarly, for the geodesic torsion $\tau_g$ we have

$$\tau_g = \sum_{i,j=1}^{2} g^i L_{ij} u_i', \tag{18}$$

where $g^1 = \frac{\mathbf{X}_2}{W}$, $g^2 = -\frac{\mathbf{X}_1}{W}$ and $g^{ij} = g^i.g^j$. On the other hand, the square geodesic curvature can be obtained as

$$\kappa_g^2 = \sum_{i,j=1}^{2} g_{ij} \gamma_i \gamma_j, \tag{19}$$

where

$$\gamma_i = \ddot{u}_i + \sum_{k,l=1}^{2} \Gamma_{kl}^i \dot{u}_k \dot{u}_l, i = 1,2 \tag{20}$$

and the quantities $\Gamma_{kl}^i$ are the Christoffel symbols of the second kind; available formulas are expressed as functions $g_{ij}$, and their first partial derivatives according to $u_i$. Furthermore, the Christoffel symbols can be written as follows:

$$\Gamma_{kl}^1 = \frac{\mathbf{X}_2 \psi_{ij} - \mathbf{X}_{ij} \psi_2}{W} \cdot Q,\ \Gamma_{kl}^2 = \frac{\mathbf{X}_1 \psi_{ij} - \mathbf{X}_{ij} \psi_1}{W} \cdot Q,\ k, l = 1,2. \tag{21}$$

Hence, the equations of a geodesic curve, which is characterized by identically vanishing $\kappa_g$, must be given by $\gamma_i = 0$, and indeed they are [11].

## 2. The incomplete and complete variational problems

### 2.1. The incomplete variational problem

**Definition 2.1.**
Let $\alpha$ be a $C^2$-curve with parametrized by arc-length $x$, $0 \le x \le \ell$, on an oriented surface $S$ in $G_3$. A relaxed elastic line of length $\ell$ is defined as a curve with associated energy

$$K = \int_0^\ell \kappa^2 dx, \tag{22}$$

where $\kappa^2$ is the square curvature of the curve $\alpha$ [1]. The integral $K$ is called the *total square curvature*.

If $\alpha$ is geodesic line, then the geodesic curvature $\kappa_g$ identically vanishes. When we substitute this into the following equation ,

$$\kappa^2 = \kappa_g^2 + \kappa_n^2$$

we obtain

$$K = \int_0^\ell \kappa^2 dx = \int_0^\ell \kappa_n^2 dx = K_n. \tag{23}$$

We call this problem *incomplete* because it seeks to minimize $K_n$, not the total square curvature $K$. If a curve that minimizes $K_n$ is a geodesic, then the curve minimizes $K$. Therefore, the relaxed elastic curve lie on a geodesic trajectory. Through $\kappa_n^2$ has a dependence on $u_1, u_2, \dot{u}_1$ and $\dot{u}_2$. Therefore, the Euler-Lagrange equations for the incomplete variational problem are

$$H_{u_1} - (H_{\dot{u}_1})^{\cdot} = 0, \tag{24}$$
$$H_{u_2} - (H_{\dot{u}_2})^{\cdot} = 0, \tag{25}$$

where

$$H = \kappa_n^2 + \lambda(g-1) \tag{26}$$

and $\lambda = \lambda(x)$ is a Lagrange multiplier function. Equations (24),(25) and (26) are a system of three equations to determine the three functions $u_1(x)$, $u_2(x)$ and $\lambda(x)$. Equations (24),(25) and (26) are second order in $u_1$, $u_2$ and first order in $\lambda$. Therefore, the system is fifth order. To get the system in normal form, Eq. (14) must be differentiated one times with respect to $x$. Reintegration of $g' = 0$ gives that the constant of integration equal unity. The other four constants of integrations in the general solution are determined by the boundary conditions.
We turn now to the boundary conditions at $x = 0$ and $x = \ell$ that determine the other four constants of integration of the equations. This conditions are

$$H_{\dot{u}_1} = H_{\dot{u}_2} = 0, \ (x = 0, \ell) \tag{27}$$

If the initial point of the elastic line is fixed say,

$$u_1(0) = u_{1_0}, \ u_2(0) = u_{2_0} \tag{28}$$

but the end of the elastic line is free, then

$$H_{\dot{u}_1} = H_{\dot{u}_2} = 0, \ (x = \ell). \tag{29}$$

Therefore, the four constants of integration are now determined by above two equations.

