Dynamics of one dimensional mouse map

Research Article

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Abstract: In this paper, we have discussed the dynamics of 1-D Mouse map which is also known as Gaussian Iterated map or simply Gauss map and analyzed their chaotic behaviors in several senses. We have mainly focused on several chaotic dynamics like Orbit Analysis, Time Series Analysis, Lyapunov Exponent Analysis, Sensitivity to Initial Conditions, Bifurcation Diagram, Cobweb Diagram, Histogram, Trajectories and Sensitivity to Numerical Inaccuracies of this map. Finally we have shown its graphical analysis and found that this map is chaotic in the mentioned senses. We have performed all graphical activities by using Mathematica and MATLAB.

Keywords: Chaos • Sensitivity • Bifurcation • Cobweb • Lyapunov • Trajectories

1. Introduction

During the last few decades, dynamical system [1] has made long strides. Dynamical system is the study of the long-term behavior of an evolving system. It is observed that in various models of economics, biology and various other sciences the chaotic non-linear dynamical system has made its presence felt. The phenomenon of chaos [2] has been studied extensively and it has attracted increasing interests from mathematicians, physicists, engineers, and so on. Since chaotic systems not only admit abundant complex and interesting dynamical behaviors but also have many potential practical applications, great efforts have been devoted to investigation related to these systems. Nnumerous articles are written on chaos and chaos control. Some of the pioneer articles in this direction are those of Chirikov [2], Lorenz [3], Sharkovskii [4], Smale [5], May [6], Feigenbaum [7], Devany [8], Grassberger and Procaccia [9], Moon [10], Gleick [11], Stewart [12], Mandelbrot [13], HaoBai-Lin [14], Henon [15] and many others.In Mathematics, researchers deal with various map to study the different qualitative features related to it.The objective of the present work is to investigate the chaotic properties of certain real system represented by one dimensional discrete mathematical model.In this paper, we intend to explore a one-dimensional map, the Mouse map, which has drawn less attention due to its similarity with the logistic map. However, this map exhibits some features which make it distinguishable from logistic map [16].

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2.   Definitions

2.1.   Mouse map

The non-linear one dimensional map defined by $f(x) = e^{-bx^2} + c$ Where 'b' and 'c' are parameters is termed as the mouse map or Gauss map. The parameters 'b' and 'c' are related to the width and the height of the curve respectively. Below we have shown the Mouse map for different values of the parameters $c \in [-1, 1]$ (keeping b fixed) in Fig. 1. In this paper we use $b = 7.5$.
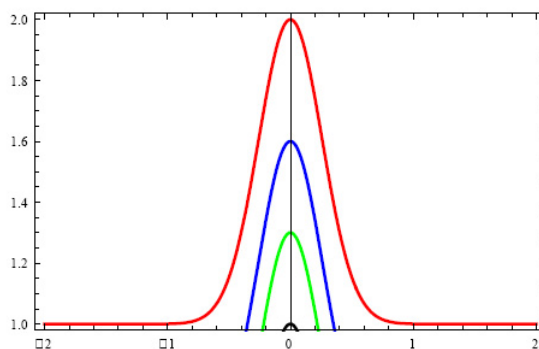

Fig. 1. Diagram of Mouse map for fixed $b = 7.5$

In Figure we have plotted the Mouse map for different sets of parameters b and c. It is clear that the map is symmetric about $x = 0$, and has maximum value (always at $x = 0$) equals to $c + 1$. For large values of $|x|$, the function approaches to the minimum value equals to c.

2.2.   Cobweb diagram

Given a map f and a point x in the domain of f, we call the set of points $\{x, f(x), f^2(x), f^3(x), \cdots\}$ the forward orbit of x under f and denote it by $O^+(x)$. As x is the starting point of the orbit, we call it the initial value or seed of the orbit.

2.3.   Trajectories of the map

In dynamical systems, a trajectory is the set of points in state space that are the future states resulting from a given initial state. In a discrete dynamical system, a trajectory is a set of isolated points in state space. In a continuous dynamical system, a trajectory is a curve in state space.

