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Unsteady flow of a Newtonian Fluid through a cylindrical collapsible tube

Research Article

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Abstract: In this research, unsteady flow of a Newtonian fluid flowing through a cylindrical collapsible tube has been investigated. The Governing equations are the equation of continuity, momentum and the tube law relation. This study has formulated a mathematical model of a Newtonian fluid in collapsible tube to simulate physiological flows such as flow of blood and urine within the human body system. The study has considered a two-dimensional model. A non-linear equation has been obtained from the governing equations. Finite difference approximation techniques have been employed to solve numerically this non-linear equation since they cannot be solved analytically. A computer program code has been used to obtain results which has been presented inform of graphs. The effects of varying different flow parameters on various flow variables has been determined numerically. It was noted that a change in flow parameters leads to either increase or decrease on flow variables. This study is very important because of its widespread, especially in the medical field where it will help the clinician to predict and prevent diseases such as cardiovascular diseases (heart stroke) or lung disease (asthma).

MSC: 74Bxx • 92Cxx

Keywords: Collapsible tube • Flow variables • Newtonian fluid

1. Introduction

Many physiological and medical applications have permitted a variety of researchers to study the effects of unsteady flow in collapsible tubes. The study of flow in elastic tubes has many applications within the human body system since such flows mainly occurs in many biological, medical and industrial environments. As noted by Ashraf et al [1] and Heil [3] that the study of unsteady flow in elastic tubes is important in interpreting the nature of blood flow in veins, arteries and airways. It is also important in interpreting the nature of flow of urine in the urethra. Jensen [4] suggested that, these vessels can collapse when the pressure outside is greater than the pressure inside. Important exceptions of such vessels are coronary arteries, placed in the muscular wall of the heart, which can remarkably constrict as the heart contracts and the brachia artery. This vessel is flattened by a cuff distended on the upper arm in the course of blood-pressure measurement. A research done by Cancelli [5] showed that hydrostatic pressure variations within the human body system can induce collapse since veins operates under much lower transmural pressure than arteries. This limit the flow of blood returning to the heart or flowing in the major organs as suggested by Ananthararaman et al [2]. Pulmonary system and urethra also display collapse, for instance, pulmonary airway can experience

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collapse during the closure and reopening and during forced or rapid expiration. Urethra encounters collapse during the malnutrition, where flow limitation is again common place. These observations are very important in the medical field whereby it helps clinical officers in studying and predicting diseases such as cardiovascular diseases (heart stroke) or lung disease (asthma). Numerical work in the laboratory have generated a rich variety of self-excited oscillation of fluid flow in elastic tubes. For example, Marchandise and Flaud [6] studied an accurate modelling of unsteady flow in collapsible tubes whereby the study constructed a numerical haemodynamic tool that was used by clinicians and researchers to investigate the effects of flow in elastic conduits. A study of the behavior of the wall thickness on post buckling behavior of the tube was done by Kozlovsky et al [7] in order to evaluate the real geometry of the deformed cross-sectional area due to negative pressure difference, where a computational model was utilized. The study came up with a new general tube law that included the physical contribution of wall thickness for description of the pressure area relationship for a wide range of thickness to radius ratio. Odejide [8] studied an incompressible viscous fluid flow and heat transfer in a collapsible tube with heat source or sink, where a perturbation series was used to evaluate the solution of the nonlinear equation which was obtained from the model which was in agreement with the study done by Alam et al [16]. The study found that Prandtl number was largely dependent on the rate of heat transfer. A generalized Newtonian fluid flow characterized by a power-law model, through a channel consisting of a wall with a flexible membrane under longitudinal tension was developed by Goswami et al [9]. The temporal behavior of membrane was found to be chaotic for shear-thinning fluid whereas it was stable for shear-thickening fluids at the parametric regime. A one-dimensional model for the unsteady fluid structure interaction between a soft-walled micro-channel and viscous fluid flow within it was developed by Inamdar and Christov [11]. An internal fluid structure interaction was employed to handle the stiff nonlinear algebraic problems within each time step. The dimensionless Young's modulus and Reynolds number were varied while the Strouhal number was fixed at a unit to explore the unsteady fluid structure interactions. Shabbir et al [12] carried out a theoretical study of unsteady non-Newtonian flow of blood through a rigid artery in the presence of multi-irregular stenosis. It was found that, the plug core radius was inversely proportional the stenosis height and the amplitude of the flow. The study also found that the wall sheer stress was directly proportional to the yield stress and the amplitude of the flow. Finally, any increment in the yield stress causes an increase in the steady-state pressure gradient. These studies have reviewed a very complicated Phenomena which allows the interaction of the flowing fluid and the tube wall. These interactions have motivated many researchers because of its desire of understanding flow behaviors within the human body system. A model for a Newtonian and powder law fluids in flexible circular-symmetric conduits was developed by Sochi [13] with the conduits based on a lubrication estimation where the velocity of the flow was taken to have its axially-dependent shape behavior at each cross-section for the given rheology and cross-sectional size. The effects of flow parameters of a Newtonian fluid through a cylindrical collapsible tube was analyzed by Kanyiri et al [14] whereby the flow parameters ware found to depend on flow variables. Tube stiffness and longitudinal tension was considered as the flow parameter which was increasing with the cross-sectional area of the tube and pressure drop. From the previous work much numerical and experimental progress has been made but the analytical modelling of a Newtonian fluid flowing through a collapsible tube remains to be discovered since it is not absolutely understood. This study therefore has focused on analyzing a two-dimensional model for a Newtonian fluid flowing unsteadily through a cylindrical collapsible tube. In this study, finite difference method has been used to solve the nonlinear equations. MATLAB software has been used to generate results which are present inform of graphs.