### 2.2. The complete variational problem

If a curve that minimises $K_n$ is not a geodesic, then the curve does or does not minimize $K$. Therefore, the incomplete variational problem is not anything say about the relaxed elastic curve that minimizes K. In this case, we have no choice but is to find the curve that minimises the integral

$$K = \int_0^\ell \kappa^2 dx = \int_0^\ell (\kappa_n^2 + \kappa_g^2) dx$$

The relaxed elastic curve that is not a geodesic curve is one of solutions of the complete variational problem. Through $\kappa_n^2 + \kappa_g^2$ has a dependence on $u_1, u_2, \dot{u}_1, \dot{u}_2, \ddot{u}_1$ and $\ddot{u}_2$. Therefore, the Euler-Lagrange equations for the complete variational problem are

$$H_{u_1} - (H_{\dot{u}_1})^{\cdot} + (H_{\ddot{u}_1})^{\cdot\cdot} = 0, \tag{30}$$
$$H_{u_2} - (H_{\dot{u}_2})^{\cdot} + (H_{\ddot{u}_2})^{\cdot\cdot} = 0, \tag{31}$$

where

$$H = \kappa^2 + \lambda\left(g-1\right) \tag{32}$$

We nevertheless find the following six boundary terms if Eq. (14) can be solved for $\dot{u}_2$ as a function of $u_1, u_2$, and $\dot{u}_1$ to give $\dot{u}_2 = U_2(u_1, u_2, \dot{u}_1)$:

$$\left\{H_{\ddot{u}_1} + H_{\ddot{u}_2}(U_2)_{\dot{u}_1}\right\}\delta\dot{u}_1 \qquad (s = 0, \ell) \tag{33}$$

$$\left\{H_{\dot{u}_1} - (H_{\ddot{u}_1})^{\cdot} + H_{\ddot{u}_2}(U_2)_{u_1}\right\}\delta u_1 \qquad (s = 0, \ell) \tag{34}$$

$$\left\{H_{\dot{u}_2} - (H_{\ddot{u}_2})^{\cdot} + H_{\ddot{u}_2}(U_2)_{u_2}\right\}\delta u_2 \qquad (s = 0, \ell) \tag{35}$$

On the other hand, if $U_{2\dot{u}_1}$ is singular, then we must use a different set of six boundary terms, after solving Eq. (14) for $\dot{u}_1 = U_1(u_1, u_2, \dot{u}_2)$:

$$\left\{H_{\ddot{u}_2} + H_{\ddot{u}_1}(U_1)_{\dot{u}_2}\right\}\delta\dot{u}_2 \qquad (s = 0, \ell) \tag{36}$$

$$\left\{H_{\dot{u}_2} - (H_{\ddot{u}_2})^{\cdot} + H_{\ddot{u}_1}(U_1)_{u_2}\right\}\delta u_2 \qquad (s = 0, \ell) \tag{37}$$

$$\left\{H_{\dot{u}_1} - (H_{\ddot{u}_1})^{\cdot} + H_{\ddot{u}_1}(U_1)_{u_1}\right\}\delta u_1 \qquad (s = 0, \ell) \tag{38}$$

Suppose that we place no restrictions at all on the elastic line other than confinement to the surface. Then six integration constants are determined by setting equal to zero each of the six factors multiplying $\delta u_1, \delta u_2$, and $\delta\dot{u}_1$ in the "$U_1$ boundary terms," or those multiplying $\delta u_1, \delta u_2$, and $\delta\dot{u}_2$ in the "$U_2$ boundary terms." These boundary conditions are *completely natural (free)* [1].

## 3. Elastic lines on some surfaces in Galilean space

### 3.1. Elastic lines on Galilean plane

Let $S$ be a Galilean plane. Then, for the coefficients of the first and second fundamental forms we have

$$g_{11} = 1, g_{12} = 0, g_{22} = 1$$
$$L_{11} = L_{12} = L_{22} = 0.$$

Thus, we obtain $\Gamma^j_{kl} = 0, 1 \le i, k, l \le 2$. Therefore, we can say that the normal curvature $\kappa_n$ and the geodesic torsion $\tau_g$ of any curve on $S$ vanish in all points of this curve from Eq.(16). In particular, any geodesic line on the plane minimizes $K_n$ (giving to it the value zero), Hence, the curve minimizes K. Therefore, the relaxed elastic lines on Galilean plane lies along a geodesic line.

It follows from Eq.(20) that the differential equations of geodesic curves on a plane become $u_1'' = u_2'' = 0$. The solutions of differential equations are straight lines $u_1 = ax + b, u_2 = cx + d, a^2 + c^2 = 1$. The relaxed elastic line on a plane assumes the form of a straight line.