2.4.   Histogram

A histogram is an accurate representation of the distribution of numerical data. It is an estimate of the probability distribution of a continuous variable (quantitative variable). It differs from a bar graph, in the sense that a bar graph relates two variables, but a histogram relates only one. To construct a histogram, the first step is to 'bin' (or 'bucket') the range of values - that is, divide the entire range of values into a series of intervals—and then count how many values fall into each interval. As the adjacent bins leave no gaps, the rectangles of a histogram touch each other to indicate that the original variable is scattered.

3.   Behaviour of Mouse map taking initial seeds

In this section we iterate the Mouse map $f(x) = e^{-7.5x^2} + c$ for the following c and x_0 values (initial seed) and investigate whether the orbit of $f(x)$ forms a cycle or converges to any fixed point or becomes chaotic (random).Now we iterate the given function considering the following case: Taking $c = -0.36$;

(i) For $x_0 = 0.30$, The orbit is: { 0.30, 0.15, 0.48, - 0.19, 0.40, - 0.06, 0.61, - 0.29, 0.15, 0.48, $\cdots\cdots$}

(ii) For $x_0 = 0.35$, The orbit is: { 0.35, 0.04, 0.63, - 0.31, 0.13, 0.52, - 0.23, 0.31, 0.12, 0.54, $\cdots\cdots$}

(iii) For $x_0 = 0.45$, The orbit is: { 0.45, - 0.14, 0.50, - 0.21, 0.36, 0.01, 0.64, - 0.31, 0.12, 0.54, $\cdots\cdots$}

(iv) For $x_0 = 0.50$, The orbit is: { 0.50, - 0.20, 0.36, 0.01, 0.64, - 0.31, 0.12, 0.54, - 0.25, 0.27, $\cdots\cdots$}

Obviously no pattern forms in the above orbits. The orbits approach randomly. The initial seeds 0.30, 0.35, 0.40, 0.45 and 0.50 are neither fixed/periodic points nor eventually fixed/periodic points of $f(x)$. We conclude that the dynamical behaviour of the given function for $c = -0.36$ is chaotic [15], [23], [**?**]. The graphical representation of above orbit analysis is shown below:

(i)Considering initial seed $x_0 = 0.30$

(ii) Considering initial seed $x_0 = 0.35$

(iii) Considering initial seed $x_0 = 0.45$

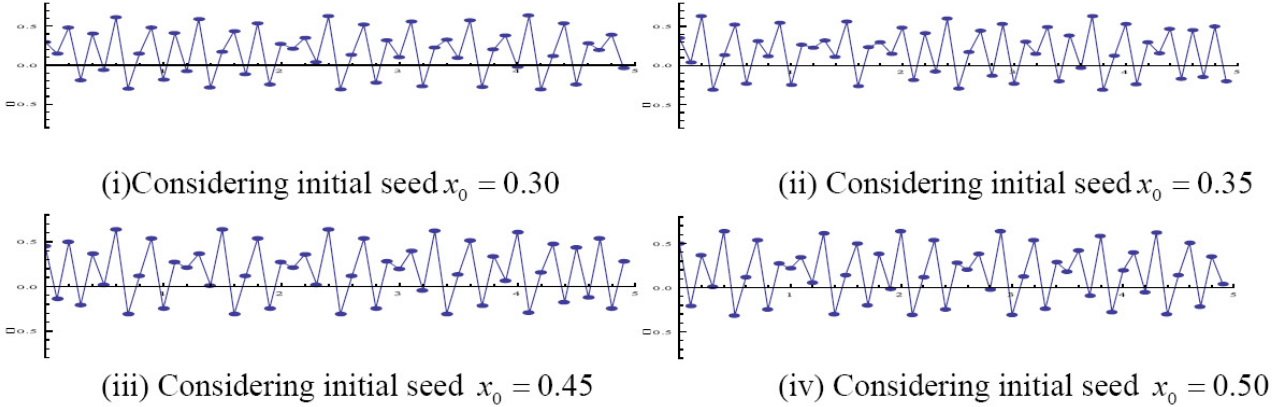(iv) Considering initial seed $x_0 = 0.50$

Fig. 2. Orbit of Mouse map for $c = -0.36$

Here we clearly visualize that taking different initial seeds the orbits of given dynamical system changes its nature randomly. So from graphical view of the orbits our decision is Mouse map is chaotic for some $c \in [-1, 1]$ [17], [18],[19].