2. Mathematical Model

In the recent years, the analysis of flow in collapsible tube has attracted thorough attention. This is due to its occurrence for a diversity practice in many industrial systems and its capability to generate a variety of instabilities when using a rigid wall. Although much numerical and experimental progress has been made during the past decades, analytical representations of a Newtonian fluid flowing past collapsible tube remains to be discovered because it is not fully understood. In spite of some studies been done on Newtonian fluid flowing unsteadily through a collapsible tube over the past years, a clear understanding of it is not yet discovered. This study therefore has focused on analysing a unidirectional flow of a Newtonian fluid flowing unsteadily through a cylindrical collapsible tube. The collapsible tube has been taken to collapse in the transverse direction which is perpendicular to the direction of the main flow. The direction of the main flow is taken to be along the x direction velocity u.

2.1. Equations governing the flow

2.1.1. Continuity equation

$$\frac{\partial A}{\partial t}+\frac{\partial (Au)}{\partial x}+\frac{\partial (Av)}{\partial y}=0. \tag{1}$$

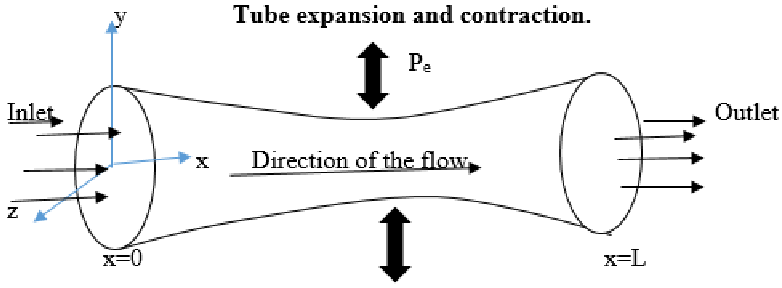

Fig. 1. Geometry of the problem

Considering the unidirectional flow and the precise relation, $Q = Au$, (1) becomes,

$$\frac{\partial A}{\partial t}+\frac{\partial Q}{\partial x}=0 \tag{2}$$

2.1.2. Equation of conservation of momentum

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}\right)=-\frac{\partial P}{\partial x}-\frac{8\pi\mu}{\rho A}u. \tag{3}$$

From the relation $Q = Au$, equation (3.) can be written as,

$$\rho\left(\frac{\partial\left(\frac{Q}{A}\right)}{\partial t}+\frac{Q}{A}\frac{\partial\left(\frac{Q}{A}\right)}{\partial x}\right)=-\frac{\partial p}{\partial x}-\frac{8\pi\mu}{\rho A}\frac{Q}{A}. \tag{4}$$

Equation (4) can be written as;

$$\frac{\rho}{A}\frac{\partial Q}{\partial t}+2\rho\frac{Q}{A^2}\frac{\partial Q}{\partial x}-\rho\frac{Q^2}{A^3}\frac{\partial A}{\partial x}=-\frac{\partial p}{\partial x}-\frac{8\pi\mu Q}{\rho A^2}. \tag{5}$$

