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Solving nonlinear Volterra integral equations by using numerical techniques

Research Article

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Abstract: We present new numerical techniques to discover a new solution of nonlinear Volterra integral equations. The considered technique utilizes the homotopy perturbation method, Adomain decomposition and the variational iteration method. The explained techniques are delineated with a numerical case to demonstrate the benefit of the technique used by us.

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Keywords: Volterra integral equation • Adomain decomposition method • homotopy perturbation method • variational iteration method.

1. Introduction

In this paper, we shall be concerned with the non-homogeneous Volterra integral equations of the second kind of the form:

$$u(x) = f(x) + \lambda \int_a^x K(x,t)F(u(t))dt, \tag{1}$$

where a and x are the limits of integration, λ is a constant parameter, and $K(x,t)$ is a function of two variables x and t called the kernel of the integral equation. The function $u(x)$ that will be determined appears under the integral sign, and it appears inside the integral sign and outside the integral sign as well. The functions $f(x)$ and $K(x,t)$ are given in advance.

Volterra integral equations arise in many scientific applications, also Volterra integral equations can be derived from initial value problem [5, 24]. It is well known that linear and nonlinear Volterra integral equations arise in many scientific fields such as the population dynamics, spread of epidemics, and semi-conductor devices [18]. Volterra started working on integral equations in 1884, but his serious study began in 1896. The name integral equation was given by du Bois-Reymond in 1888 [10]. However, the name Volterra integral equation was first coined by Lalesco in 1908. Abel considered the problem of determining the equation of a curve in a vertical plane. In this problem, the time taken by a mass point to slide under the influence of gravity along this curve, from a given positive height, to the horizontal axis is equal to a prescribed function of the height [1, 6, 7, 11, 12, 23, 24].

The main objective of the present paper is to study the behavior of the solutions that can be formally determined by semi-analytical approximated methods as the Adomian decomposition method, variational iteration method and homotopy perturbation method.

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2. Description of the Methods

Some powerful methods have been focusing on the development of more advanced and efficient methods for Volterra integral equations such as the Adomian decomposition method, variational iteration method and homotopy perturbation method [2–4, 8, 9, 13–17].

2.1. Adomian Decomposition Method (ADM)

Adomian decomposition method [4, 8, 9] defines the unknown function $u(x)$ by an infinite series

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{2}$$

where the components $u_n(x)$ are usually determined recurrently. The nonlinear operator $F(u)$ can be decomposed into an infinite series of polynomials given by

$$F(u) = \sum_{n=0}^{\infty} A_n \tag{3}$$

where A_n are the so-called Adomian polynomials of u_0, u_1,..., u_n defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[F\Big(\sum_{i=0}^{n}\lambda^i u_i\Big)\right]_{\lambda=0} \quad , n = 0, 1, 2, \ldots \tag{4}$$

or equivalently

$$\begin{aligned}
&A_0 = F(u_0)\\
&A_1 = u_1 F'(u_0)\\
&A_2 = u_2 F'(u_0) + \frac{1}{2}u_1^2 F(u_0)\\
&A_3 = u_3 F'(u_0) + u_1 u_2 F''(u_o) + \frac{1}{3}u_1^3 F'''(u_0)\\
&\ldots.
\end{aligned} \tag{5}$$

It is now well known that these polynomials can be generated for all classes of nonlinearity according to specific algorithms defined by [10]. Recently, an alternative algorithm for constructing Adomian polynomials has been developed by Wazwaz [23].

$$\sum_{i=0}^{\infty} u_i(x) = f(x) + \lambda \sum_{i=0}^{\infty}\left(\int_a^x K(x,t) A_i(t)dt\right).$$

The components $u_0, u_1, u_2, \cdots$ are usually determined recursively by

$$\begin{aligned}
u_0 &= f(x)\\
u_1 &= \lambda \int_a^x K(x,t) A_0(t)dt,\\
u_n &= \lambda \int_a^x K(x,t) A_{n-1}(t)dt, \ n \geq 1.
\end{aligned} \tag{6}$$

Then, $u(x) = \sum_{i=0}^{n} u_i(x)$ as the approximate solution.

