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Griffith crack in an orthotropic strip by successive approximation method

Research Article

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Abstract: The angular distribution of the circumferential stress and the strain energy density of a static Griffith crack situated in a mid plane of elastic symmetry of an infinite orthotropic strip of finite thickness have been investigated using integral representation of the displacement. Two problems have been considered in this paper. In the first problem, we consider the boundaries to be rigid and in the second problem, boundaries are assumed to be stress free. For large value of strip depth, asymptotic expression of the stresses are obtained. Expression for stress intensity factor at the crack tip is determined and computed numerically for two different orthotropic materials.

MSC: 74G15 • 45B05

Keywords: Griffith crack • Asymptotic expression • Fredholm integral equation • Circumferential stress • Strain energy density • Stress intensity factor

1. Introduction

From the past few decades various types of research works have been done in finding the solutions of static problems of Griffith crack in an isotropic strip. However, the static problem of Griffith crack in an anisotropic strip acquired less attention due to the mathematical complexity. Atkinson [1] find out the energy release rates for cracks propagating in media with spatially varying elastic moduli where the variation is in the perpendicular direction to the crack growth. In another paper, he [2] applied Cauchy integral formula to solve the problem of a semi-infinite crack in an aeolotropic material. Kassir and Tse [3] in their paper also employed integral transform technique to obtain the solution of an elastodynamic crack problem of finite length propagating with a constant speed in a uniformly stressed orthotropic medium. Li [4] considered a moving problem of finite impermeable crack in a piezoelectric ceramic strip when uniform anti-plane shear stress and uniform electric field are active. A finite crack in a functionally graded orthotropic strip under the plane loading is observed by Ma, Wu and Guo [5]. Under the action of impact load Sih [6–8] derived the dynamic SIF for finite crack and penny shaped crack in an infinite medium. Manfred and Ayatollahi [9] presented a theoretical and numerical description of the dynamic stress intensity factors of multiple cracks in a functionally graded orthotropic half plane. Xiao and Chen [10] analysed the SIF of a Griffith crack in a great detail. Parihar and Sowdamini [11] investigated the problem of a collinear crack in a two dimensional stressed anisotropic medium. Use of extended finite element method to a two dimensional interfacial crack problem in an anisotropic magneto electro elastic bimaterial was applied in the work of Ma, Su, Feng [12]. Peng and Jones [13] worked on the

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problem involving two unequal cracks in a finite width plate. Bayat, Ayatollahi and Bagheri [14] investigated multiple defects on an orthotropic strip with piezoelectric coating. Bhargava and Verma [15] obtained an analytic solution of two semi-permeable collinear cracks in a poled piezoelectric strip by using Fourier series method and integral equation method. Crack problem in one dimensional hexagonal quasi-crystal (Qcs) with the effect of piezoelectricity has been discussed by Yang and Li [16]. Recently works in crack problem are also available in the research articles of Patra et. al. [17], Sur and Kanoria [18] and Hitti et. al. [19].

In this paper, a static problem of Griffith crack in an infinite orthotropic strip is investigated by applying successive approximation method. Two cases, one is for rigid boundary and another is for stress free boundary are considered. The expression of stress intensity factor, the circumferential stress and the strain energy density have been obtained and their numerical simulation are also presented for both the Problem I and II. Considering suitable integral representation of the displacement fields, a closed form solution is obtained for large strip depth.

2. Formulation of the problem

We consider the static plane strain problem parallel to the xy-plane of a Griffith crack located along the x-axis from $-l$ to $+l$, in the mid-plane on infinite orthotropic strip of finite thickness $2d$. It is assumed that the principal axes of the orthotropic strip are aligned with the Cartesian co-ordinates (x, y, z) with the elastic constants C_{ij}.

The displacement equations of equilibrium are

$$C_{11}\frac{\partial^2 u}{\partial x^2}+C_{66}\frac{\partial^2 u}{\partial y^2}+(C_{12}+C_{66})\frac{\partial^2 v}{\partial x\partial y}=0 \tag{1}$$

$$C_{22}\frac{\partial^2 v}{\partial y^2}+C_{66}\frac{\partial^2 v}{\partial x^2}+(C_{12}+C_{66})\frac{\partial^2 u}{\partial x\partial y}=0 \tag{2}$$

where u and v are displacement in x, y directions respectively and the corresponding stresses are

$$\sigma_{xx}=C_{11}\frac{\partial u}{\partial x}+C_{12}\frac{\partial v}{\partial y} \tag{3}$$

$$\sigma_{yy}=C_{22}\frac{\partial v}{\partial y}+C_{12}\frac{\partial u}{\partial x} \tag{4}$$

$$\tau_{xy}=C_{66}(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}). \tag{5}$$

