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Approximation with a Kantorovich type Ibragimov-Gadjiev operator

Research Article

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Abstract: In this study, a Kantorovich type generalization of an Ibragimov-Gadjiev type operator on a variable bounded interval will be defined and important approximation properties will be examined. Since it contains visual and numerical elements, it can be considered as a remarkable study for real-life application areas that progress by approaching to functions.

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Approaching to functions with basic and useful operators includes a method that facilitates studies not only in mathematics but also in many fields of science. For this purpose, many operators have been defined and made available to researchers. Here are a few examples of such operators: [1], [2], [3], [4], [5], [6], [7], [8], [9].

By transferring the studies in the field of q-analysis to operator theory, q versions of many known operators were constructed. [10] also investigated the approximation properties of q-Laguerre type operators. Herdem and Buyukyazici give the important properties of the classical q-Ibragimov-Gadjiev operator in [11]. [12] also investigated the q modification of Bernstein-Chlodowsky polynomials in three-dimensional space.

Later, it was defined in (p,q) operators and brought to the literature. [13], [14], [15], [16] are a few examples of these operators.

Ibragimov-Gadjiev operator, one of the important operators in the literature, is one of the operators that researchers continue to work with. As an example of the studies on the operator, which was first defined in [17]: [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28].

After proving the necessary equations provided by the operator in the study and showing that it provides a Korovkin type theorem, the study was made applicable with the application part such as the rate of convergence calculation of the operator.

1. Preliminaries

Definition 1.1.

Let a_n and β_n be sequences of real numbers provides the following features

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$\lim_{n\to\infty}\beta_n = \infty$, $\lim_{n\to\infty}\dfrac{a_n}{\beta_n} = 0$, $\lim_{n\to\infty}\dfrac{n+1}{\beta_n+n+2} = 0$ and $\lim_{n\to\infty}\dfrac{a_n}{\beta_n+n+2}n = 1$. Also, let $M_{n,\theta}(x)$ be a function that satisfies the following conditions depending on θ and n parameters:

$i)$ For all $n,\theta \in \mathbb{N}_0$ and all $x \in \left[0, \frac{n+1}{n+2}\right]$,

$(-1)^{\theta} M_{n,\theta}(x) \geq 0.$

$ii)$ For all $x \in \left[0, \frac{n+1}{n+2}\right]$,

$\sum\limits_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^{\theta}}{\theta!} = 1.$

$iii)$ For every $x \in \left[0, \frac{n+1}{n+2}\right]$,

$M_{n,\theta}(x) = -nxM_{n+m,\theta-1}(x)$

Here, m is an integer where $n+m$ is a natural number.

With the help of this information, a generalization of the Gadjiev Ibragimov operator is defined by [29] as:

$\widetilde{G}_n(f,x) = \sum\limits_{\theta=0}^{\infty} f\left(\dfrac{\theta+n+1}{\beta_n+n+2}\right) M_{n,\theta}(x)\frac{(-a_n)^{\theta}}{\theta!}.$

Lemma 1.1.

The following equations are valid for the $\widetilde{G}_n(f,x)$ operator [29].

$i)$ $\widetilde{G}_n(1,x) = 1,$

$ii)$ $\widetilde{G}_n(t,x) = \frac{na_n x}{\beta_n+n+2} + \frac{n+1}{\beta_n+n+2},$

$iii)$ $\widetilde{G}_n(t^2,x) = \frac{a_n^2 n(n+m)}{(\beta_n+n+2)^2}x^2 + \frac{na_n(2n+3)}{(\beta_n+n+2)^2}x + \frac{n+1}{(\beta_n+n+2)^2} + \frac{(n+1)^2}{(\beta_n+n+2)^2}.$

Theorem 1.1.

For all $f \in \left[0, \frac{n+1}{n+2}\right]$ and

$\widetilde{G}_n(f,x) = \sum\limits_{\theta=0}^{\infty} f\left(\dfrac{\theta+n+1}{\beta_n+n+2}\right) M_{n,\theta}(x)\frac{(-a_n)^{\theta}}{\theta!},$

$\lim_{n\to\infty} ||\widetilde{G}_n(f,x) - f(x)||_{C\left[0,\frac{n+1}{n+2}\right]} = 0.$

Theorem 1.2.

For all $f \in \left[0, \frac{n+1}{n+2}\right]$, for a sufficiently large number of n and a constant M independent of the sequences $(a_n), (\beta_n)$,

$||\widetilde{G}_n(f,x) - f(x)||_{C\left[0,\frac{n+1}{n+2}\right]} \leq M\omega\{f;\delta_n\}$

inequality is valid. Here,

$\delta_n := \sqrt{\left(\frac{a_n n}{n} - 1\right)^2 + \left(\frac{n+1}{n+2} - 1\right)^2}.$

2. Materials and Methods

Definition 2.1.

Let $f : L_1\left[0, \frac{n+1}{n+2}\right] \to C\left[0, \frac{n+1}{n+2}\right]$.The operator defined as

$$\widetilde{\widetilde{G}}_n(f,x) = (\beta_n+n+2)\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^{\theta}}{\theta!}\int_{\frac{\theta+n+1}{\beta_n+n+2}}^{\frac{\theta+n+2}{\beta_n+n+2}} f(p)\,dp$$

is called the Kantorovich generalization of the Gadjiev-Ibragimov type an operator. Here $M_{n,\theta}(x)$; satisfies the conditions in Definition 1.1

Theorem 2.1.