Approximating pressure derivative using (5) yields,

$$\frac{\partial p}{\partial x}=\rho\frac{Q^2}{A^3}\frac{\partial A}{\partial x}-2\rho\frac{Q}{A^2}\frac{\partial Q}{\partial x}-\frac{\rho}{A}\frac{\partial Q}{\partial t}-\frac{8\pi\nu Q}{A^2}. \tag{6}$$

2.1.3. Tube Law

$$p-p_e=Z\left(\frac{A}{A_0}\right) \tag{7}$$

Considering the longitudinal tension and viscous damping force, the walls of the above equation can be written as,

$$p-p_e=K_{pe}\left[\left(\frac{A}{A_0}\right)^{10}-\left(\frac{A}{A_0}\right)^{-\frac{3}{2}}\right]-T\frac{\partial^2 A}{\partial x^2}+\gamma\frac{\partial A}{\partial t} \tag{8}$$

$$T\frac{\partial^2 A}{\partial x^2}-\gamma\frac{\partial A}{\partial t}=p_e-p+K_{pe}\left[\left(\frac{A}{A_0}\right)^{10}-\left(\frac{A}{A_0}\right)^{-\frac{3}{2}}\right] \tag{9}$$

2.2. Method of solution

2.2.1. Finite difference method

The principle of finite difference technique has been employed to solve the governing equations above. The final set of equation in finite difference form are given below,

$$\begin{aligned}\frac{p_{i+1}^m-P_i^m}{\Delta x}&=\rho\frac{\left(Q_i^m\right)^2}{\left(A_i^m\right)^3}\left(\frac{A_{i+1}^m-A_i^m}{\Delta x}\right)-2\rho\frac{Q_i^m}{\left(A_i^m\right)^2}\left(\frac{Q_{i+1}^m-Q_i^m}{\Delta x}\right)\\&\quad-\frac{\rho}{A_i^m}\left(\frac{Q_i^{m+1}-Q_i^m}{\Delta t}\right)-\frac{8\nu\pi Q_i^m}{\left(A_i^m\right)^2}.\end{aligned} \tag{10}$$

$$
\begin{aligned}
p_{i+1}^m &= p_i^m + \frac{\rho\left(Q_i^m\right)^2}{\left(A_i^m\right)^3}\left(A_{i+1}^m - A_i^m\right) - \frac{2\rho Q_i^m}{\left(A_i^m\right)^2}\left(Q_{i+1}^m - Q_i^m\right)\\
&\quad - \frac{\rho}{A_i^m}\left(Q_i^{m+1} - Q_i^m\right)\frac{\Delta x}{\Delta t} - \frac{8\Delta x \nu \pi Q_i^m}{\left(A_i^m\right)^2}.
\end{aligned}
\tag{11}
$$

$$
T\left(\frac{A_{i+1}^m - 2A_i^m + A_{i-1}^m}{(\Delta x)^2}\right) - \gamma\left(\frac{A_i^{m+1} - A_i^m}{\Delta t}\right) = p_e - p_i^m + K_{pe}\left[\left(\frac{A_i^m}{A_0}\right)^{10} - \left(\frac{A_i^m}{A_0}\right)^{-\frac{3}{2}}\right].
\tag{12}
$$

The right hand side of (12) is non-linear and can be linearised using the expansion of Taylor series of the term $K_{pe}\left[\left(\frac{A_i^m}{A_0}\right)^{10} - \left(\frac{A_i^m}{A_0}\right)^{-\frac{3}{2}}\right]$. This term is expanded about the point $A_i^m = c$ to get,

$$
\begin{aligned}
K_{pe}\left[\left(\frac{A_i^m}{A_0}\right)^{10} - \left(\frac{A_i^m}{A_0}\right)^{-\frac{3}{2}}\right] &\approx K_{pe}\left[\left(\frac{c}{A_0}\right)^{10} - \left(\frac{c}{A_0}\right)^{\frac{-3}{2}}\right]\\
&+ \left(A_i^m - c\right)K_{pe}\left[\frac{10}{c}\left(\frac{c}{A_0}\right)^{10} - \frac{3}{2c}\left(\frac{c}{A_0}\right)^{\frac{-3}{2}}\right]
\end{aligned}
\tag{13}
$$