3. Boundary conditions

We consider the static problem of a Griffith crack situated on the line segment $-l$ to l in an infinite orthotropic strip of thickness $2d$. We assume that the crack is opened up by applying normal pressure $T(x)$ to it's faces. Due to symmetry it is sufficient to consider the half strip, $0\le y\le d$. Therefore, the boundary conditions of the problem may be taken in the form on $y=0$,

$$\tau_{xy}=0, \quad -\infty<x<\infty \tag{6a}$$

$$\sigma_{yy}=T(x), \quad |x|\le l \tag{6b}$$

$$v=0, \quad |x|>l \tag{6c}$$

We consider two problems

3.1. Problem I

In the first problem we assume that the boundary is rigid ie; on $y=d$,

$$u=0 \quad -\infty<x<\infty \tag{7a}$$

$$v=0 \quad -\infty<x<\infty. \tag{7b}$$

3.2. problem II

In the second problem we assume that the boundary is stress free ie; on $y = d$,

$$\tau_{xy} = 0 \qquad -\infty < x < \infty \tag{8a}$$

$$\sigma_{yy} = 0 \qquad -\infty < x < \infty. \tag{8b}$$

4. Solution of the problems

The integral solution of the equations (1) and (2) can be written in the form

$$u(x, y) = \int_0^\infty M(z, y)\sin(zx)dz \tag{9a}$$

$$v(x, y) = \int_0^\infty N(z, y)\cos(zx)dz \tag{9b}$$

where M and N are arbitrary functions to be determined. Substituting (9a) and (9b) into (1) and (2) we have the following equations:

$$C_{11}Mz^2 - C_{66}\frac{d^2M}{dy^2} + (C_{12} + C_{66})z\frac{dN}{dy} = 0 \tag{10a}$$

$$C_{66}Nz^2 - C_{22}\frac{d^2N}{dy^2} - (C_{12} + C_{66})z\frac{dM}{dy} = 0. \tag{10b}$$

The solution of (10a) and (10b) can be written as

$$M(z, y) = M_1(z)\cosh(\eta_1 zy) + M_2(z)\cosh(\eta_2 zy) + P_1(z)\sinh(\eta_1 zy) + P_2(z)\sinh(\eta_2 zy) \tag{11a}$$

$$N(z, y) = N_1(z)\sinh(\eta_1 zy) + N_2(z)\sinh(\eta_2 zy) + Q_1(z)\cosh(\eta_1 zy) + Q_2(z)\cosh(\eta_2 zy) \tag{11b}$$

where $M_i(z), N_i(z), P_i(z), Q_i(z), i = 1,2$ are arbitrary functions to be determined and η_1^2 and η_2^2 are positive roots of the equation

$$C_{66}C_{22}\eta^4 + [(C_{12} + C_{66})^2 - C_{11}C_{22} - C_{66}^2]\eta^2 + C_{66}C_{11} = 0. \tag{12}$$

Using (11a) and (11b) we obtain from (10a) the following relations:

$$N_i(z) = -\frac{\zeta_i}{\eta_i}M_i(z), i = 1,2$$

$$Q_i(z) = -\frac{\zeta_i}{\eta_i}P_i(z), i = 1,2$$

where

$$\zeta_i = \frac{C_{11} - {\eta_i}^2 C_{66}}{C_{12} + C_{66}}. \tag{13}$$

Now using the above relations the shearing stress can be obtained as

$$\tau_{xy}(x, y) = C_{66}\int_0^\infty \sum_{1=1}^{2} \frac{\gamma_i}{\eta_i}\{M_i(z)\sinh(\eta_i zy) + P_i(z)\cosh(\eta_i zy)\}z\sin(zx)dz \tag{14}$$

where

$$\gamma_i = \zeta_i + {\eta_i}^2, \qquad i = 1,2. \tag{15}$$

Using the condition (6a) in (14) we have that

$$P_2(z) = -\frac{\gamma_1\eta_2}{\eta_1\gamma_2}P_1(z). \tag{16}$$

Thus the solution can be determined completely if the three unknown functions $M_1(z)$, $M_2(z)$ and $P_1(z)$ can be determined.
Now we consider the two problem separately.