### 3.2. Elastic lines on Galilean sphere

Non-isotropic Galilean unit sphere is defined a set of non-isotropic unit vectors in Galilean space. Therefore, Non-isotropic Galilean unit sphere is described as set of points $(\pm 1, y, z)$, i.e. two parallel plane given by $x = \pm 1$. The non-isotropic Galilean unit sphere with parametrization is obtained by $\varphi(u_1, u_2) = (\pm 1, u_1, u_2)$. Its first fundamental form is given by

$$ds^2 = du_1^2 + du_2^2.$$

Then, the coefficients of the first and second fundamental forms are

$$g_{11} = 1, g_{12} = 0, g_{22} = 1,$$
$$L_{11} = L_{12} = L_{22} = 0.$$

Thus, we obtain $\Gamma^j_{kl} = 0, 1 \le i, k, l \le 2$.

Therefore, we can say that the relaxed elastic line on the Galilean sphere assumes the form of a straight line.

### 3.3. Elastic lines on a cylinder

Let $M$ be the cylinder over a curve $C : f(y, z) = c$ in the $yz$ plane. If $\alpha = (0, \alpha_1, \alpha_2)$ is a parametrization of $C$, we assert that

$$\chi(u, v) = (u, \alpha_1(v), \alpha_2(v))$$

is a parametrization of $M$. If $\alpha = (0, R\cos(v/R), R\sin(v/R))$ is a parametrization of $C$, we have

$$\chi(u, v) = (u, R\cos(v/R), R\sin(v/R))$$

From Eq.(10 – 21), we obtain the following coefficients

$$g_{11} = 1, g_{12} = g_{22} = 0,$$
$$L_{11} = L_{12} = 0, L_{22} = R^{-1},$$
$$\Gamma^{j}_{kl} = 0.$$

where $1 \leq i, k, l \leq 2$. From Eq.(19), (20) and (21) the geodesic curvature along any curve $[u(x), v(x)]$ on cylinder is given by $\kappa_g = |\ddot{u}|$. It is a characteristic of geodesic curves that the geodesic curvature $\kappa_g$ vanishes identically along them. Therefore, the geodesic on a cylinder have $u(x) = ax + b$, $a^2 = 1$. Consequently, any curve $[u(x), v(x)]$ on cylinder is geodesic if and only if $u(x) = \pm x + b$.

From Eq.(16) the normal curvature along any curve $[u(x), v(x)]$ on the cylinder is given by

$$\kappa_n(x) = R^{-1}\dot{v}^2 \tag{39}$$

We obtain following $H$ for the incomplete variational problem from Eq.(15) and (26)

$$H = R^{-2}\dot{v}^4 + \lambda(\dot{u}^2 - 1).$$

Therefore, we get from the Euler equations (24) and (25), and also (27)

$$H_{\dot{u}} = H_{\dot{v}} = 0.$$

Therefore, we have the system of three differential equations for $u(x)$, $v(x)$, and $\lambda(x)$ is

$$2\lambda\dot{u} = 0, \tag{40}$$
$$4R^{-2}\dot{v}^3 = 0, \tag{41}$$
$$\dot{u}^2 = 1, \tag{42}$$

where Eq.(42) is side condition Eq.(14) for any curve on a cylinder. Hence, any solution of this system automatically satisfies the boundary conditions. From Eq.(42), we have $u(x) = \pm x + b$. We must choose $\lambda = 0$ and $v = const.$ to satisfy Eq.(40) and (41). Hence, all solutions of this system are any vertical generator $u(x) = \pm x + b$, $v = const.$ together with $\lambda = 0$. From Eq.(39) the total normal curvature is zero for a vertical generator, and this solution provides a minimum for $K_N$. In the same time, the vertical generator is a geodesic on cylinder. Consequently, we can say that minimum-energy trajectory on a cylinder adopts the form of a vertical generator. That is, a relaxed elastic line on cylinder is lying on a geodesic.

Now, we examine the complete variational problem. In this case, we have following $H$ for the complete variational problem from Eq.(32)

$$H = R^{-2}\dot{v}^4 + \ddot{u}^2 + \lambda(\dot{u}^2 - 1).$$

Therefore, we get from the Euler equations (30) and (31)

$$H_{\dot{u}} - (H_{\ddot{u}})' = const,$$
$$H_{\dot{v}} - (H_{\ddot{v}})' = const.$$

Therefore, we have the system of three differential equations for $u(x)$, $v(x)$, and $\lambda(x)$ is

$$\lambda\dot{u} - \dddot{u} = 0, \tag{43}$$
$$4R^{-2}\dot{v}^3 = 0, \tag{44}$$
$$\dot{u}^2 = 1. \tag{45}$$

## 4. Conclusion

In this article, we investigate the classical variational problem related to total square curvature of a curve on an oriented surface in the Galilean space. we also construct the Euler-Lagrange equations for a elastic line on an oriented surface in $G_3$.. Finally, we present some characterizations related to elastic curves on some sample surfaces and provide some elaborated examples.

## Acknowledgements

This study was supported financially by the Research Centre of Amasya University (Project No: FMB-BAP 18-0356).

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