Substituting (13) to equation (12) yields,

$$
\begin{aligned}
T\left(\frac{A_{i+1}^m - 2A_i^m + A_{i-1}^m}{(\Delta x)^2}\right) - \gamma\left(\frac{A_i^{m+1} - A_i^m}{\Delta t}\right) &= p_e - p_i^m + K_{pe}\left[\left(\frac{c}{A_0}\right)^{10} - \left(\frac{c}{A_0}\right)^{\frac{-3}{2}}\right]\\
&+ \left(A_i^m - c\right)K_{pe}\left[\frac{10}{c}\left(\frac{c}{A_0}\right)^{10} - \frac{3}{2c}\left(\frac{c}{A_0}\right)^{\frac{-3}{2}}\right]
\end{aligned}
\tag{14}
$$

2.2.2. Boundary conditions

$$
t > \begin{cases}
p(0) = p_0 & \\
A(0) = A_0 & x = 0\\
u(0) = u_0 & \\
 & \\
p(L) = p_0 & \\
A(L) = A_0 & x = L\\
u(L) = u_0 &
\end{cases}
\tag{15}
$$

Equation (14) can be written as $D\vec{A} = \vec{R}$ where D is the tridiagonal matrix and R is a vector.

$$
\begin{pmatrix}
\beta & -1 & 0 & 0 & \dots & 0\\
-1 & \beta & -1 & 0 & \dots & 0\\
0 & -1 & \beta & -1 & 0 & 0\\
\vdots & & \ddots & \ddots & & \\
\vdots & & & 1 & \beta & 1\\
0 & \dots & & & -1 & \beta
\end{pmatrix}
\begin{pmatrix}
A_2^m\\ A_3^m\\ A_4^m\\ \vdots\\ A_{N-2}^m\\ A_{N-1}^m
\end{pmatrix}
=
\begin{pmatrix}
\alpha_0 p_2^m - \alpha_0\gamma\frac{A_2^{m+1}}{\Delta t} + \lambda + A_0\\
\alpha_0 p_3^m - \alpha_0\gamma\frac{A_3^{m+1}}{\Delta t} + \lambda\\
\alpha_0 p_4^m - \alpha_0\gamma\frac{A_4^{m+1}}{\Delta t} + \lambda\\
\vdots\\
\alpha_0 p_{N-2}^m - \alpha_0\gamma\frac{A_{N-2}^{m+1}}{\Delta t} + \lambda\\
\alpha_0 p_{N-1}^m - \alpha_0\gamma\frac{A_{N-1}^{m+1}}{\Delta t} + \lambda + A_0
\end{pmatrix}
\tag{16}
$$

where

$$
\frac{(\Delta x)^2}{T} = \alpha_0
\tag{17}
$$

$$\beta = \alpha_0 K_{pe} \left[\frac{10}{c} \left(\frac{c}{A_0} \right)^{10} - \frac{3}{2c} \left(\frac{c}{A_0} \right)^{-\frac{3}{2}} \right] - \alpha_0 \frac{\gamma}{\Delta t} + 2$$

and (18)

$$\lambda = \alpha_0 p_e + \alpha_0 K_{pe} \left[\left(\frac{c}{A_0} \right)^{10} - \left(\frac{c}{A_0} \right)^{\frac{-3}{2}} \right] - \alpha_0 K_{pe} \left[10 \left(\frac{c}{A_0} \right)^{10} - \frac{3}{2} \left(\frac{c}{A_0} \right)^{\frac{-3}{2}} \right]$$

The above equations were iterated in MATLAB software to generate the following results.