4.1. Problem I

Boundary conditions (7a) and (7b) will be fulfilled if we take

$$\sum_{i=1}^{2}[M_i(z)\cosh(\eta_i zd)+P_i(z)\sinh(\eta_i zd)]=0 \tag{17a}$$

and

$$\sum_{i=1}^{2}\frac{\zeta_i}{\eta_i}[M_i(z)\sinh(\eta_i zd)+P_i(z)\cosh(\eta_i zd)]=0. \tag{17b}$$

From (17a) and (17b) with the help of (16) we get

$$M_1(z)=\Delta_1(z)P_1(z) \tag{18a}$$

$$M_2(z)=\Delta_2(z)P_1(z) \tag{18b}$$

where

$$\Delta_1(z)=\frac{\frac{\zeta_2\gamma_1}{\gamma_2\eta_1}+\frac{\zeta_2}{\eta_2}\sinh(\eta_1 zd)\sinh(\eta_2 zd)-\frac{\zeta_1}{\eta_1}\cosh(\eta_1 zd)\cosh(\eta_2 zd)}{\frac{\zeta_1}{\eta_1}\sinh(\eta_1 zd)\cosh(\eta_2 zd)-\frac{\zeta_2}{\eta_2}\sinh(\eta_2 zd)\cosh(\eta_1 zd)} \tag{19a}$$

$$\Delta_2(z)=\frac{\frac{\zeta_1}{\eta_1}+\frac{\zeta_1\gamma_1\eta_2}{\gamma_2\eta_1^2}\sinh(\eta_1 zd)\sinh(\eta_2 zd)-\frac{\zeta_2\gamma_1}{\gamma_2\eta_1}\cosh(\eta_1 zd)\cosh(\eta_2 zd)}{\frac{\zeta_1}{\eta_1}\sinh(\eta_1 zd)\cosh(\eta_2 zd)-\frac{\zeta_2}{\eta_2}\sinh(\eta_2 zd)\cosh(\eta_1 zd)}. \tag{19b}$$

Hence only one function $P_1(z)$ is to be determined for the complete solution of the problem. From the boundary conditions (6b) and (6c) we get the following pair of dual integral equations for $P_1(z)$

$$\int_0^\infty zP_1(z)\cos(zx)dz=g_1(x), \qquad |x|\le l \tag{20a}$$

$$\int_0^\infty P_1(z)\cos(zx)dz=0, \qquad |x|>l \tag{20b}$$

where

$$g_1(x)=\frac{1}{A}[\int_0^\infty zP_1(z)\{A+w(z)\}\cos(zx)dz-T(x)] \tag{21a}$$

in which

$$A=C_{12}-\zeta_1C_{22}-\frac{\gamma_1\eta_2}{\gamma_2\eta_1}(C_{12}-\zeta_2C_{22}) \tag{21b}$$

and

$$w(z)=\sum_{i=1}^{2}(C_{12}-\zeta_iC_{22})\Delta_i(z). \tag{21c}$$

Solution of the pair of dual integral equations (20a) and (20b), is [20]

$$P_1(z)=\frac{2}{\pi}\int_0^l rJ_0(zr)dr\int_0^r\frac{g_1(x)dx}{(r^2-x^2)^{\frac{1}{2}}} \tag{22}$$

where J_0 is the Bessel function of first kind of order zero.
In general it is very hard to determine $g_1(x)$ as well as $P_1(z)$, as $g_1(z)$ contains the function $w(z)$ that involves the functions $\Delta_1(z)$ and $\Delta_2(z)$.
However, the asymptotic expansion for $w(z)$ for large d can be determined and is given by

$$w(z)\sim\frac{B-Ae^{2\eta_2 zd}}{e^{2\eta_2 zd}} \tag{23a}$$

where

$$B=\frac{2\gamma_1}{\gamma_2\eta_1^2}(\zeta_1\eta_2+\zeta_2\eta_1)(\zeta_2C_{22}-C_{12}). \tag{23b}$$

From the equation (21a) and (22) and using (23a) we have,

$$P_1(z) = \frac{1}{A}[B\int_0^{\infty} P_1(v)dv \int_0^l vrJ_0(vr)e^{-2\eta_2 vd}dr - \frac{2}{\pi}\int_0^l rJ_0(zr)dr\int_0^r \frac{T(x)}{\sqrt{r^2-x^2}}dx]. \tag{24}$$

We consider a constant transverse function $T(x) = -q_0$ and equation (24) gives

$$P_1(z) = \frac{lq_0 J_1(lz)}{Az} + \int_0^{\infty} P_1(v)K(z,v)dv \tag{25}$$

where,

$$K(z,v) = \frac{B}{A}\int_0^l vrJ_0(zr)J_0(vr)e^{-2\eta_2 vd}dr \tag{26}$$

Equation (25) is a Fredholm integral equation with the kernel $K(z,v)$ given in the equation (26). let us now solve the Fredholm integral equation by Iterative method. For this we take the zero order approximation ${P_1}^0(z)$ of $P_1(z)$ as

$${P_1}^0(z) = \frac{lq_0 J_1(lz)}{Az} \tag{27}$$

Substituting equation (27) in the right side of equation (25) we have the first order approximation [21]

$${P_1}^1(z) = \frac{lq_0 J_1(lz)}{Az}(1+\delta) \tag{28}$$

in which,

$$\delta = [Bl^2/8A{\eta_2}^2 d^2] \tag{29}$$

where we have neglected the terms of the order of $(l/d)^4$. The function ${P_1}^1(z)$ when substituted in the right side of equation (25), we have the second approximation ${P_1}^2(z)$, given by,

$${P_1}^2(z) = \frac{lq_0 J_1(lz)}{Az}(1+\delta+\delta^2). \tag{30}$$

Repeating the same procedure we have the successive approximation and the function $P_1(z)$ can be written in the form ,

$$P_1(z) = \frac{lq_0 J_1(lz)}{Az(1-\delta)} \quad \text{for } |\delta| < 1. \tag{31}$$

