Int. J. Adv. Appl. Math. and Mech. 7(1) (2019) 20 – 40 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

Symmetry Classification of First Integrals for Scalar Linearizable

Research Article

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Received 13 June 2019; accepted (in revised version) 19 August 2019

Abstract: In this research,we study Lie algebraic properties of first integrals of simplest second-, third and higher-order ordinary differential equations. Symmetries of the first integrals for simplest second-order ODEs which are linear or linearizable by point transformations have already been obtained. Firstly we show how one can determine the relationship between the point symmetries and the first integrals of linear or linearizable simplest ODEs of second order. Secondly, a complete classification of point symmetries of first integrals of such linear ODEs is studied. As a consequence, we provide a counting theorem for the point symmetries of first integrals of scalar linearizable second-order ODEs.We show that there exists the 0, 1 and 2 point symmetry cases. By use of Lie symmetry group methods we further analyze the relationship between the first integrals of the simplest linear third-order ODEs and their point symmetries. The simplest scalar linear third-order equation has seven point symmetries. We obtain the classifying relation between the symmetry and the first integral for the simplest equation. In the case of sub-maximal linear higher-order ODEs, we show that their full Lie algebras can be generated by the subalgebras of certain basic integrals. For the $n+2$ symmetry class, the symmetries of the first integral I_2 and a two-dimensional subalgebra of I_1 generate the symmetry algebra and for the $n+1$ symmetry class, the full algebra is generated by the symmetries of I_1 and a two-dimensional subalgebra of the quotient I_3/I_2.

MSC: 26A33 • 34A12

Keywords: Symmetries • First integral • Lie algebra

1. Introduction

The Lie algebraic classification of such differential equations is now well-known from the works of Lie [7] as well as recently Mahomed and Leach [8]. However, the algebraic properties of first integrals are not known except in the maximal cases for the basic first integrals and some of their quotients. In this research we investigate the complete problem for simplest second-order and maximal symmetry classes of higher-order ODEs using Lie algebras and Lie symmetry methods. More than a century ago, the Norwegian mathematician Sophus Lie put forward many of the fundamental ideas behind symmetry methods. This method is very successfully used in several branches of physics such as quantum filed theory,classical mechanics and physical chemistry. In our research we give complete classification of point symmetries for the first integrals of scalar linear second-order ODEs and the relationship between the symmetries and first integrals. For this purpose we use the projective transformations to find the different cases of symmetries for the first integrals of simplest second-order ODEs are linear or linearizable by point transformations. Since all scalar second-order ODEs which are linear or linearizable by point transformations are transformable to the free particle equation, we utilize this as our base ODE. We find that there are: the no symmetry, one symmetry, two

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symmetry.
It is well -known of first integrals for scalar linearizable second -order ODE [1, 6]

$$\phi(x, y, y', y'') = 0, \tag{1}$$

is invariant under the infinitesimal generator :

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} \tag{2}$$

if and only if

$$X^{[2]}\phi\mid_{\phi=0} = 0, \tag{3}$$

where $X^{[2]}$ is the second-order extended of Eq.(2) namely

$$\begin{aligned} X^{[2]} &= \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \zeta^{(1)}\frac{\partial}{\partial y'} + \zeta^{(2)}\frac{\partial}{\partial y''} \\ &= X + \zeta^{(1)}\frac{\partial}{\partial y'} + \zeta^{(2)}\frac{\partial}{\partial y''}, \end{aligned} \tag{4}$$

in which

$$\begin{aligned} \zeta^{(1)} &= D_x(\eta) - y' D_x(\xi) \\ &= \eta_x + (\eta_y - \xi_x)y' - \xi_y (y')^2, \end{aligned}$$

$$\begin{aligned} \zeta^{(2)} &= D_x(\zeta^{(1)}) - y'' D_x(\xi) \\ &= \eta_{xx} + y'\left(2\eta_{xy} - \xi_{xx}\right) + (y')^2\left(\eta_{yy} - 2\xi_{xy}\right) - \xi_{yy}(y')^3 + \left(\eta_y - 2\xi_x - 3\xi_y y'\right) y'', \end{aligned}$$

such that $\zeta^{(1)}$ is the linearized symmetry condition of second-order ODE and $\zeta^{(2)}$ is the linearized symmetry condition of second-order ODE and D_x is the total differentiation operator.
Now Eq.(2) can be considered as a point symmetry of Eq.(1) while in the case of first integrals, the first integral

$$I = h(x, y, y'),$$

of the ODE (1) , is exterminate by X, such that X is the symmetry generator of

$$I = h(x, y, y')$$

if and only if

$$X^{[1]} I = 0,$$

where $X^{[1]}$ is the first-order extended of Eq.(2) namely

$$X^{[1]} = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y} + \zeta^{(1)}\frac{\partial}{\partial y'}.$$