3. Results and Discussion

3.1. Effects of varying longitudinal tension on flow variables

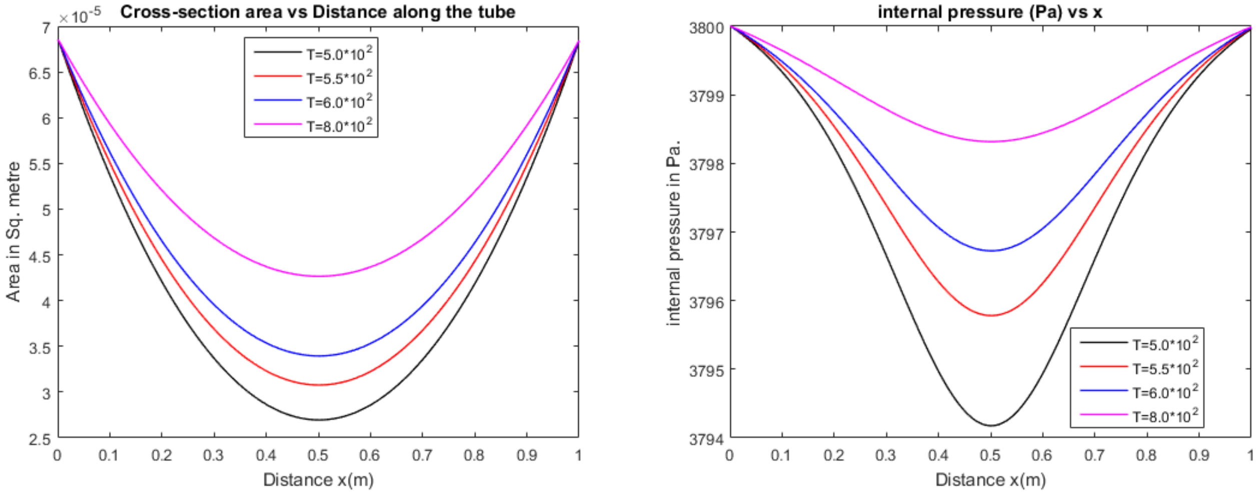

Fig. 2. Graphs showing the effects of varying longitudinal tensions on flow variables

Fig. 2 shows that longitudinal tension increases with an increase in cross-sectional area and internal pressure. An increase of longitudinal tension reduces the tube's tendency to collapse which leads to an increase in cross-sectional area. Since cross-sectional area has increased, then the flow velocity decreases causing an increase in internal pressure. When the fluid flow past a collapsible tube there is an impact between molecules which lowers the kinetic energy. Since the flow velocity has already decreased, this impact between molecules also decreases leading to a gain in kinetic energy causing an increase in internal pressure.

3.2. Effects of varying tube stiffness on flow variables

Fig. 3 shows that, tube stiffness increases with the increase of cross-sectional area. This leads to decrease in the collapsibility of the tube hence increasing the cross sectional area of the tube. Again, tube stiffness increases with a decrease in flow velocity. This is due to the already increased cross sectional area which resulted from the decrease of collapse. This can also be explained by an explicit relationship between the flow rate and velocity, Q = Av. In order to maintain this relationship, flow velocity must decrease since the cross-sectional area tube have already increased.

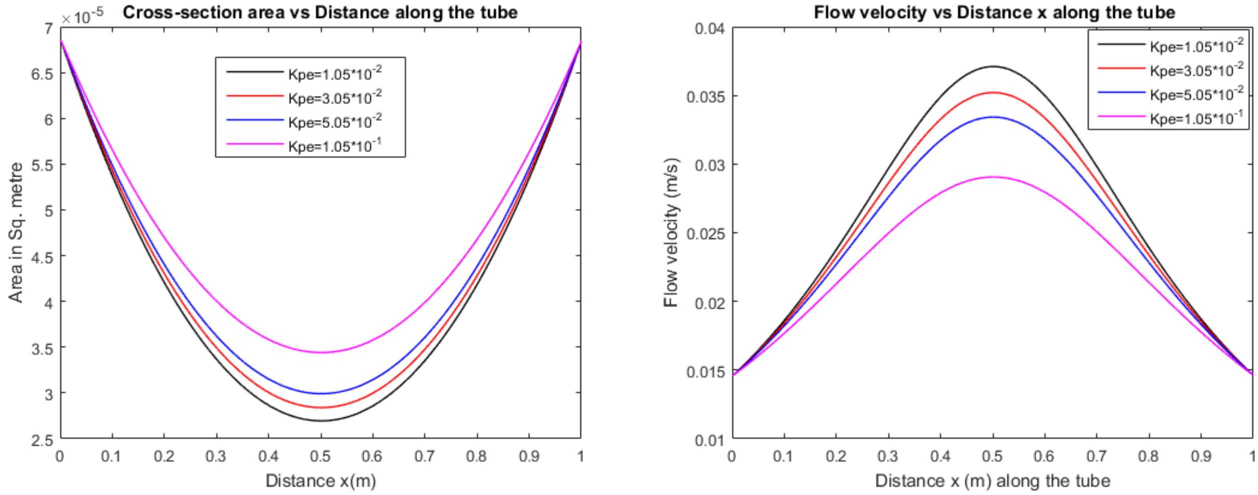

Fig. 3. Graphs showing the effect of tube stiffness on flow variables.