We will choose d in such a way so that$(l/d) << 1$ and $|\delta| < 1$. When $d \to \infty$, the expression for $P_1(z)$ in (31) becomes ,

$$P_1(z) = \frac{lq_0 J_1(lz)}{Az} \tag{32}$$

which being the corresponding solution for a half-plane problem. Since the method of solution is straight forward we will neglect the higher order approximations.

Using the identity [21]

$$\int_0^{\infty} e^{-\alpha x}J_1(\beta x)dx = \frac{1}{\beta}[1 - \frac{\alpha}{(\alpha^2+\beta^2)^{\frac{1}{2}}}] \tag{33}$$

and introducing the abbreviation ,

$$F(x_j,y_j) + iH(x_j,y_j) = \frac{1}{l}[1 + \frac{i\epsilon_j}{(l^2-{\epsilon_j}^2)^{\frac{1}{2}}}] \quad j = 1,2,...,6 \tag{34}$$

where

$$\epsilon_j = x + i\eta_j y, (j = 1,2)$$

$$\epsilon_3 = x + i[(\eta_1+\eta_2)d + \eta_1 y]$$

$$\epsilon_4 = x + i[(\eta_1+\eta_2)d - \eta_1 y]$$

$$\epsilon_5 = x + i(2\eta_2 d + \eta_2 y)$$

$$\epsilon_6 = x + i(2\eta_2 d - \eta_2 y) \tag{35}$$

and $x_j = Re(\epsilon_j), y_j = Im(\epsilon_j)(j = 1,2,...,6), i = \sqrt{-1}$. Following the above equations the stresses at any point of solid become,

$$\begin{aligned}\sigma_{xx}(x,y) = -\frac{lq_0}{A(1-\delta)}[(C_{11}-\zeta_1 C_{12})\{F(x_1,y_1) - \frac{2\zeta_2\gamma_1\eta_2}{\gamma_2(\zeta_1\eta_2-\zeta_2\eta_1)}(F(x_3,y_3)+F(x_4,y_4))\}\\ -(C_{11}-\zeta_2 C_{12})\frac{\gamma_1\eta_2}{\gamma_2\eta_1}\{F(x_2,y_2) - \frac{\zeta_1\eta_2+\zeta_2\eta_1}{\zeta_1\eta_2-\zeta_2\eta_1}(F(x_5,y_5)+F(x_6,y_6))\}]\end{aligned} \tag{36}$$

$$\begin{aligned}\sigma_{yy}(x,y) = -\frac{lq_0}{A(1-\delta)}[(C_{12}-\zeta_1 C_{22})\{F(x_1,y_1) - \frac{2\zeta_2\gamma_1\eta_2}{\gamma_2(\zeta_1\eta_2-\zeta_2\eta_1)}(F(x_3,y_3)+F(x_4,y_4))\}\\ -(C_{12}-\zeta_2 C_{22})\frac{\gamma_1\eta_2}{\gamma_2\eta_1}\{F(x_2,y_2) - \frac{\zeta_1\eta_2+\zeta_2\eta_1}{\zeta_1\eta_2-\zeta_2\eta_1}(F(x_5,y_5)+F(x_6,y_6))\}]\end{aligned} \tag{37}$$

$$\begin{aligned}\tau_{xy}(x,y) = \frac{lq_0}{A(1-\delta)}\frac{C_{66}\gamma_1}{\eta_1}[H(x_1,y_1)-H(x_2,y_2)+\frac{2\zeta_2\gamma_1\eta_2}{\gamma_2(\zeta_1\eta_2-\zeta_2\eta_1)}\times\{H(x_4,y_4)-H(x_3,y_3)\}\\ -\frac{(\zeta_1\eta_2+\zeta_2\eta_1)}{(\zeta_1\eta_2-\zeta_2\eta_1)}\{H(x_6,y_6)-H(x_5,y_5)\}].\end{aligned} \tag{38}$$