2. Symmetries of the first integrals

We consider the simplest second-order ODE as follows :

$$y'' = 0, \tag{5}$$

now we show that the Eq.(5) has the maximum number of symmetries by using the Linearized symmetry condition for Eq.(5) is

$$\zeta^{(2)} = 0 \text{ when } y'' = 0,$$

that leave us with

$$\zeta^{(2)} = \eta_{xx} + y'\left(2\eta_{xy} - \xi_{xx}\right) + (y')^2\left(\eta_{yy} - 2\xi_{xy}\right) - \xi_{yy}(y')^3 + \left(\eta_y - 2\xi_x - 3\xi_y y'\right) y'' = 0,$$

since $y'' = 0$ so this part

$$\left(\eta_y - 2\xi_x - 3\xi_y y'\right),$$

is canceled. So we get

$$\zeta^{(2)} = \eta_{xx} + y'\left(2\eta_{xy} - \xi_{xx}\right) + (y')^2\left(\eta_{yy} - 2\xi_{xy}\right) - \xi_{yy}(y')^3 = 0,$$

then

$$\eta_{xx} + y'\left(2\eta_{xy} - \xi_{xx}\right) + (y')^2\left(\eta_{yy} - 2\xi_{xy}\right) - \xi_{yy}(y')^3 = 0.$$

As $\xi(x,y)$ and $\eta(x,y)$ independent of y' then we can be split into a system of four equations

$$\eta_{xx} = 0, \tag{6}$$

$$2\eta_{xy} - \xi_{xx} = 0, \tag{7}$$

$$\eta_{yy} - 2\xi xy = 0, \tag{8}$$

$$\xi_{yy} = 0, \tag{9}$$

we integrate Eq.(9) with respect to y twice result in

$$\xi(x,y) = A(x)y + B(x),$$

since

$$\xi_{xy} = A'(x),$$

then substitute $\xi_{xy} = A'(x)$ into Eq.(8) yields

$$\eta_{yy} = 2A'(x), \tag{10}$$

integrate Eq.(10) with respect to y twice result in

$$\eta(x,y) = A'(X)y^2 + C(x)y + D(x).$$

We need to find η_{xy},ξ_{xx} and η_{xx}

$$\eta_{xy} = 2A''(x)y + C'(x),$$
$$\xi_{xx} = A''(x)y + B''(x),$$
$$\eta_{xx} = A'''(x)y^2 + C''(x)y + D''(x).$$

Now substituting $\xi(x,y)$ and $\eta(x,y)$ into Eqs.(6),(7) yields

$$\begin{aligned} 2\eta_{xy} - \xi_{xx} &= 2\left(2A''(x)y + C'(x)\right) - \left(A''(x)y + B''(x)\right) \\ &= 3A''(x)y + 2C'(x) - B''(x) = 0. \end{aligned} \tag{11}$$

Since $\eta_{xx} = 0$ then

$$A'''(x)y^2 + C''(x)y + D''(x) = 0. \tag{12}$$

Then from Eqs.(11),(12) we can determine that $A''(x) = 0$, $C''(x) = 0$, $D''(x) = 0$ and $B''(x) = 2C'(x)$ by integrate $A''(x) = 0$, $C''(x) = 0$ and $D''(x) = 0$ twice respectively we get the general solutions for $A(x)$, $B(x)$ and $C(x)$

$$A(x) = c_1x + c_2,$$
$$C(x) = c_3x + c_4,$$
$$D(x) = c_5x + c_6,$$

where $c_1, \cdots, c_6$ are constant.
Since $B''(x) = 2C'(x)$ and $C'(x) = C_3$ then