4.   Mouse map in the sense of Lyapunov exponent

Definition 4.1.
A defining feature of a chaotic system is sensitivity to initial conditions. If two trajectories which start off close to each other deviate more and more with increasing time, the system is said to be chaotic. The rate at which nearby trajectories deviate from each other with time is characterized by quantity called the Lyapunov exponent.

Proposition 4.1.
Show that $f(x) = e^{-7.5x^2} + c$ the Mouse map is chaotic in the interval $-1 \le c \le 1$.

Proof.   Here we discuss the Lyapunov exponent for the Mouse map.

Consider two iterations of the Mouse map starting from two values of x which are close together. Let the two starting values be x_0 and $x_0 + \delta x_0$. These map to x_1 and $x_1 + \delta x_1, \cdots\cdots, x_n + \delta x_n$. Expanding $f(x)$ about x_n we have $\delta x_n = f'(x_{n-1})\delta x_{n-1}$. Assuming that δx_n is sufficiently small. Hence the separation of two trajectories after n steps, δx_n is related to their initial separation δx_0 by $\left|\frac{\delta x_n}{\delta x_0}\right| = \prod_{i=0}^{n-1}\left|f'(x_i)\right|$. We expect that this will vary exponentially at large n like $\left|\frac{\delta x_n}{\delta x_0}\right| = e^{c_L}$ (Large n).

And so we define the Lyapunov exponent c_L by $c_L = \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \ln\left|f'(x_i)\right|$

If $c_L > 0$ neighbouring trajectories diverge from each other at large n this corresponds to chaos. However if the trajectories converge to a fixed point or limit cycle they will get closer together, which corresponds to $c_L < 0$.

Hence we can determine whether or not the system is chaotic by the sign of the Lyapunov exponent.

Below we calculate the lyapunov exponent for some values of parameter c (not to be confused with the Lyapunov exponent c_L). For $c = -0.5653$ we get $c_L = 0.458141$. The positive value of c_L indicates that $c = -0.5653$ is in a region of chaos. By contrast if we specify $c = -0.2453$ we get $c_L = $ - 0.523585which is a negative value, indicating that the trajectory of points x_i $(i = 0, 1, 2, \cdots)$ converges to an attractor, which in this case we already know is a length of 2 limit cycle. Next lets plot the Lyapunov exponent for a range of values of c [19].

We see several points (the first is at $c = 0.4$ and $c = -0.211$) where the Lyapunov exponent hits 0 and then goes negative again. These are the period doubling bifurcations. Precisely at the period doubling point the system is at the limit of chaos, but then becomes non-chaotic when the period doubles. However, at the end of the period doubling regime, at c about -0.2655, c_L crosses the axis and the system enters a chaotic regime. Note that for c in the range greater than the point where c_L first goes positive, there are many regions where c_L is negative, "island of stability" where the behavior is fixed point or limit cycle. We see that chaos emerges (i.e.$c_L > 0$) for c between -0.2655 and -0.4054. It is again become non chaotic for the values of c between -0.4054 and -0.4785 then it enters into the chaotic
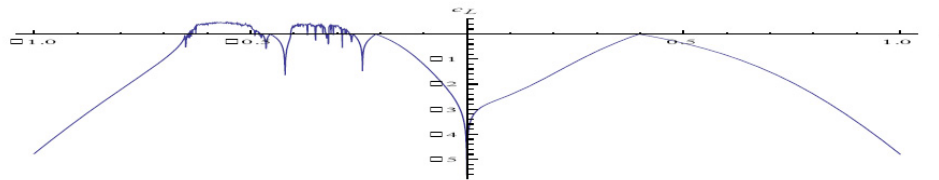
Fig. 3. Lyapunov exponent for $-1 \leq c \leq 1$

region. Again it's become chaotic between -0.4785 and -0.6425 then it enters to the non-chaotic region finally. Hence the Mouse map is chaotic for $c \in [-1, 1]$ [18, 20, 21].   □