Polar co-ordinate, (r,θ) is introduced to compute the stress and strain in the region ahead of the crack point, such that

$$\begin{aligned} x = l + r\cos\theta, \qquad y = r\sin\theta,\\ x + iy = l(1+\Delta e^{i\theta})\end{aligned} \tag{39}$$

where $\Delta = (r/l)$, is a small dimensionless quantity. Taking

$$\epsilon_j = l(1+\Delta_j e^{i\theta_j}), j = 1,2 \tag{40}$$

it follows that,

$$\begin{aligned}\Delta_j = \Delta(\cos^2\theta + \eta_j^2\sin^2\theta)^{\frac{1}{2}}\\ \tan\theta_j = \eta_j\tan\theta.\end{aligned} \tag{41}$$

Illustrating the expressions in equations (36-38) for small values of Δ, it has been found that the asymptotic expression for circumferential stress

$$\sigma_\theta = \sigma_{xx}\sin^2\theta + \sigma_{yy}\cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

is given by,

$$\begin{aligned}\sigma_\theta = \frac{q_0}{A(1-\delta)(2\Delta)^{\frac{1}{2}}}[\{(C_{11}-\zeta_1 C_{12})\sin^2\theta + (C_{12}-\zeta_1 C_{22})\cos^2\theta\}\times\frac{\cos(\frac{\theta_1}{2})}{(\cos^2\theta+\eta_1^2\sin^2\theta)^{\frac{1}{4}}}\\ -\frac{\gamma_1\eta_2}{\gamma_2\eta_1}\{(C_{11}-\zeta_2 C_{12})\sin^2\theta + (C_{12}-\zeta_2 C_{22})\cos^2\theta\}\times\frac{cos(\frac{\theta_2}{2})}{(\cos^2\theta+\eta_2^2\sin^2\theta)^{\frac{1}{4}}} - \frac{2C_{66}\gamma_1}{\eta_1}\sin\theta\cos\theta\times\\ \{\frac{cos(\frac{\theta_1}{2})}{(\cos^2\theta+\eta_1^2\sin^2\theta)^{\frac{1}{4}}} - \frac{cos(\frac{\theta_2}{2})}{(\cos^2\theta+\eta_2^2\sin^2\theta)^{\frac{1}{4}}}\}] + o(\Delta^0)\end{aligned} \tag{42}$$

where terms of higher orders have been neglected . In the similar manner the expression of the strain energy density

$$S = \frac{1}{2}(C_{11}\epsilon_X^2 + 2C_{12}\epsilon_x\epsilon_y + C_{22}\epsilon_y^2 + C_{66}\gamma_{xy}^2)$$

assume the form,

$$\begin{aligned}S = \frac{q_0^2}{2A^2(1-\delta)^2}[C_{11}\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1)^{\frac{1}{2}}} - \frac{\gamma_1\eta_2}{\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2)^{\frac{1}{2}}}\}^2 + \zeta_1^2 C_{22}\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1)^{\frac{1}{2}}} - \frac{\zeta_2\gamma_1\eta_2}{\zeta_1\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2)^{\frac{1}{2}}}\}^2\\ -2C_{12}\zeta_1\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1)^{\frac{1}{2}}} - \frac{\gamma_1\eta_2}{\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2)^{\frac{1}{2}}}\}\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1)^{\frac{1}{2}}} - \frac{\zeta_2\gamma_1\eta_2}{\zeta_1\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2)^{\frac{1}{2}}}\} + C_{66}\frac{\gamma_1^2}{\eta_1^2}\{\frac{\cos\frac{\theta_1}{2}}{(2\Delta_1)^{\frac{1}{2}}} - \frac{\cos\frac{\theta_1}{2}}{(2\Delta_2)^{\frac{1}{2}}}\}^2].\end{aligned} \tag{43}$$

We have neglected higher order terms than $(1/\Delta)$ in equation (43). We can see that the angular distribution in expressions (42) and (43) are independent of crack length.

The Stress intensity factor at the crack tip $x = l$ is ,

$$k = \lim_{x\to l^+}\sqrt{(x-l)}\sigma_{yy}(x,0) = \frac{lq_0}{A(1-\delta)}\times\sqrt{\frac{\pi}{l}}\{(C_{12}-\zeta_1 C_{22}) - \frac{\gamma_1\eta_2}{\gamma_2\eta_1}(C_{12}-\zeta_2 C_{22})\}. \tag{44}$$

4.2. Problem II

Boundary conditions (8a) and (8b) will be fulfilled if we take

$$\sum_{i=1}^{2}\frac{\gamma_i}{\eta_i}[M_i(z)\sinh(\eta_i zd)+P_i(z)\cosh(\eta_i zd)]=0 \tag{45a}$$

$$C_{22}\sum_{i=1}^{2}[\gamma_i\eta_i z\cosh(\eta_i zd)+Q_i\eta_i z\sinh(\eta_i zd)]+C_{12}\sum_{i=1}^{2}[M_i\cosh(\eta_i zd)+P_i\sinh(\eta_i zd)]=0 \tag{45b}$$