Let $f : L_1\left[0, \frac{n+1}{n+2}\right] \to C\left[0, \frac{n+1}{n+2}\right]$ and for operators

$$\widetilde{\widetilde{G}}_n(f,x) = (\beta_n+n+2)\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^{\theta}}{\theta!}\int_{\frac{\theta+n+1}{\beta_n+n+2}}^{\frac{\theta+n+2}{\beta_n+n+2}} f(p)\,dp$$

the following equations are provided.

$i)$ $\widetilde{\widetilde{G}}_n(1,x) = 1,$

$ii)$ $\widetilde{\widetilde{G}}_n(t,x) = \frac{na_n x}{\beta_n+n+2} + \frac{2n+3}{2(\beta_n+n+2)},$

$iii)$ $\widetilde{\widetilde{G}}_n(t^2,x) = \frac{a_n^2 n(n+m)}{(\beta_n+n+2)^2}x^2 + \frac{(2n^2a_n+4na_n)}{(\beta_n+n+2)^2}x + \frac{3n^2+9n+7}{3(\beta_n+n+2)^2}.$

Proof. From the definition of the operator

$i)\ \tilde{\tilde{G}}_n(1,x) = (\beta_n+n+2)\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\int_{\frac{\theta+n+1}{\beta_n+n+2}}^{\frac{\theta+n+2}{\beta_n+n+2}} 1dp$

$= (\beta_n+n+2)\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\left(\frac{1}{\beta_n+n+2}\right) = 1$

equation is obtained.

$ii)$ Description of the operator and from Lemma 1.1 $i)$ and $ii)$ using the $(\theta\to\theta+1)$ transform

$$\tilde{\tilde{G}}_n(t,x) = (\beta_n+n+2)\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\int_{\frac{\theta+n+1}{\beta_n+n+2}}^{\frac{\theta+n+2}{\beta_n+n+2}} pdp$$

$$= \frac{(\beta_n+n+2)}{2}\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\left[\left(\frac{\theta+n+2}{\beta_n+n+2}\right)^2-\left(\frac{\theta+n+1}{\beta_n+n+2}\right)^2\right]$$

$$= \sum_{\theta=0}^{\infty}\left(\frac{2\theta+2n+3}{2(\beta_n+n+2)}\right) M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}$$

$$= \sum_{\theta=1}^{\infty}\left(\frac{-nx}{\beta_n+n+2}\right) M_{n+m,\theta-1}(x)\frac{(-a_n)^\theta}{(\theta-1)!}+\left(\frac{2n+3}{2(\beta_n+n+2)}\right) \qquad (\theta\to\theta+1)$$

$$= \frac{na_nx}{\beta_n+n+2}+\frac{2n+3}{2(\beta_n+n+2)}$$

is obtained.

$$iii)\ \tilde{\tilde{G}}_n(t^2,x) = (\beta_n+n+2)\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\int_{\frac{\theta+n+1}{\beta_n+n+2}}^{\frac{\theta+n+2}{\beta_n+n+2}} p^2dp$$

$$= \frac{(\beta_n+n+2)}{3}\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\left[\left(\frac{\theta+n+2}{\beta_n+n+2}\right)^3-\left(\frac{\theta+n+1}{\beta_n+n+2}\right)^3\right]$$

$$= \frac{(\beta_n+n+2)}{3}\sum_{\theta=0}^{\infty} M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\left(\frac{1}{\beta_n+n+2}\right)\left[\left(\frac{3\theta^2+6\theta n+9\theta+3n^2+9n+7}{(\beta_n+n+2)^2}\right)\right]$$

$$= \sum_{\theta=0}^{\infty}\frac{\theta(\theta-1)}{(\beta_n+n+2)^2}M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\frac{2\theta n}{(\beta_n+n+2)^2}M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}$$
$$+\sum_{\theta=0}^{\infty}\frac{4\theta}{(\beta_n+n+2)^2}M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}$$
$$= \frac{a_n^2n(n+m)}{(\beta_n+n+2)^2}x^2+\frac{(2n^2a_n+4na_n)}{(\beta_n+n+2)^2}x+\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}$$

is achived.Thus the theorem is proven. □

Theorem 2.2.
For $\tilde{\tilde{G}}_n(f,x)$ and all $f\in[0,\frac{n+1}{n+2}]$,

$\lim_{n\to\infty}||\tilde{\tilde{G}}_n(f,x)-f(x)||_{C[0,\frac{n+1}{n+2}]}=0$

is valid.