5. Sensitivity analysis of mouse map

We want to check the difference of the orbits by taking two neighbouring initial seeds. We first define the function governing the system and then calculate the distance between two orbits for the considered neighbouring initial seeds. Here we will consider 50 iteration and calculate the distance between two orbits. After that we will analyze whether the two orbits overlap or not, whether the function is chaotic or not.Now we want to analyze the sensitivity of Mouse map $f(x) = e^{-7.5 x^2} + c$ for different c values.Now we take $c = -0.36$and two sets of neighbouring initial seeds $x_0 = 0.30$, $x_0 = 0.301$and $x_0 = 0.6$, $x_0 = 0.6000001$we get the following Table 1

Table 1. Sensitivity analysis of Mouse map

(iteration) n	$x_0 = 0.30$	$x_0 = 0.301$	Distance	$x_0 = 0.6$	$x_0 = 0.6000001$	Distance
1	0.15	0.15	0	-0.29	-0.29	0
2	0.49	0.49	0	0.16	0.16	0
3	-0.19	-0.19	0	0.45	0.45	0
4	0.4	0.39	0.01	-0.14	-0.14	0
5	-0.06	-0.04	0.02	0.49	0.49	0
.	.	.	.	.	.	.
48	0.39	0.22	0.17	0.22	0.52	0.3
49	-0.04	0.34	0.38	0.34	-0.22	0.56
50	0.63	0.06	0.57	0.06	0.32	0.26

The Table 1 tells us the 'story' of the two orbits from the 1st to 50th positions. We see that the distance between the two orbits is bouncing between 0 to 1 in an apparent erratic manner. This type of behavior tells us the system is chaotic. We can conclude that the given Mouse map is chaotic for $c = -0.36$ according to the sensitivity to initial condition. The graphical representations of the above sensitivity analysis are shown below. Take $c = -0.36$ we get,
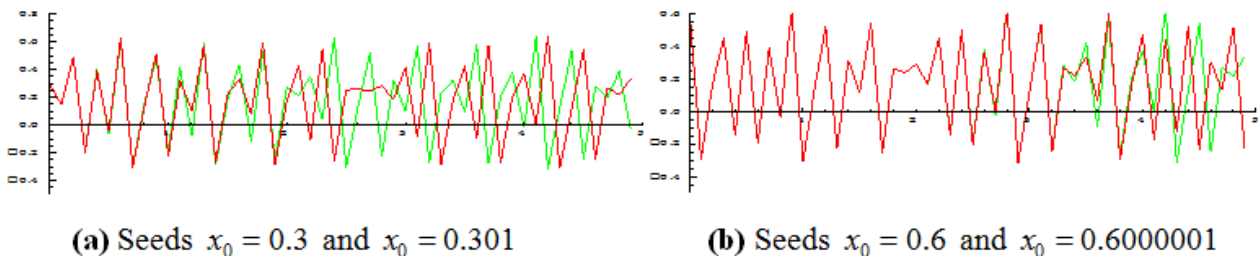

(a) Seeds $x_0 = 0.3$ and $x_0 = 0.301$ **(b)** Seeds $x_0 = 0.6$ and $x_0 = 0.6000001$

Fig. 4.

In Fig. 4(a)the orbits are same up to 5th iteration than separate and go far. In Fig. 4(b) two orbits are same up to 26th iteration than separate and go far from each other. This phenomenon is called sensitivity to initial conditions. According to the definition of chaos we can say that the given Mouse map is chaotic for $c = -0.36$.In this section we have considered the values up to two decimal places and in above figures red line represents the orbits of $x_0 = 0.30$ and 0.6 and green line represents the orbits of $x_0 = 0.301$ and 0.6000001 [18, 19, 22].