From (45a) and (45b) with the help of (16) we get,

$$M_1(z)=\Delta_1'(z)P_1(z) \tag{46a}$$

$$M_2(z)=\Delta_2'(z)P_1(z) \tag{46b}$$

where,

$$\Delta_1'(z)=\frac{[\frac{\gamma_2}{\eta_2}(C_{12}-C_{22}\zeta_1 z)\sinh(\eta_1 zd)\sinh(\eta_2 zd)-\frac{\gamma_1}{\eta_1}(C_{12}-C_{22}\zeta_2 z)\cosh(\eta_1 zd)\cosh(\eta_2 zd)-\frac{\gamma_1}{\eta_1}(C_{12}-C_{22}\zeta_2 z)]}{[\frac{\gamma_1}{\eta_1}(C_{12}-C_{22}\zeta_2 z)\sinh(\eta_1 zd)\cosh(\eta_2 zd)-\frac{\gamma_2}{\eta_2}(C_{12}-C_{22}\zeta_1 z)\sinh(\eta_2 zd)\cosh(\eta_1 zd)]} \tag{47a}$$

$$\Delta_2'(z)=\frac{[\frac{\gamma_1^2\eta_2}{\gamma_2\eta_1^2}(C_{12}-C_{22}\zeta_2 z)\sinh(\eta_1 zd)\sinh(\eta_2 zd)-\frac{\gamma_1}{\eta_1}(C_{12}-C_{22}\zeta_1 z)\cosh(\eta_1 zd)\cosh(\eta_2 zd)-\frac{\gamma_1}{\eta_1}(C_{12}-C_{22}\zeta_1 z)]}{[\frac{\gamma_1}{\eta_1}(C_{12}-C_{22}\zeta_2 z)\sinh(\eta_1 zd)\cosh(\eta_2 zd)-\frac{\gamma_2}{\eta_2}(C_{12}-C_{22}\zeta_1 z)\sinh(\eta_2 zd)\cosh(\eta_1 zd)]}. \tag{47b}$$

We now use these in the boundary conditions (6b) and (6c). Proceeding as in problem-I, we obtain the dual integral equations for $P_1(z)$ given by,

$$\int_0^\infty zP_1(z)\cos(zx)dz=g_1(x),\qquad |x|\le l \tag{48a}$$

$$\int_0^\infty P_1(z)\cos(zx)dz=0,\qquad |x|>l \tag{48b}$$

where

$$g_1(x)=\frac{1}{A}[\int_0^\infty zP_1(z)\{A+w_1(z)\}\cos(zx)dz-T(x)] \tag{49a}$$

in which

$$A=C_{12}-\zeta_1 C_{22}-\frac{\gamma_1\eta_2}{\gamma_2\eta_1}(C_{12}\zeta_2 C_{22}) \tag{49b}$$

and

$$w_1(z)=\sum_{i=1}^{2}(C_{12}-\zeta_i C_{22})\Delta_i'. \tag{49c}$$

Interestingly, it is observed that for large value of d the asymptotic expansion of $w_1(z)$ in problem-II and $w(z)$ in problem-I are same.
As we consider two different problems, analytically we would have different values of the required entities and we would get completely different expression for stresses. As it is practically impossible to solve the problem analytically, we consider asymptotic expansion and it is quite fascinating that we got same values of some entities. Again ignoring the higher order terms for small quantities we have the same expression for circumferential stress as well as stress intensity factor.
So proceeding in same way as in problem-I and replacing Δ_j by Δ_j' $(j=1,2)$ in equation (40) to (41), we will get the required expression for circumferential stress σ_θ and stress intensity factor k, which are similar as in equations (41) and (44) respectively and the expression for strain energy density S, as

$$\begin{aligned}S=&\frac{q_0^2}{2A^2(1-\delta)^2}[C_{11}\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1')^{\frac12}}-\frac{\gamma_1\eta_2}{\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2')^{\frac12}}\}^2+\zeta_1^2C_{22}\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1')^{\frac12}}-\frac{\zeta_2\gamma_1\eta_2}{\zeta_1\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2')^{\frac12}}\}^2\\&-2C_{12}\zeta_1\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1')^{\frac12}}-\frac{\gamma_1\eta_2}{\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2')^{\frac12}}\}\{\frac{\sin\frac{\theta_1}{2}}{(2\Delta_1')^{\frac12}}-\frac{\zeta_2\gamma_1\eta_2}{\zeta_1\gamma_2\eta_1}\frac{\sin\frac{\theta_2}{2}}{(2\Delta_2')^{\frac12}}\}+C_{66}\frac{\gamma_1^2}{\eta_1^2}\{\frac{\cos\frac{\theta_1}{2}}{(2\Delta_1')^{\frac12}}-\frac{\cos\frac{\theta_1}{2}}{(2\Delta_2')^{\frac12}}\}^2]\end{aligned} \tag{50}$$