$$B''(x) = 2c_3, \tag{13}$$

Integrate Eq.(13) twice we get $B(x) = c_3x^2 + c_7x + c_8$,
now substituting $A(x)$, $B(x)$,$C(x)$, and $D(x)$ into $\xi(x, y)$ and $\eta(x, y)$ yields

$$\begin{aligned}\xi(x,y) &= A(x)y + B(x)\\ &= c_1 + c_3x + c_5y + c_7x^2 + c_8xy,\end{aligned} \tag{14}$$

and

$$\begin{aligned}\eta(x,y) &= A'(x)y^2 + C(x)y + D(x)\\ &= c_2 + c_4y + c_6x + c_7xy + c_8y^2,\end{aligned} \tag{15}$$

where $c_1,\cdots,c_8$ are constant.
Hence , the infinitesimal operator

$$X = \sum_{i=1}^{8} c_i X_i,$$

then

$$X = c_1X_1 + c_2X_2 + \cdots + c_8X_8.$$

Now by substituting Eqs.(14) and (15) in generator $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ we get

$$\begin{aligned}X =&(c_1 + c_3x + c_5y + c_7x^2 + c_8xy)\frac{\partial}{\partial x} + (c_2 + c_4y + c_6x + c_7xy + c_8y^2)\frac{\partial}{\partial y}\\ =&c_1\frac{\partial}{\partial x} + c_2\frac{\partial}{\partial y} + c_3x\frac{\partial}{\partial x} + c_4y\frac{\partial}{\partial y} + c_5x\frac{\partial}{\partial y} + c_6y\frac{\partial}{\partial x} + c_7(x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y})\\ &+ c_8(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}),\end{aligned}$$

The maximum number of symmetries [3] are

$$\begin{aligned}X_1 &= \frac{\partial}{\partial x},\\ X_2 &= \frac{\partial}{\partial y},\\ X_3 &= x\frac{\partial}{\partial x},\\ X_4 &= y\frac{\partial}{\partial y},\\ X_5 &= x\frac{\partial}{\partial y},\\ X_6 &= y\frac{\partial}{\partial y},\\ X_7 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y},\\ X_8 &= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}.\end{aligned}$$

Now by reference to simplest second -order ODE (5) we note that it has two functionally independent first integral , we find this by integrate Eq.(5) with respect to y get the first integrals, termed fundamental [4]

$$I_1 = y', \tag{16}$$

and integrate again yields

$$I_1x = y + I_2,$$

then

$$\begin{aligned}I_2 &= I_1x - y\\ &= xy' - y,\end{aligned} \tag{17}$$

that means

$$X^{[1]}(I_1)=0,$$

i.e.

$$\begin{aligned}X^{[1]}(I_1)&=\left(\xi\frac{\partial I_1}{\partial X}+\eta\frac{\partial I_1}{\partial y}+\zeta^{(1)}\frac{\partial I_1}{\partial y'}\right)\\&=\left(\xi\frac{\partial y'}{\partial X}+\eta\frac{\partial y'}{\partial y}+\zeta^{(1)}\frac{\partial y'}{\partial y'}\right)\\&=\zeta^{(1)}.\end{aligned}$$

If $X_1=\frac{\partial}{\partial x}$ then $\xi(x,y)=1$ and $\eta(x,y)=0$,
now we find $\zeta^{(1)}$ when $\xi(x,y)=1$ and $\eta(x,y)=0$ then

$$\begin{aligned}\zeta^{[1]}&=D_x(\eta)-y'D_x(\xi)\\&=D_x(0)-y'D_x[1]\\&=0,\end{aligned}$$

then

$$X^{[1]}(I_1)=0.$$

Now If $X_2=\frac{\partial}{\partial y}$ then $\xi(x,y)=0$ and $\eta(x,y)=1$

$$\begin{aligned}\zeta^{[1]}&=D_x(\eta)-y'D_x(\xi)\\&=D_x(1)-y'D_x(0)\\&=0,\end{aligned}$$

then

$$X^{[1]}(I_1)=0.$$

If $X_3=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ then $\xi(x,y)=x$ and $\eta(x,y)=y$

$$\begin{aligned}\zeta^{[1]}&=D_x(\eta)-y'D_x(\xi)\\&=D_x(y)-y'D_x(x)\\&=0,\end{aligned}$$