6.   Time series analysis of the Mouse map graphically

The orbits listed in orbit analysis seem to be wandering around the interval $0 \leqslant x \leqslant 1$ rather aimlessly. Let's see if we can detect a pattern from the time series for one of these orbits. Here is the time series graph for the seed $x_0 = 0.2, 0.2001$ with 100 iteration.
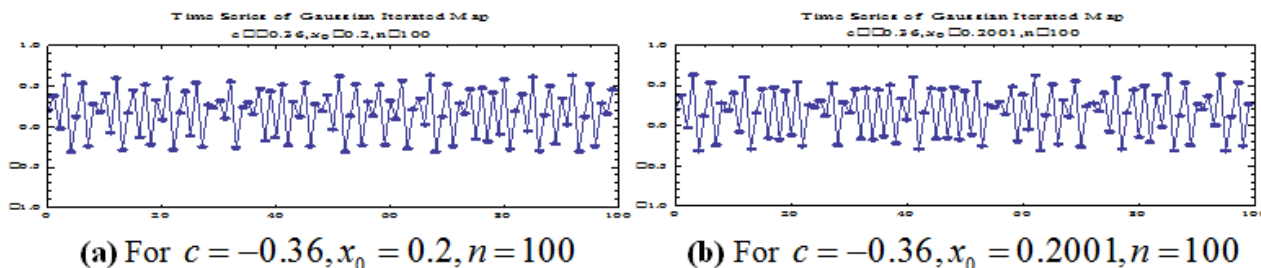
(a) For $c = -0.36, x_0 = 0.2, n = 100$ **(b)** For $c = -0.36, x_0 = 0.2001, n = 100$

Fig. 5.

Just when we think we are beginning to see a pattern in the above picture, the time series graphs begins to do something else and a new pattern emerges after some iteration we observe that there is no pattern in the above picture. This is called unpredictability which is another meaning if chaos [19, 23].

7.   Mathematical Analysis by Newton's Iteration

Consider the Mouse map $f(x) = e^{-7.5x^2} + c$ and choose $c = -0.36$. The Newton's iteration function associated with $f(x)$ is defined by

$$N(x) = x - \frac{f(x)}{f'(x)} = x - \frac{e^{-7.5x^2} - 0.36}{-15xe^{-7.5x^2}} = x + \frac{e^{-7.5x^2} - 0.36}{15xe^{-7.5x^2}} = \frac{15x^2e^{-7.5x^2} + e^{-7.5x^2} - 0.36}{15xe^{-7.5x^2}} = \frac{e^{-7.5x^2}(15x^2 + 1) - 0.36}{15xe^{-7.5x^2}}$$

Using the Mathematica program (Appendix), we get the following orbits for the points

$x = -0.20, -0.40, -0.60, 0.20, 0.40 \, \text{and} \, 0.60 \, \text{under} \, N.$

$-0.20 \rightarrow -0.3713 \rightarrow -0.36907 \rightarrow -0.36908 \rightarrow -0.36908 \rightarrow -0.36908$

$-0.40 \rightarrow -0.3674 \rightarrow -0.36907 \rightarrow -0.36908 \rightarrow -0.36908 \rightarrow -0.36908$

$-0.60 \rightarrow -0.1159 \rightarrow -0.46203 \rightarrow -0.34877 \rightarrow -0.36857 \rightarrow -0.36908 \rightarrow -0.36908$

$0.20 \rightarrow 0.3713 \rightarrow 0.36907 \rightarrow 0.36908 \rightarrow 0.36908 \rightarrow 0.36908$

$0.40 \rightarrow 0.3674 \rightarrow 0.36907 \rightarrow 0.36908 \rightarrow 0.36908 \rightarrow 0.36908$

$0.60 \rightarrow 0.1159 \rightarrow 0.46203 \rightarrow 0.34877 \rightarrow 0.36857 \rightarrow 0.36908 \rightarrow 0.36908$