Table 1. Material constant in orthotropic media (x 10^4 MPa)

Material	C_{11}	C_{22}	C_{12}	C_{66}
Boron Epoxy Type-I	22.49	1.27	0.33	0.44
Steel-Mylar	18.70	2.92	1.30	0.62

5. Numerical results and discussion

The expressions (42) and (43) can be used to compute $\sigma_\theta/[q_0/\sqrt{2\Delta}]$ and $S/[q_0{}^2/2\Delta]$ at various strip depths and positions near crack tip. For numerical computation, we take the material constants for orthotropic media $C_{11}, C_{12}, C_{22}, C_{66}$ from [3] for Boron Epoxy Type-I and Steel Mylar, which are given in Table 1
The circumferential stress, strain energy density, stress intensity factor have been plotted graphically for various values of strip depth for the material boron epoxy type-1 as well as steel mylar.
Fig. 1 and Fig. 2 display the variation of $\sigma_\theta/[q_0/\sqrt{2\Delta}]$ with various values of l/d for the materials Boron-Epoxy Type-I and Steel Mylar respectively. It is observed from Fig. 1 and Fig. 2 that the circumferential stress decreases gradually with the increasing value of θ. From Fig. 1 we found that the value of $\sigma_\theta/[q_0/\sqrt{2\Delta}]$ for l/d= 0.1, 0.3 and 0.6 are approximately same around θ=0.6. However, for steel mylar $\sigma_\theta/[q_0/\sqrt{2\Delta}]$ have the same value around θ= 0.4 for the same values of l/d= 0.1, 0.3 and 0.6. Nature of the graphs are similar for both the materials.
Fig. 3 and Fig. 4 represent the graph for $S/[q_0{}^2/2\Delta]$ for Boron-Epoxy Type-I and Steel Mylar respectively when various values of l/d is considered. For Fig. 3 and Fig. 4 a general feature of curve is observed: the strain energy density ascends rapidly, grasp a peak value, reduces gradually and then approaches to their constant value. It is observed from both the figures that strain energy density always decreases with the increasing value of l/d. Peak values for various values of l/d are shown in Table 2.

Table 2. Peak values of $S/[q_0{}^2/2\Delta]$:

l/d	Boron-Epoxy Type-I	Steel Mylar
0.5	0.0662	0.0195
0.6	0.0319	0.0094
0.9	0.0063	0.0019

Taking numerical values of the crack length to be fixed, the effect of the depth of the strip on the SIF is displayed in Fig. 5. It is observed from Fig. 5 that though the values of stress intensity factor decreases with the increasing values of l/d, yet rapid decreasing is observed for l/d <0.1.

All the numerical computations and graphs have been considered for the case of problem-I ie; when rigid boundary is considered. Similar behaviour of the plots of circumferential stress, strain energy density and SIF to that in problem-I is observed in the case of problem-II.