Proof. The equations in Theorem 2.1 will be used for proof, we have

$\lim_{n\to\infty}\left\|\tilde{\tilde{G}}_n(1,x)-1\right\|_{C[0,\frac{n+1}{n+2}]}=0.$

From Theorem 2.1 ii)

$$\left|\tilde{\tilde{G}}_n(t,x)-x\right| = \left|x\left(\frac{na_n}{\beta_n+n+2}-1\right)\right|+\left|\frac{2n+3}{2(\beta_n+n+2)}\right|$$

can be written. Here since $x\in[0,\frac{n+1}{n+2}]$ and $\lim_{n\to\infty}\frac{a_n}{\beta_n+n+2}n=1$, then

$\max_{x\in[0,\frac{n+1}{n+2}]}\left|\tilde{\tilde{G}}_n(t,x)-x\right| \le \left|\frac{n+1}{n+2}\right|\left|\frac{na_n}{\beta_n+n+2}-1\right|+\left|\frac{2n+3}{2(\beta_n+n+2)}\right|$

inequality is valid. If the limit of both sides is taken since

$$\lim_{n\to\infty}\left\|\overset{\approx}{G}_n(t,x)-x\right\|_{C[0,\frac{n+1}{n+2}]}\le\lim_{n\to\infty}\left|\frac{n+1}{n+2}\right|\lim_{n\to\infty}\left|\frac{na_n}{\beta_n+n+2}-1\right|+\lim_{n\to\infty}\left|\frac{2n+3}{2(\beta_n+n+2)}\right|$$

$$\lim_{n\to\infty}\left\|\overset{\approx}{G}_n(t,x)-x\right\|_{C[0,\frac{n+1}{n+2}]}=0$$

is written. Here, if we use

$$\left|\overset{\approx}{G}_n(t^2,x)-x^2\right|=\left|\frac{a_n^2n(n+m)x^2}{(\beta_n+n+2)^2}+\frac{(2n^2a_n+4na_n)}{(\beta_n+n+2)^2}x+\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}-x^2\right|,$$

then

$$\max_{x\in[0,\frac{n+1}{n+2}]}\left|\overset{\approx}{G}_n(t^2,x)-x^2\right|\le\left(\frac{n+1}{n+2}\right)^2\left|\frac{a_n^2n(n+m)}{(\beta_n+n+2)^2}-1\right|$$

$$+\left|\frac{(2n^2a_n+4na_n)}{(\beta_n+n+2)^2}\right|\left|\frac{n+1}{n+2}\right|+\left|\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}\right|$$

is obtained. Using

$$\lim_{n\to\infty}\frac{a_n^2n(n+m)}{(\beta_n+n+2)^2}=\lim_{n\to\infty}\frac{a_n^2n^2}{(\beta_n+n+2)^2}+\lim_{n\to\infty}\frac{a_n^2nm}{(\beta_n+n+2)^2}=1$$

$$\lim_{n\to\infty}\frac{2na_n(n+2)}{(\beta_n+n+2)^2}\left(\frac{n+1}{n+2}\right)=0\text{ and }\lim_{n\to\infty}\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}=0\,,$$

then we get

$$\lim_{n\to\infty}||\overset{\approx}{G}_n(t^2,x)-x^2||_{C[0,\frac{n+1}{n+2}]}=0.$$

Hence using the Korovkin Theorem, we get

$$\lim_{n\to\infty}||\overset{\approx}{G}_n(f,x)-f(x)||_{C[0,\frac{n+1}{n+2}]}=0.$$ □

Theorem 2.3.
Let $f\in C_\rho^0[0,\infty)$. Then

$$\lim_{n\to\infty}||\overset{\approx}{G}_n(f,x)-f(x)||_{\rho,[0,\frac{n+1}{n+2}]}=0$$

is valid.

Proof. Let $\frac{x}{1+x^2}<1$, the demonstration of convergence for the test functions is sufficient for proof.
Firstly, it can be easily shown that

$$||\overset{\approx}{G}_n(1,x)-1||_{\rho,[0,\frac{n+1}{n+2}]}=\lim_{n\to\infty}\sup_{x\in[0,\frac{n+1}{n+2}]}\frac{|\overset{\approx}{G}_n(1,x)-1|}{1+x^2}=0$$

Similarly, from the definition of the operator

$$\lim_{n\to\infty}||\overset{\approx}{G}_n(t,x)-x||_{\rho,[0,\frac{n+1}{n+2}]}\le\lim_{n\to\infty}\left(\frac{na_n}{\beta_n+n+2}-1+\frac{2n+3}{2(\beta_n+n+2)}\right)=0,$$

then

$$\lim_{n\to\infty}||\overset{\approx}{G}_n(t,x)-x||_{\rho,[0,\frac{n+1}{n+2}]}=0$$

is valid. Finally, if we take the limit of both sides of the

$$||\overset{\approx}{G}_n(t^2,x)-x^2||_{\rho,[0,\frac{n+1}{n+2}]}\le\frac{a_n^2n(n+m)}{(\beta_n+n+2)^2}+\frac{(2n^2a_n+4na_n)}{(\beta_n+n+2)^2}+\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}.$$

So, it will be that

$$\lim_{n\to\infty}||\overset{\approx}{G}_n(t^2,x)-x^2||_{\rho,[0,\frac{n+1}{n+2}]}=\lim_{n\to\infty}\left(\frac{a_n^2n(n+m)}{(\beta_n+n+2)^2}\right)$$

$$+\lim_{n\to\infty}\left(\frac{(2n^2a_n+4na_n)}{(\beta_n+n+2)^2}\right)+\lim_{n\to\infty}\left(\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}\right)$$

Therefore,

$$\lim_{n\to\infty}||\overset{\approx}{G}_n(f,x)-f(x)||_{\rho,[0,\frac{n+1}{n+2}]}=0$$

is obtained. Thus, the proof is complete. □

Lemma 2.1.