Thus we see that orbit of any positive or negative real point under approaches to two fixed point for all real value of [17, 24]. The graphical analysis of N is shown in the following Fig.
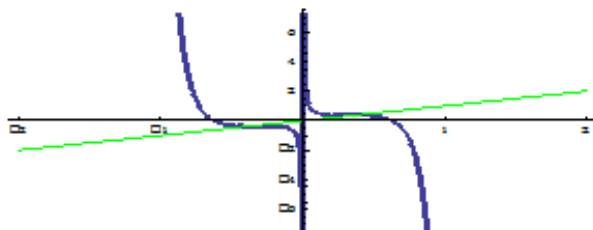
Fig. 6. : Newton's iteration function for $f(x) = e^{-7.5x^2} - 0.36$

8.   Cobweb diagram

Analysis: While all of this vocabulary is helpful, a visual presentation of orbits helps solidify the concept. We call these diagram cobweb plots and construct them as follows: Let x_0 be the seed of our orbit. In our plot we graph both

our function $f(x)$ and the line $g(x) = x$, in red. With these guidelines, we first trace a line, in black, from (x_0, x_0) to $(x_0, f(x_0))$, then from $(x_0, f(x_0))$ to $(f(x_0), f(x_0))$ (this is where plotting $g(x) = x$ is useful). From there we can trace a line to $(f(x_0), f^2(x_0))$, then to $(f^2(x_0), f^2(x_0))$ and so on. With these plots, we can find $f^n(x)$ for any n and perhaps more importantly, see how the orbit of x got to $f^n(x)$.

8.1. Cobweb diagram of Mouse map

With a basic understanding of cobweb plots, we can start to visual the behavior of Gaussian iterated map $f(x) = e^{-7.5\,x^2} + c$ with different values of c for larger values of n.
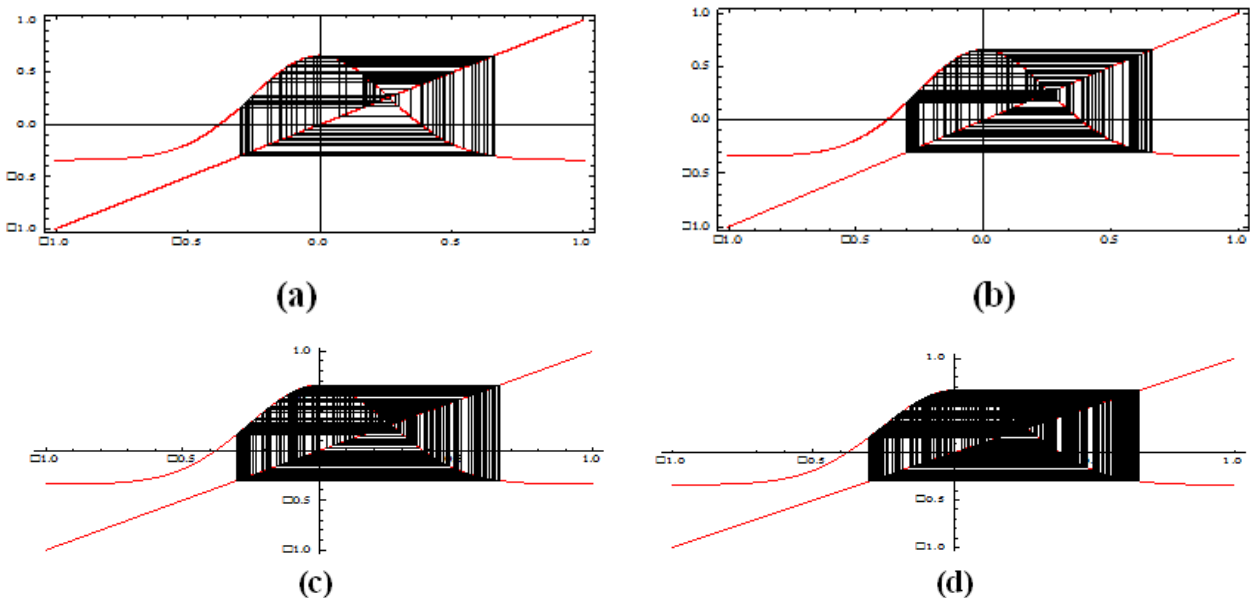
Fig. 7. **(a)** The cobweb plot of $x_0 = 0.1$ under the map $f(x) = e^{-7.5\,x^2} + c$ with $c = -0.34$ up to 50 iterations. Clearly the orbit of 0.1 is covering a fair amount of the interval $[-0.35, 0.7]$ **(b)** The cobweb plot of $x_0 = 0.1$ under the map $f(x) = e^{-7.5\,x^2} + c$ with $c = -0.34$ up to 100 iterations. We see that the orbit of 0.1 continues to hit new points **(c)** The cobweb plot of $x_0 = 0.1$ under the map $f(x) = e^{-7.5\,x^2} + c$ with $c = -0.34$ up to 200 iterations. The orbits still covering new ground. **(d)** The cobweb plot of $x_0 = 0.1$ under the map $f(x) = e^{-7.5\,x^2} + c$ with $c = -0.34$ up to 300 iterations