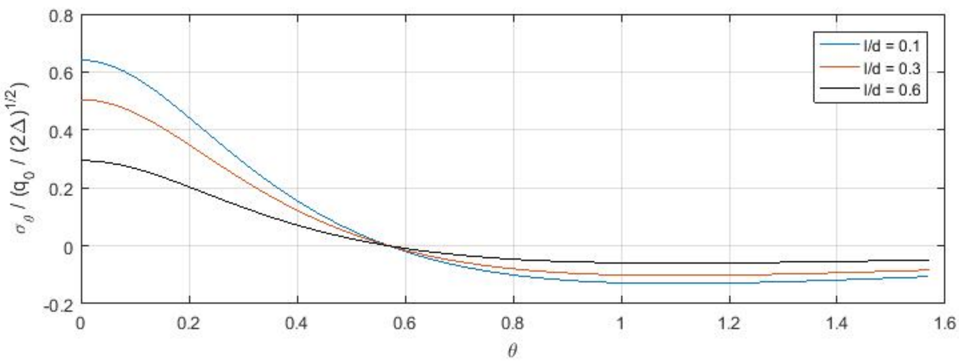

Fig. 1. variation of σ_θ with l/d for Boron Epoxy Type-I

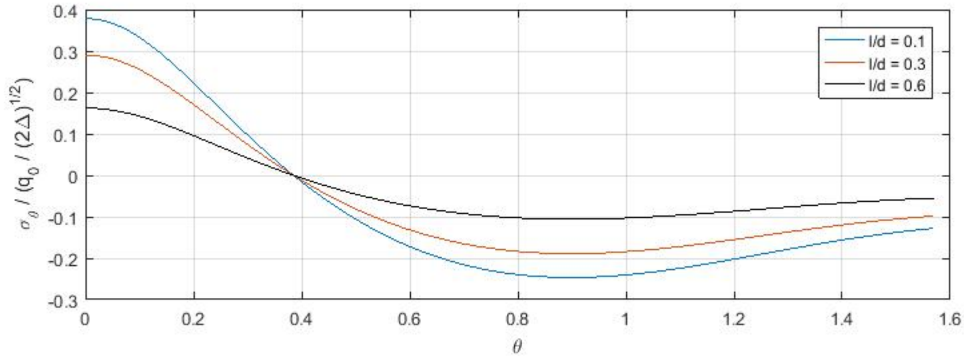

Fig. 2. variation of σ_θ with l/d for Steel Mylar

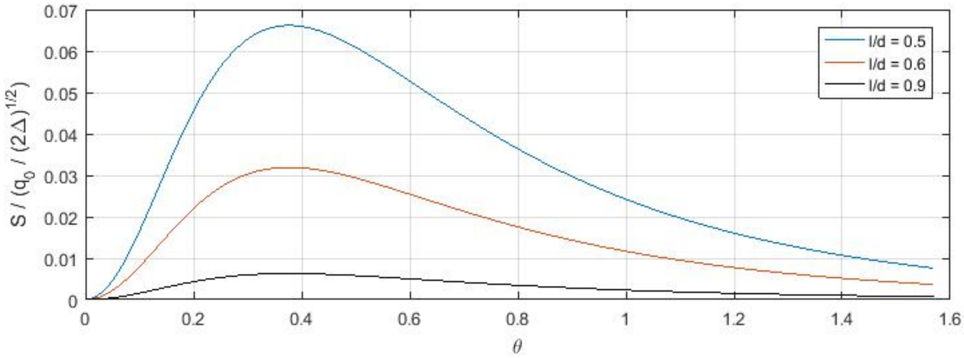

Fig. 3. Strain energy density vs angle θ for Boron Epoxy Type-I

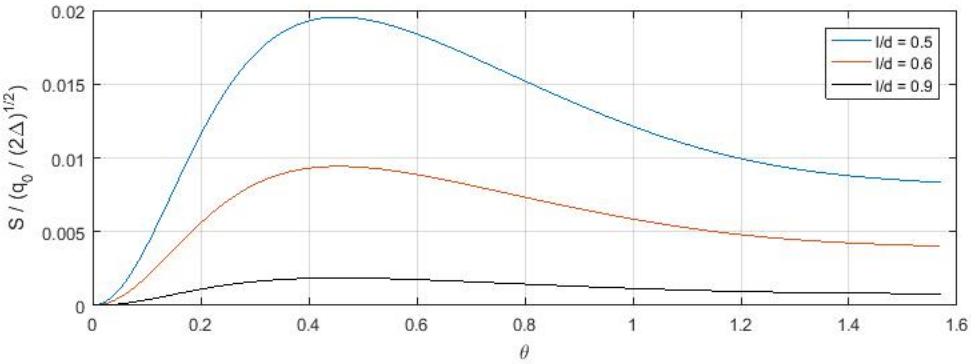

Fig. 4. Strain energy density vs angle θ for Steel Mylar

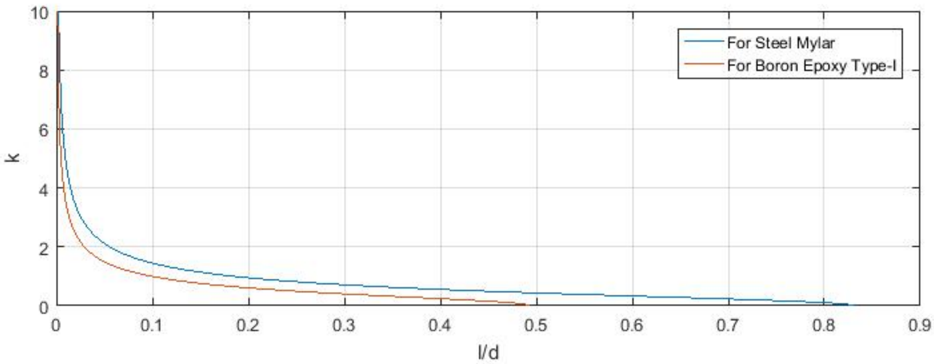

Fig. 5. SIF vs l/d boron epoxy steel Mylar

6. Conclusion

In this problem a crack in an orthotropic strip has been considered. An attempt has been made to solve the problem by considering an integral representation of the displacement component. We observed that using the boundary conditions this problem reduces to a problem of dual integral equations. Finally iterative method is used to solve the problem for a special case. We noticed that angular distribution of circumferential stress is independent of crack length. It is also observed that strain energy density always increases with the decreasing value of l/d.

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