therefore

$$\begin{aligned}X_1&=x\frac{\partial}{\partial x},\\X_2&=x\frac{\partial}{\partial y},\\X_3&=x\frac{\partial}{\partial}+y\frac{\partial}{\partial y},\end{aligned}\tag{18}$$

can be considered a symmetry of I_1.
Now we find the symmetry of I_2. So we must check the condition

$$X^{[1]}(I_2)=0,$$

since

$$X^{[1]}(I_2)=\left(\xi\frac{\partial}{\partial x}+\eta\frac{\partial}{\partial y}+\zeta^{(1)}\frac{\partial}{\partial y'}\right)(xy'-y)=\xi y'+x\zeta^{(1)}-\eta$$

I_2 has three symmetries [1, 3]

$$\begin{aligned}H_1&=x\frac{\partial}{\partial x},\\H_2&=x\frac{\partial}{\partial y},\\H_3&=x^2\frac{\partial}{\partial}+xy\frac{\partial}{\partial y},\end{aligned}\tag{19}$$

if $H_1 = x\frac{\partial}{\partial x}$ then $\xi = x$ and $\eta = 0$ we get

$$\begin{aligned}\zeta^{(1)} &= D_x(\eta) - D_x(\xi)\\ &= D_x(0) - D_x(x)\\ &= -y,\end{aligned}$$

then

$$\begin{aligned}X^{[1]}(I_2) &= \xi y' + x\zeta^{(1)} - \eta\\ &= xy' - xy' - 0\\ &= 0.\end{aligned}$$

If $H_2 = x\frac{\partial}{\partial x}$ then $\xi = 0$ and $\eta = x$ and we get

$$\begin{aligned}\zeta^{(1)} &= D_x(\eta) - D_x(\xi)\\ &= D_x(x) - D_x(0)\\ &= 1,\end{aligned}$$

then

$$\begin{aligned}X^{[1]}(I_2) &= \xi y' + x\zeta^{(1)} - \eta\\ &= (0)y' + x(1) - x\\ &= x - x\\ &= 0.\end{aligned}$$

If $H_3 = x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}$ then $\xi(x,y) = x^2$ and $\eta(x,y) = xy$ this leads directly to

$$\begin{aligned}\zeta^{[1]} &= D_x(\eta) - y'D_x(\xi)\\ &= D_x(xy) - y'D_x(x^2)\\ &= y + xy' - 2xy'\\ &= y - xy',\end{aligned}$$

since

$$X^{[1]}(I_2) = \xi y' + x\zeta^{(1)} - \eta$$

then

$$\begin{aligned}X^{[1]}(I_2) &= x^2y' + x(y - xy') - xy\\ &= x^2y' + xy - x^2y' - xy\\ &= 0,\end{aligned}$$

therefore H_1 , H_2 , H_3 can be considered a symmetry of I_2.
Its clear that the symmetry of the first integral of I_1 are the same as that of I_2 if we multiply the symmetry of I_1 by x. Also the quotient of the first integrals I_1, I_2 has symmetries, which means

$$\frac{I_2}{I_1} = x - \frac{y}{y'}, \tag{20}$$

the quotient Eq.(20) of the first integrals I_1 and I_2 has three symmetries:

$$\begin{aligned}Y_1 &= y\frac{\partial}{\partial x},\\ Y_2 &= y\frac{\partial}{\partial y},\\ Y_3 &= xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y},\end{aligned} \tag{21}$$

which are the same as the symmetries Eq.(18) if we multiply the symmetries of I_1 by y that the Lie algebras of the symmetries of the first integrals I_1, I_2 and their quotient I_2/I_1 are isomorphic.

3. Classifying relation for the symmetries

In section 2 we finding the symmetries of the functionally independent first integrals I_1 and I_2 or their quotient of the simplest second-order ODE. Now we look at the study of symmetry properties of product $I_1 I_2$. So we need to find them by using symmetry condition. Instead of doing this each time by using symmetry condition A relationship can be found between symmetries and first integrations. The benefit of having such a relation enables us to also classify the first integrals of simplest second-order ODE equation in terms of their point symmetries.