The following equations are valid for $\widetilde{\widetilde{G}}_n(f,x)$.

$$i)\widetilde{\widetilde{G}}_n(t^3,x)=\frac{a_n^3 n(n+m)(n+2m)}{(\beta_n+n+2)^3}x^3+\left(\frac{3n^2a_n^2(n+m)}{(\beta_n+n+2)^3}+\frac{30a_n^2 n(n+m)}{4(\beta_n+n+2)^3}\right)x^2$$
$$+\left(\frac{(3n^3a_n+12n^2a_n)}{(\beta_n+n+2)^3}+\frac{50a_n n}{4(\beta_n+n+2)^3}\right)x+\frac{4n^3+18n^2+28n+15}{4(\beta_n+n+2)^3},$$

$$ii)\widetilde{\widetilde{G}}(t^4,x)=\frac{a_n^4 n(n+m)(n+2m)(n+3m)}{(\beta_n+n+2)^4}x^4+\frac{(4n+12)a_n^3 n(n+m)(n+2m)}{(\beta_n+n+2)^4}x^3$$
$$+\frac{(6n^2+30n+39)a_n^2 n(n+m)}{(\beta_n+n+2)^4}x^2+\frac{(4n^3+24n^2+50n+36)a_n n}{(\beta_n+n+2)^4}x$$
$$+\frac{(5n^4+30n^3+70n^2+75n+31)}{5(\beta_n+n+2)^4}.$$

Using the equations
$a^4-b^4=(a-b)(a+b)(a^2+b^2),\theta^3=\theta(\theta-1)(\theta-2)+3\theta^2-2\theta$ *and*
$\theta^2=\theta(\theta-1)+\theta$

then we write

$$\widetilde{\widetilde{G}}_n(t^3,x)=\sum_{\theta=0}^{\infty}M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\left[\frac{(4\theta(\theta-1)(\theta-2)+12n\theta(\theta-1))+30\theta(\theta-1)}{4(\beta_n+n+2)^3}\right]$$
$$+\sum_{\theta=0}^{\infty}M_{n,\theta}(x)\frac{(-a_n)^\theta}{\theta!}\left[\frac{12\theta n^2+48\theta n+50\theta+4n^3+18n^2+28n+15}{4(\beta_n+n+2)^3}\right]$$
$$=\frac{a_n^3 n(n+m)(n+2m)}{(\beta_n+n+2)^3}x^3+\left(\frac{3n^2a_n^2(n+m)}{(\beta_n+n+2)^3}+\frac{30a_n^2 n(n+m)}{4(\beta_n+n+2)^3}\right)x^2$$
$$+\left(\frac{(3n^3a_n+12n^2a_n)}{(\beta_n+n+2)^3}+\frac{50a_n n}{4(\beta_n+n+2)^3}\right)x+\frac{4n^3+18n^2+28n+15}{4(\beta_n+n+2)^3}$$

is written.
$c^5-d^5=(c+d)(c^4-d^4)-cd(c^3-d^3)$,
$\theta^4=\theta(\theta-1)(\theta-2)(\theta-3)+6\theta^3-11\theta^2+6\theta$,

$\theta^3=\theta(\theta-1)(\theta-2)+3\theta^2-2\theta$,
$\theta^2=\theta(\theta-1)+\theta$.
equations and using the equation

$$5\theta^4+20\theta^3n+30\theta^3+30\theta^2n^2+90\theta^2n+70\theta^2+20\theta n^3+90\theta n^2+140\theta n+75\theta$$
$$+5n^4+30n^3+70n^2+75n+31=5\theta(\theta-3)(\theta-2)(\theta-1)+60\theta(\theta-1)(\theta-2)$$
$$+195\theta(\theta-1)+180\theta+30n^2\theta(\theta-1)+20n\theta(\theta-1)(\theta-2)+150n\theta(\theta-1)$$
$$+20n^3\theta+120n^2\theta+250n\theta+5n^4+30n^3+70n^2+75n+31$$

then

$$\widetilde{\widetilde{G}}_n(t^4,x)=\sum_{\theta=0}^{\infty}\tfrac{\theta(\theta-1)(\theta-2)(\theta-3)}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}$$
$$+\sum_{\theta=0}^{\infty}\tfrac{4n\theta(\theta-1)(\theta-2)}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\tfrac{12\theta(\theta-1)(\theta-2)}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}$$
$$+\sum_{\theta=0}^{\infty}\tfrac{6n^2\theta(\theta-1)}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\tfrac{30n\theta(\theta-1)}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}$$
$$+\sum_{\theta=0}^{\infty}\tfrac{39\theta(\theta-1)}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\tfrac{4n^3\theta}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}$$
$$+\sum_{\theta=0}^{\infty}\tfrac{24n^2\theta}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\tfrac{50n\theta}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}$$
$$+\sum_{\theta=0}^{\infty}\tfrac{36\theta}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}+\sum_{\theta=0}^{\infty}\tfrac{5n^4+30n^3+70n^2+75n+31}{(\beta_n+n+2)^4}M_{n,\theta}(x)\tfrac{(-a_n)^\theta}{\theta!}$$

The desired equality is obtained from the equations.