2.2. Variational Iteration Method (VIM)

The variational iteration method was recently developed by He, the method is widely used in the literature in a variety of scientific applications in and the references therein, there is an important advantage is that uses the initial conditions only and does not require the specific transformations for nonlinear terms as required by some existing techniques [19, 25]. the main goal from VIM to obtain the approximate soluations to the integral equations, also to be clear we will explain the basic idea of VIM, we consider following general form:

$$Lu(x) + N(x) = g(x), \tag{7}$$

where L is a linear operator, N is a non-linear operator and $g(t)$ is a known analytical function. According to the variational iteration method, we can construct the following correction functional can be constructed as

$$u_{n+1}(x) = u_n(x) + \int_0^x \mu(\xi)\Big[L\tilde{u}_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\Big]d\xi \tag{8}$$

where μ is a general Lagrange multiplier and the term $\tilde{u}_n$ is considered as a restricted variation, i.e. $\delta\tilde{u}_n = 0$. Making the above correction functional stationary, we get

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \mu(\xi)\Big[L\tilde{u}_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\Big]d\xi \tag{9}$$

In order to identify the Lagrange multiplier, from Eq. (9) we have

$$\delta u_{n+1}(x) = \delta u_n(x) + \delta\int_0^x \mu(\xi)\Big[L\tilde{u}_n(\xi) - g(\xi)\Big]d\xi \tag{10}$$

In general, the Lagrange multiplier μ, can be readily identified by imposing the stationary condition $\delta u_{n+1}(x) = 0$. on the correction functional (10). After determining the Lagrange multiplier μ and selecting an appropriate initial function u_0, the successive approximations u_n of the solution u can be readily obtained. successive approximations u_n of the solution u can be readily obtained. Consequently, the solution of Eq.(7) is given by

$$u(x) = \lim_{n\longrightarrow\infty} u_n(x) \tag{11}$$

In other words, the correction functional (8) will give successive approximations, and therefore the exact solution is obtained as the limit of the resulting successive approximations.

2.3. Homotopy Perturbation Method (HPM)

In recent years, many scientists and engineers were improved the homotopy perturbation method, it used to solve linear and nonlinear problems and it has been applied to many problems, specially in integral equation [20]. To illustrate the basic idea of this method, we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega, \tag{12}$$

under the boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{13}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω.
In general, the operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Eq. (12) therefore can be rewritten as follows [21]:

$$L(u) + N(u) - f(r) = 0. \tag{14}$$

By the homotopy technique (Liao 1992, 1997) [22]. $v(r, p) : \omega \times [0,1] \longrightarrow R$ which satisfies

$$H(v, p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, p \in [0,1]. \tag{15}$$

or

$$H(v, p) = L(v) - L(u_0) + pL(u_0)] + p[N(v) - f(r)] = 0, \tag{16}$$

where $p \in [0,1]$ is an embedding parameter, u_0 is an initial approximation of Eq.(12) which satisfies the boundary conditions. From Eqs.(15), (16) we have

$$\begin{aligned} H(v,0) &= L(v) - L(u_0) = 0, \\ H(v,1) &= A(v) - f(r) = 0. \end{aligned} \tag{17}$$

The changing in the process of p from zero to unity is just that of $v(r, p)$ from $u_0(r)$ to $u(r)$. In topology this is called deformation and $L(v) - L(u_0)$, and $A(v) - f(r)$ are called homotopic.

Now, assume that the solution of Eqs. (15), (16) can be expressed as

$$v = v_0 + pv_1 + p^2v_2 + \cdots. \tag{18}$$

The approximate solution of Eq.(12) can be obtained by setting $p = 1$.

$$u = \lim_{p\to 1} v = v_0 + v_1 + v_2 + \cdots. \tag{19}$$

3. Numerical Results

This section is devoted to reviewing ADM, VIM and HPM for solving the Volterra integral equation.