Let F be an arbitrary function of I_1 and I_2, in other words $F = (I_1, I_2)$. The symmetry of this general function of the first integrals is

$$X^{[1]}F = X^{[1]}I_1 \frac{\partial F}{\partial I_1} + X^{[1]}I_2 \frac{\partial F}{\partial I_2} = 0, \tag{22}$$

in which

$$\begin{aligned} X^{[1]}(I_1) &= \left(\xi \frac{\partial}{\partial X} + \eta \frac{\partial}{\partial y} + \zeta^{(1)} \frac{\partial}{\partial y'}\right) y' = \zeta^{(1)}, \\ X^{[1]}(I_2) &= \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta^{(1)} \frac{\partial}{\partial y'}\right) (xy' - y) = \xi y' + x\zeta^{(1)} - \eta, \end{aligned} \tag{23}$$

from section 2 since

$$\begin{aligned} \xi(x, y) &= c_1 + c_3 x + c_6 y + c_7 x^2 + c_8 xy, \\ \eta(x, y) &= c_2 + c_4 y + c_5 x + c_7 xy + c_8 y^2, \end{aligned}$$

then

$$\begin{aligned} \zeta^{(1)} &= D_x(\eta) - y' D_x(\xi) \\ &= D_x\left(c_2 + c_4 y + c_6 x + c_7 xy + c_8 y^2\right) - y' D_x\left(c_1 + c_3 x + c_5 y + c_7 x^2 + c_8 xy\right) \\ &= \frac{\partial}{\partial x}\left(c_2 + c_4 y + c_6 x + c_7 xy + c_8 y^2\right) + y' \frac{\partial}{\partial y}\left(c_2 + c_4 y + c_6 x + c_7 xy + c_8 y^2\right) \\ &\quad - y'\left[\frac{\partial}{\partial x}\left(c_1 + c_3 x + c_5 y + c_7 x^2 + c_8 xy\right) + y' \frac{\partial}{\partial y}\left(c_1 + c_3 x + c_5 y + c_7 x^2 + c_8 xy\right)\right] \\ &= -y' c_3 + y' c_4 + c_5 - (y')^2 c_6 + \left(y - xy'\right) c_7 + \left(yy' - x(y')^2\right) c_8, \end{aligned}$$

Now substituting the values of $X^{[1]}(I_2)$, $X^{[1]}(I_2)$ as in Eq.(23) with ξ, η and $\zeta^{(1)}$ in Eq.(22) yields

$$\zeta^{(1)} \frac{\partial F}{\partial I_1} + \left(\xi y' + x\zeta^{(1)} - \eta\right) \frac{\partial F}{\partial I_2} = 0,$$

then

$$\begin{aligned} &\left[-y' c_3 + y' c_4 + c_5 - (y')^2 c_6 + \left(y - xy'\right) c_7 + \left(yy' - x(y')^2\right) c_8\right] \frac{\partial F}{\partial I_1} \\ &+ \Big[\left(c_1 + c_3 x + c_6 y + c_7 x^2 + c_8 xy\right) y' + \left(-y' c_3 + y' c_4 + c_5 - (y')^2 c_6 + \left(y - xy'\right) c_7 + \left(yy' - x(y')^2\right) c_8\right) x \\ &\qquad - \left(c_2 + y c_4 + x c_5 + xy c_7 + y^2 c_8\right)\Big] \frac{\partial F}{\partial I_2} = 0, \end{aligned} \tag{24}$$

Simplify the relationship of Eq.(24) more we get

$$\begin{aligned} &\left[-y' c_3 + y' c_4 + c_5 - (y')^2 c_6 - \left(xy' - y\right) c_7 - y'\left(xy' - y\right) c_8\right] \frac{\partial F}{\partial I_1} \\ &+ \Big[c_1 y' - c_2 + c_3 xy' - c_3 xy' + c_4 xy' - c_4 y + c_5 x - c_5 x + c_6 yy' - c_6 x(y')^2 \\ &\qquad + c_7 x^2 y' + c_7 xy - c_7 x^2 y' - c_7 xy + c_8 xyy' + c_8 xyy' - c_8 x^2 (y')^2 - c_8 y^2\Big] \frac{\partial F}{\partial I_2} = 0, \end{aligned} \tag{25}$$