In the following Lemma, some of the central moments of the operator $\widetilde{\widetilde{G}}_n(f,x)$ *are calculated.*

Lemma 2.2.

The following equations are valid for the first four central moments of the operator $\tilde{\tilde{G}}_n(f,x)$.

$i)$ $\tilde{\tilde{G}}_n\left((t-x)^0,x\right)=1$

$ii)$ $\tilde{\tilde{G}}_n\left((t-x)^1,x\right)=\frac{na_n-\beta_n+2+n}{\beta_n+2+n}x+\frac{2n+3}{2(\beta_n+2+n)}$

$iii)$ $\tilde{\tilde{G}}_n\left((t-x)^2,x\right)=\left(\frac{a_n^2n(n+m)}{(\beta_n+n+2)^2}-\frac{2na_n}{\beta_n+n+2}+1\right)x^2+\left(\frac{(2n^2a_n+4na_n)}{(\beta_n+n+2)^2}-\frac{2n+3}{\beta_n+n+2}\right)x+\frac{3n^2+9n+7}{3(\beta_n+n+2)^2}$

$$iv)\tilde{\tilde{G}}_n\left((t-x)^3,x\right)=\left(\frac{a_n^3n(n+m)(n+2m)}{(\beta_n+n+2)^3}-\frac{3na_n}{\beta_n+n+2}+\frac{3a_n^2n(n+m)}{(\beta_n+n+2)^2}-1\right)x^3$$
$$+\left(\frac{3n^2a_n^2(n+m)}{(\beta_n+n+2)^3}+\frac{30a_n^2n(n+m)}{4(\beta_n+n+2)^3}+\frac{9+6n}{2(\beta_n+2+n)}+\frac{(12na_n+6n^2a_n)}{(\beta_n+2+n)^2}\right)x^2$$
$$+\left(\frac{(3n^3a_n+12n^2a_n)}{(\beta_n+n+2)^3}+\frac{50a_nn}{4(\beta_n+n+2)^3}+\frac{3n^2+9n+7}{(\beta_n+n+2)^2}\right)x+\frac{4n^3+18n^2+28n+15}{4(\beta_n+n+2)^3}$$

$$v)\tilde{\tilde{G}}_n\left((t-x)^4,x\right)=\left(\frac{a_n^4n(n+m)(n+2m)(n+3m)}{(\beta_n+n+2)^4}-\frac{4a_n^3n(n+m)(n+2m)}{(\beta_n+n+2)^3}+\frac{6a_n^2n(n+m)}{(\beta_n+n+2)^2}-\frac{4na_n}{\beta_n+n+2}+1\right)x^4$$
$$+\left(\frac{(4n+12)a_n^3n(n+m)(n+2m)}{(n+\beta_n+2)^4}-\frac{12n^2a_n^2(n+m)+30a_n^2n(n+m)}{(n+\beta_n+2)^3}+\frac{(12n^2a_n+24na_n)}{(n+\beta_n+2)^2}-\frac{4n+6}{n+\beta_n+2}\right)x^3$$
$$+\left(\frac{(6n^2+30n+39)a_n^2n(n+m)}{(n+\beta_n+2)^4}-\frac{(12n^3a_n+48n^2a_n)+50a_nn}{(n+\beta_n+2)^3}+\frac{6n^2+18n+14}{(n+\beta_n+2)^2}\right)x^2$$
$$+\left(\frac{(4n^3+24n^2+50n+36)a_nn}{(n+\beta_n+2)^4}-\frac{4n^3+18n^2+28n+15}{(n+\beta_n+2)^3}\right)x$$
$$+\frac{(5n^4+30n^3+70n^2+75n+31)}{5(\beta_n+n+2)^4}.$$

Theorem 2.4.

For all $f\in C_\rho[0,\infty]$,

$$\sup_{x\in[0,\frac{n+1}{n+2}]}\frac{\left|\tilde{\tilde{G}}_n(f,x)-f(x)\right|}{(1+x^2)^2}\le M\Omega(f;\delta_n)$$

inequality is provided. Here δ_n *is defined as* $\delta_n=\sqrt[4]{\left(\frac{a_nn}{\beta_n+n+2}-1\right)^4}$.

Proof. From the linearity and monotony
$\left|\tilde{\tilde{G}}_n(f,x)-f(x)\right|\le\tilde{\tilde{G}}_n\left(\left|f(t)-f(x)\right|;x\right)$
is valid. If it is defined as
$T_n(t,x)=\left(1+\frac{|t-x|}{\delta_n}\right)\left(1+(t-x)^2\right)$,
then
$\left|f(t)-f(x\right|\le4\left(1+\delta_n^2\right)\Omega\left(f,\delta_n\right)\left(1+x^2\right)T_n(t,x)$
is written. For $T_n(t,x)$,

$$T_n(t,x)\le\begin{cases}4\left(1+\delta_n^2\right),|t-x|<\delta_n\\4\left(1+\delta_n^2\right)\frac{(t-x)^4}{\delta_n^4},|t-x|\ge\delta_n\end{cases}$$