3.3. Effects of varying viscous damping on flow variables

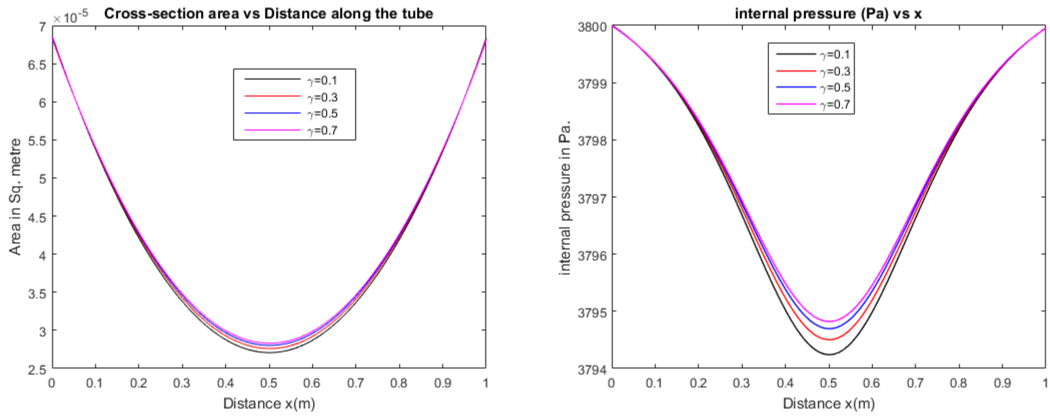

Fig. 4. Graphs showing the effect of damping viscous on flow variables

Fig. 4 shows that viscous damping increases with an increase in the cross-sectional area of the tube and decrease with a decrease in cross-sectional area. This is because an increase in viscous damping increases the viscosity of the fluid. Since viscosity is a measure to its resistance to deformation, an increase of it tends to resist the rate at which the tube will collapse hence causing an increase in the cross-sectional area of the tube. viscous damping also increases with an increase in internal pressure. This is because of the already decreased flow velocity. A decrease in flow velocity decreases the collision between molecules which in turn causes a gain in kinetic energy. This gain in kinetic energy causes the internal pressure to increase. Since an increase in viscous damping resists the deformation of the tube, the cross-sectional area of the tube increases causing the flow velocity to decrease which in turns increases the internal pressure.

3.4. Validation of results

Comparing the present study with the study done by Table [15] it is found that there exist a lot of similarities. In both studies longitudinal tension and tube stiffness increases with an increase in cross-sectional area and internal pressure but increase with a decrease in flow velocity. The results for tube stiffness for the work of [15] and the present study are tabulated below:

4. Conclusion

This study has formulated a mathematical model of a Newtonian fluid flowing unsteadily through a cylindrical collapsible tube where the effects of various flow parameters on flow variables have been determined. Finite differ-

Table 1. Internal pressure terms for various values of Tube stiffness K_{pe}

K_{pe}	pressure (Pa) [15]	pressure (Pa) [Present study]
1.05×10^{-2}	3993.9	3994.2
5.05×10^{-2}	3995.4	3995.5
1.05×10^{-1}	3997.2	3996.9

ence technique has been employed to find the numerical solution of the Governing equations. It was found that, an increase in longitudinal tension leads to an increase in cross-sectional area of the tube and internal pressure but leads to a decrease in flow velocity. Similar findings were also obtained when the tube stiffness was varied. Also, it was found that an increase in viscous damping leads to an increase in cross-sectional area of the tube and internal pressure but leads to a decrease in flow velocity. Both longitudinal tension and tube stiffness parameter reduces the tendency of the tube to collapse, which makes the cross-sectional area of the tube to increase. Viscous damping resists the rate of deformation making it difficult for the tube to collapse hence increasing the cross-sectional area of the tube. This study is very applicable in many biological systems and industrial areas. Within the human body system almost all the conduits are elastic and can accommodate distortion when the internal pressure is lower than the external pressure. This study is very important in the medical field since it can help clinicians to study and predict diseases such as cardiovascular diseases (heart stroke) or lung disease (asthma). The study is also important in the industries where it will help in predicting and preventing any possible collapse occurring in the industry.

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