We see that the orbit of 0.1 continues to hit new points. Fig. 7(a), Fig. 7(b), Fig. 7(c) and Fig. 7(d) reveal that the orbit of $x_0 = 0.1$ under f seems to travel all over the interval $[-0.35, 0.70]$. This phenomenon is called chaotic behavior. So we can say that the given Mouse map is chaotic for $c = -0.34$ [18, 19].

9.   Trajectories of the Mouse map

In this section we want to draw trajectories of mouse map for different values of parameters and analyze the behavior of the stated map. We first write the equation in the form $x_{n+1} - x_n = (e^{-7.5\,x_n^2} + c) - x_n$ and plot the trajectories for different value of $c \in [-1, 1]$. To plot the sets of trajectories of Mouse map we first calculate a solution set by starting from various points and iterating the equation n times. The starting points are chosen between $x_0 1$ and $x_0 2$ in steps of dx_0. We get the following trajectories:

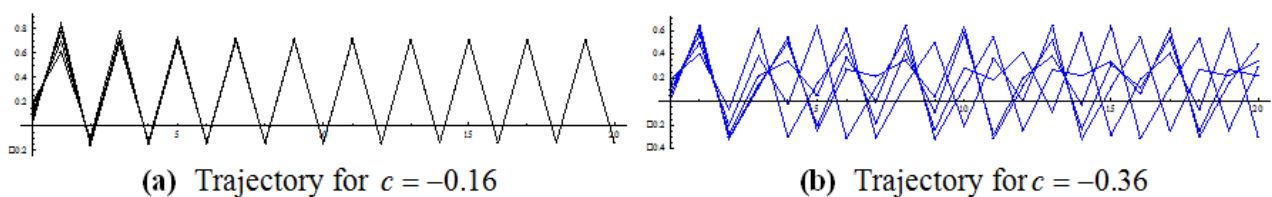(a) Trajectory for $c = -0.16$ (b) Trajectory for $c = -0.36$

Fig. 8.

In Fig. 8(a) all trajectories seem to form a cycle with two points. Again in Fig. 8(b) when $c = -0.36$, the trajectories appear to be chaotic. From the above trajectories of Mouse map we can find that it is chaotic for some $c \in [-1, 1]$ [17, 25].

10.   Histogram analysis of Mouse map

In this section we consider the Mouse map $f(x) = e^{-7.5\,x^2} + c$ and investigate the dynamical behavior of the given function by analyzing the histogram image.
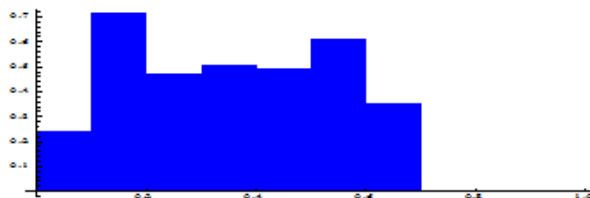

Fig. 9. Histogram of Mouse map for $c = -0.36$

From histogram of Fig. 9 we notice that the variable values of the given function fall into different bins or buckets and each bin touches each other. The variable of the given Mouse map is scattered and hence the function is chaotic for $c = -0.36$ [19, 25].