inequality is valid. From here using $\left[\frac{(t-x)^4}{\delta_n^4}\right]\ge1$, it is written
$T_n(t,x)\le4\left(1+\delta_n^2\right)\left[\frac{(t-x)^4}{\delta_n^4}\right]$
So,
$$\left|\tilde{\tilde{G}}_n(f,x)-f(x)\right|\le\tilde{\tilde{G}}_n\left(\left|f(t)-f(x)\right|;x\right)\le4\left(1+\delta_n^2\right)\Omega\left(f,\delta_n\right)\left(1+x^2\right)\tilde{\tilde{G}}_n\left(T_n(t,x),x\right)$$
$$\le4\left(1+\delta_n^2\right)\Omega\left(f,\delta_n\right)\left(1+x^2\right)\left[1+\frac{1}{\delta_n^4}\tilde{\tilde{G}}_n\left((t-x)^4,x\right)\right]$$
is obtained. If both sides are divided by $\left(1+x^2\right)^2$ and the supremum is taken on $[0,\frac{n+1}{n+2}]$
$$\sup_{x\in[0,\frac{n+1}{n+2}]}\frac{\left|\tilde{\tilde{G}}_n(f,x)-f(x)\right|}{(1+x^2)^2}\le4\left(1+\delta_n^2\right)\sup_{x\in[0,\frac{n+1}{n+2}]}\frac{\Omega\left(f,\delta_n\right)}{1+x^2}\left[\sup_{x\in[0,\frac{n+1}{n+2}]}\frac{1}{(1+x^2)^2}+\sup_{x\in[0,\frac{n+1}{n+2}]}\frac{\left|\tilde{\tilde{G}}_n((t-x)^4;x)\right|}{(1+x^2)^2}\right]$$
is found. Since the equation for $\tilde{\tilde{G}}_n\left((t-x)^4,x\right)$ is valid,

$$\sup_{x\in\left[0,\frac{n+1}{n+2}\right]}\frac{\left|\tilde{\tilde{G}}_n\left((t-x)^4;x\right)\right|}{\left(1+x^2\right)^2}\le\Big(\frac{a_n^4 n(n+m)(n+2m)(n+3m)}{\left(\beta_n+n+2\right)^4}-\frac{4a_n^3 n(n+m)(n+2m)}{(\beta_n+n+2)^3}$$
$$+\frac{6a_n^2 n(n+m)}{(\beta_n+n+2)^2}-\frac{4na_n}{\beta_n+n+2}+1\Big)+\Big(\frac{(4n+12)\,a_n^3 n(n+m)(n+2m)}{(\beta_n+n+2)^4}$$
$$-\frac{12n^2a_n^2(n+m)+30a_n^2 n(n+m)}{(\beta_n+n+2)^3}+\frac{\left(12n^2a_n+24na_n\right)}{(\beta_n+n+2)^2}-\frac{4n+6}{\beta_n+n+2}\Big)$$
$$+\left(\frac{\left(6n^2+30n+39\right)a_n^2 n(n+m)}{(\beta_n+n+2)^4}-\frac{\left(12n^3a_n+48n^2a_n\right)+50a_n n}{(\beta_n+n+2)^3}+\frac{6n^2+18n+14}{\left(\beta_n+n+2\right)^2}\right)$$
$$+\left(\frac{\left(4n^3+24n^2+50n+36\right)a_n n}{(\beta_n+n+2)^4}-\frac{4n^3+18n^2+28n+15}{(\beta_n+n+2)^3}\right)$$
$$+\frac{\left(5n^4+30n^3+70n^2+75n+31\right)}{5(\beta_n+n+2)^4}$$
$$\le\frac{a_n^4(n^4+6n^3m+11n^2m^2+6nm^3)}{\left(\beta_n+n+2\right)^4}-\frac{4a_n^3\left(n^3+3n^2m+2nm^2\right)}{(\beta_n+n+2)^3}+\frac{6a_n^2\left(n^2+nm\right)}{\left(\beta_n+n+2\right)^2}$$
$$-\frac{4na_n}{\beta_n+n+2}+1\Big)+\Big(\frac{a_n^3\left(12n^3+36n^2m+24nm^2\right)}{(\beta_n+n+2)^4}+\frac{a_n^3\left(4n^4+12n^3m+8n^2m^2\right)}{\left(\beta_n+n+2\right)^4}$$
$$-\frac{12a_n^2\left(n^3+n^2m\right)+30a_n^2\left(n^2+nm\right)}{(\beta_n+n+2)^3}+\frac{\left(12n^2a_n+24na_n\right)}{(\beta_n+n+2)^2}-\frac{4n+6}{\beta_n+n+2}\Big)$$
$$+\Big(\frac{\left(6n^4+30n^3+39n^2\right)a_n^2}{(\beta_n+n+2)^4}+\frac{\left(6n^3m+30n^2m+39nm\right)a_n^2}{(\beta_n+n+2)^4}\frac{\left(12n^3a_n+48n^2a_n\right)+50a_n n}{(\beta_n+n+2)^3}$$
$$+\frac{6n^2+18n+14}{(\beta_n+n+2)^2}\Big)+\left(\frac{\left(4n^4+24n^3+50n^2+36n\right)a_n}{(\beta_n+n+2)^4}-\frac{4n^3+18n^2+28n+15}{(\beta_n+n+2)^3}\right)$$
$$+\frac{\left(5n^4+30n^3+70n^2+75n+31\right)}{5(\beta_n+n+2)^4}$$