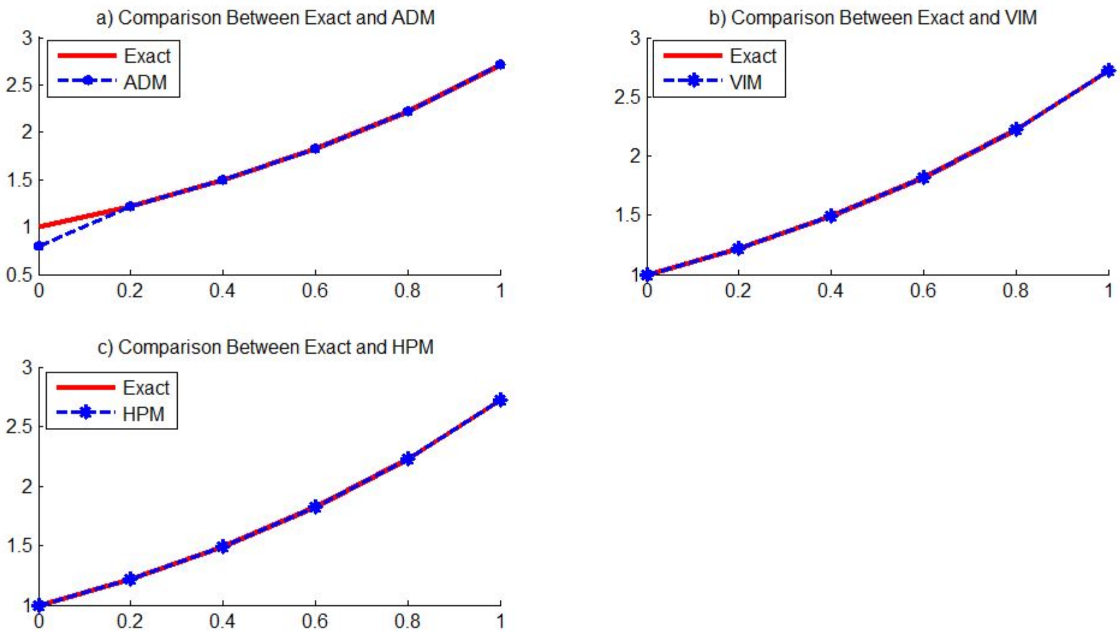
Example 3.1.
Consider the nonlinear Volterra integral equation:

$$u(x) = e^x - \frac{1}{3}e^{3x} + \frac{1}{3} + \int_0^x [u(t)]^3 dt,$$

with the exact solution $u(x) = e^x$.

Table 1.

x	Exact Solution	*ADM*	*VIM*	*HPM*
0	1	0.79486	1	1
0.2	1.22140	1.22052	1.22136	1.22140
0.4	1.49182	1.49163	1.49181	1.49182
0.6	1.82211	1.82204	1.82211	1.82211
0.8	2.22554	2.22513	2.22536	2.22554
1	2.71828	2.71666	2.71796	2.71828



The Table 1 shows comparison between the approximate solutions by using ADM, VIM and HPM for results of the Example 3.1.

4. Conclusion

We discussed different methods for solving Volterra integral equations, namely, Adomian decomposition method, variational iteration method and homotopy perturbation method. To assess the accuracy of each method, the test example with known exact solution is used. The study outlines important features of these methods as well as sheds some light on advantages of one method over the other. In this work, the above methods have been successfully employed to obtain the approximate solutions of a Volterra integral equations. The results show that these methods are very efficient, convenient and can be adapted to fit a larger class of problems. The comparison reveals that although the numerical results of these methods are similar approximately, HPM is the easiest, the most efficient and convenient.

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