11.   Numerical inaccuracy of the Mouse map

For some values of the parameter c, the Mouse model

$$x_{n+1} = e^{-7.5\,x_n^2} + c$$

is very sensitive to numerical inaccuracies. To see this, we calculate 90 values from the model with $c = -0.56$, first by using normal decimal numbers and then by using high-precision numbers. In the latter case, we start with numbers that have a precision of 45 digits:

$$values1 = NestList[(\exp[-7.5*\#\textasciicircum 2] - .56)\&, 0.4, 90]; \quad values2 = NestList\left[(\exp[-7.5*\#\textasciicircum 2] - .56)\&, 0.4\grave{}45, 90\right]$$

Values corresponding to values2 are thick. From approximately iteration 69 on, the values differ greatly. In calculating values2, we started with numbers having 45 digits of precision. During the calculation, many digits were lost so that the last value -0.0199857 only has a precision of approximately 13.4587. Look at some elements of values2.
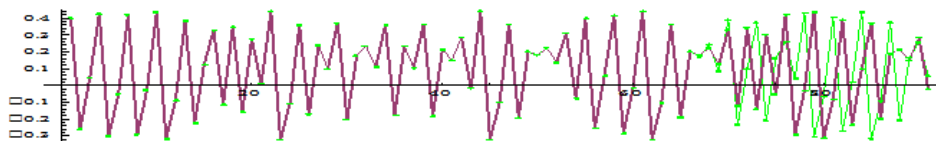

Fig. 10. Sensitivity to Numerical inaccuracies of the Mouse Map

We know that all the digits of values2are correct which means that the values in values1are incorrect from approximately iteration 69 on. This demonstrates the sensitivity to numerical inaccuracies of the Mouse map for some values of the parameter c. Thus, if we calculate long sequences from the Mouse function, it is important to use a high enough precision during the calculation. From the plot of values2, we see that the series behaves quite chaotically. It is known that chaotic models are very sensitive to numerical inaccuracies [8, 17, 22].

12.   Bifurcation diagram of Mouse map

The next figure shows the bifurcation diagram of the Mouse map, c along the $x-$axis.
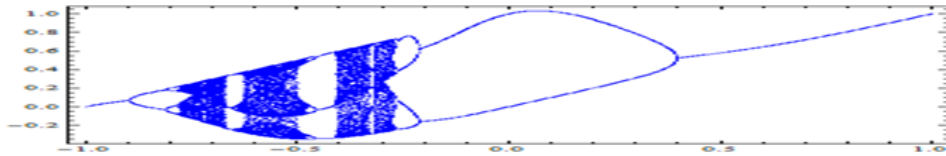
Fig. 11. Bifurcation diagram ofMouse map for $-1 \le c \le 1$

We observe that for the Mouse map

$$f(x) = e^{-7.5\,x^2} + c$$

, starting from $c = -0.90$ (approximately) period doubling orbits appear with the increase in parameter value that ends at the accumulation point -0.79(approximately), which is the point corresponding to an orbit of period 2^∞. Another accumulation point for the Mouse can be found to exist at -0.26(approximately) in which case the repeated period doubling scenario starts at $c = 0.39$(approximately) and continues with the decrease in the parameter value.From the above figure, we can see that the bifurcation diagram of the Mouse map reveals some more interesting behaviors. This shows that period-one behavior exists for two ranges of the parameter values which in fact is supported by the fact that the map has two points of inflection. This in turn means that a period-one point can make a transition from being stable to unstable and back again. As the parameter c is increased from $c = -1$, a fixed point of period one becomes unstable and a sequence of period doubling occurs through period-two, period-four and back to period-two again. As the parameter is increased still further, the unstable fixed point of period one becomes stable again and a single branch appears once more. Thus the bifurcation displays period doubling and period-undoubling bifurcations [18, 19, 21].

13. Conclusion

In this research, we have tried to show the 1-DMouse mapis chaotic for some parameter values in unit interval. We have shown all properties numerically and graphically. To perform such kind of activities we have used software's named Mathematica and MATLAB. In future we will try to establish new theorems taking these dynamics which control chaos.

Acknowledgement

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