is written. So,

$$\sup_{x\in\left[0,\frac{n+1}{n+2}\right]}\frac{\left|\tilde{\tilde{G}}_n\left((t-x)^4;x\right)\right|}{\left(1+x^2\right)^2}\le\left(\frac{a_n n}{\beta_n+n+2}\right)^4-4\left(\frac{a_n n}{\beta_n+n+2}\right)^3+6\left(\frac{a_n n}{\beta_n+n+2}\right)^2$$
$$-\frac{4a_n n}{\beta_n+n+2}+1\le\left(\frac{a_n n}{\beta_n+n+2}-1\right)^4$$

is obtained. Here, if we get $\delta_n=\sqrt[4]{\left(\frac{a_n n}{\beta_n+n+2}-1\right)^4}$ then,

$$\sup_{x\in\left[0,\frac{n+1}{n+2}\right]}\frac{\left|\tilde{\tilde{G}}_n\left(f,x\right)-f(x)\right|}{\left(1+x^2\right)^2}\le 4\left(1+\delta_n^2\right)\Omega\left(f,\delta_n\right)\left[1+\sup_{x\in\left[0,\frac{n+1}{n+2}\right]}\frac{\left|\tilde{\tilde{G}}_n\left((t-x)^4;x\right)\right|}{\left(1+x^2\right)^2}\right].$$

Thus, since

$$\sup_{x\in\left[0,\frac{n+1}{n+2}\right]}\frac{\left|\tilde{\tilde{G}}_n\left(f,x\right)-f(x)\right|}{\left(1+x^2\right)^2}\le M\Omega\left(f;\sqrt[4]{\left(\frac{a_n n}{\beta_n+n+2}-1\right)^4}\right)$$

proof is complete. □

Example 2.1.

$$\tilde{\tilde{G}}_n\left(f,x\right)=\left(\beta_n+n+2\right)\sum_{\theta=0}^{\infty}M_{n,\theta}(x)\,\frac{(-a_n)^\theta}{\theta!}\int_{\frac{\theta+n+1}{\beta_n+n+2}}^{\frac{\theta+n+2}{\beta_n+n+2}}f\left(p\right)dp$$

$n=15$, $m=20$ and for $x\in\left[0,\frac{n+1}{n+2}\right]$, let $M_{n,\theta}(x)=(-1)^\theta(nx)^\theta e^{-nxa_n}$, $(a_n)=n$ and $\left(\beta_n\right)=n^2$.In this case, the graph of the operator's approximation to the $f(x)=\frac{1}{20}\frac{\sin(x^2)}{(x+1)^2\sqrt{x^2+1}}$ is given in Fig. 1. The drawing is made for $\tilde{\tilde{G}}_9\left(f,x\right)$ in green, $\tilde{\tilde{G}}_{10}\left(f,x\right)$ in red, $\tilde{\tilde{G}}_{11}\left(f,x\right)$ in magenta, $\tilde{\tilde{G}}_{12}\left(f,x\right)$ in cyan, $f(x)$ in blue.

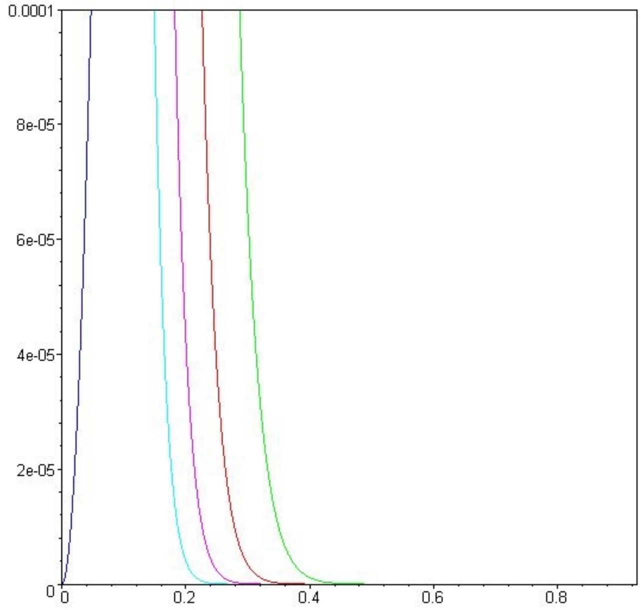

Fig. 1. Approximation to the $f(x) = \frac{1}{20}\frac{\sin(x^2)}{(x+1)^2\sqrt{x^2+1}}$

3. Conclusions

In this study, which is defined on a mobile range and examined the important approach features of the operator; It is thought that it will shape their studies by approaching the functions and will guide the researchers who cannot work on a fixed interval.

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Conflict of Interests

The authors stated that there are no conflict of interest in this article.

Authors Contribution

All authors read and approved the final manuscript.

References

[1] I. Buyukyazici, Direct and converse theorems for generalized Bernstein polynomials based on the q-integers, Journal of Applied Functional Analysis 7 (1-2) (2012) 81-91.
[2] I. Buyukyazici, E. Ibikli, Inverse theorems for Bernstein-Chlodowsky type polynomials, Journal of Mathematics of Kyoto University 46 (1) (2006) 21-29.
[3] U.D. Kantar, I. Yuksel, Investigating (p, q)-hybrid Durrmeyer-type operators in terms of their approximation properties, Gazi University Journal of Science Part A: Engineering and Innovation 9 (1) (2022) 1-11.
[4] I. Buyukyazici, C. Atakut, On the order of approximation of unbounded functions by the modified rational type Baskakov operators, Gazi University Journal of Science 31 (2) (2018) 569-576.
[5] S. Karateke, C. Atakut, I. Buyukyazici, Approximation by generalized integral Favard-Szász type operators involving Sheffer polynomials, Filomat 33 (7) (2019) 1921-1935.
[6] A. Kumar, D.K. Artee, Approximation properties of general gamma type operators in polynomial weighted space, Int. J. Adv. Appl. Math. and Mech. 4 (3) (2017) 7-134.

[7] I. Altıntas, K. Taskopru, M. Beyaztas, Bracket polynomials of torus links as Fibonacci polynomials, Int. J. Adv. Appl. Math. and Mech. 5 (3) (2018) 35-43.
[8] D.K. Artee, Approximation by modified Gamma type operators, Int. J. Adv. Appl. Math. Mech. 5 (4) (2018) 12-19.
[9] N.G. Bilgin, Approximation properties of modified Gadjiev Ibragimov operators in weighted spaces, ICJMS 2011 Conference, The 24th Congress of The Jangjeon Mathematical Society (2011).
[10] M. Ozarslan, q-Laguerre type linear positive operators, Studia Scientiarum Mathematicarum Hungarica 44 (1) (2007) 65-80.
[11] S. Herdem, I. Buyukyazici, Ibragimov–Gadjiev operators based on q-integers, Advances in Difference Equations 2008:304 (2018) 1-10.
[12] N.G. Bilgin, M. Cetinkaya, Approximation by three-dimensional q-Bernstein-Chlodowsky polynomials, Sakarya University Journal of Science 22 (6) (2018) 1774-1786.
[13] M. Mursaleen, A.A.H.A. Al-Abied, F. Khan, M.A. Salman, On (p, q)-Stancu-Szász-Beta operators and their approximation properties, Facta Universitatis Series: Mathematics and Informatics 35 (4) (2020) 1127-1143.
[14] V.N. Mishra, A.R. Devdhara, K.J. Ansari, S. Karateke, On a new approach by modified (p, q)-Szasz-Mirakyan operators, International Journal of Analysis and Applications 19 (3) (2021) 389-404.
[15] N.G. Bilgin, M. Eren, A generalization of two dimensional Bernstein-Stancu operators, Sinop University Journal of Science 6 (2) (2021) 130-142.
[16] N.G. Bilgin, M. Eren, Results on bivariate modified (p, q)-Bernstein type operators, Gazi University Journal of Science 36 (2) (2023) x-x.
[17] I.I. Ibragimov, A.D. Gadjiev, On a certain family of linear positive operators, Soviet Math. Dokl. English Trans 11 (1970) 1092-1095.
[18] O. Dogru, On a certain family of linear positive operators, Turkish Journal of Mathematics 21 (4) (1997) 387-399.
[19] N. Ispir, A. Aral, O. Dogru, On Kantorovich process of a sequence of the generalized linear positive operators, Numerical Functional Analysis and Optimization 29 (5-6) (2008) 574-589.
[20] N. Gonul, E. Coskun, Weighted approximation by positive linear operators, Proceedings of IMM of NAS of Azerbaijan 36 (2012) 41-50.
[21] A.M. Ghorbanalizadeh, On the order of weighted approximation of unbounded functions by some generalizations of Gadjiev–Ibragimov operators, Journal of Applied Analysis 20 (1) (2014) 87-92.
[22] N.G. Bilgin, N. Ozgur, Approximation by two dimensional Gadjiev-Ibragimov type operators, Ikonion Journal of Mathematics 1 (1) (2019) 1-10.
[23] T. Coskun, On a construction of positive linear operators for approximation of continuous functions in the weighted spaces, Journal of Computational Analysis and Applications 13 (4) (2011).
[24] N. Gonul, E. Coskun, Approximation with modified Gadjiev-Ibragimov operators in C[0,A], Journal of Computational Analysis and Applications, 15 (1) (2013) 868-879.
[25] T. Acar, Rate of convergence for Ibragimov-Gadjiev-Durrmeyer operators, Demonstratio Mathematica 50 (1) (2017) 119-129.
[26] E. Deniz, A. Aral, Convergence properties of Ibragimov-Gadjiev-Durrmeyer operators, Creat. Math. Inform. 24 (1) (2015) 17-26.
[27] N.G. Bilgin, N. Coskun, Comparison result of some Gadjiev-Ibragimov type operators, Karaelmas Journal of Science and Engineering 8 (1) (2018) 188-196.
[28] S. Herdem, I. Buyukyazici, Weighted approximation by q-Ibragimov-Gadjiev operators, Mathematical Communications 25 (2) (2020) 201-212.
[29] G. Bozma, E. Bars, On the approximation with an Ibragimov-Gadjiev type operator, Euroasia Journal of Mathematics, Engineering, Natural and Medical Sciences 9 (20) (2022) 74-83.