We deleted some terms of Eq. (25) yield

$$\begin{aligned} &\left[-y' c_3 + y' c_4 + c_5 - (y')^2 c_6 - \left(xy' - y\right) c_7 - y'\left(xy' - y\right) c_8\right] \frac{\partial F}{\partial I_1} \\ &\qquad + \left[c_1 y' - c_2 + c_4 xy' - c_4 y + c_6 yy' - c_6 x(y')^2 + c_8 xyy' + c_8 xyy' - c_8 x^2 (y')^2 - c_8 y^2\right] \frac{\partial F}{\partial I_2} = 0, \end{aligned}$$

More simplify equation above yield

$$\left[-y'c_3 + y'c_4 + c_5 - (y')^2 c_6 - \left(xy' - y\right)c_7 - y'\left(xy' - y\right)c_8\right]\frac{\partial F}{\partial I_1} + \left[\left(c_1 y' - c_2 + c_4(xy' - y) - c_6 y'(xy' - y) - c_8 (xy' - y)^2\right)\right]\frac{\partial F}{\partial I_2} = 0, \quad (26)$$

then by using the relations $I_1 = y'$ and $I_2 = xy' - y$ from Eqs.(16) and (17), we finally arrive at the classifying relation

$$\left(-I_1 c_3 + I_1 c_4 + c_5 - I_1{}^2 c_6 - I_2 c_7 - I_1 I_2 c_8\right)\frac{\partial F}{\partial I_1} + \left(I_1 c_1 - c_2 + I_2 c_4 - I_1 I_2 c_6 - I_2{}^2 c_8\right)\frac{\partial F}{\partial I_2} = 0, \quad (27)$$

the relation of Eq.(27) represent the relationship between the symmetries and first integrals of the simplest the second-order Eq.(5)

4. Symmetry cases of first integrals

We use the classifying relation (27) to determine the number of symmetries formed by the first integrations of the simplest the second-order Eq.(5).
In this research we study three cases .

Case 1. No Symmetry.

If F is any arbitrary function of I_1 and I_2 then $\partial F/\partial I_1$ and $\partial F/\partial I_2$ are not linked with each other. From the Eq.(27), since $\partial F/\partial I_1 \neq 0$ and $\partial F/\partial I_2 \neq 0$ then must be

$$\left(-I_1 c_3 + I_1 c_4 + c_5 - I_1{}^2 c_6 - I_2 c_7 - I_1 I_2 c_8\right) = 0, \quad (28)$$

and

$$\left(I_1 c_1 - c_2 + I_2 c_4 - I_1 I_2 c_6 - I_2{}^2 c_8\right) = 0, \quad (29)$$

from Eqs.(28), (29) can see that all the $c's$ equals zero. Therefore there is no symmetry in this case.
Take an illustrative example of the case, if $F = I_1 \ln I_2$, then Eq.(27) yields

$$\begin{aligned}&\left(-I_1 c_3 + I_1 c_4 + c_5 - I_1{}^2 c_6 - I_2 c_7 - I_1 I_2 c_8\right) = 0,\\ &\left(I_1 c_1 - c_2 + I_2 c_4 - I_1 I_2 c_6 - I_2{}^2 c_8\right) = 0.\end{aligned} \quad (30)$$

This leads directly results in all the $c's$ equal zero.

Case 2. One Symmetry.

Firstly we notice that if F satisfies the classifying relation (27), then X which is a linear combination of the simplest the second-order generators, is a symmetry of this classifying relation. We also observe from (27) that if one has any of the free symmetry generators X_i as a symmetry of a first integral of the equation, then one ends up with three symmetries. That is one can have more than one symmetry. Now if we take $X = \frac{\partial}{\partial x}$ then a_1 arbitrary then the relation (27) becomes

$$-I_1 c_3 + I_1 c_4 + c_5 - I_1{}^2 c_6 - I_2 c_7 - I_1 I_2 c_8 = 0,$$

because $\frac{\partial F}{\partial I_2} = 0$ and $\frac{\partial F}{\partial I_1} \neq 0$ this shows that a_2 arbitrary and $a_3 = a_4$, and a_5 , a_6 , a_7 ,a_8 are zero, if $a_3 = a_4$ this means $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial x}$. Thus we will get the three symmetries

$$X_1 = \frac{\partial}{\partial x},$$
$$X_2 = \frac{\partial}{\partial x},$$
$$X_3 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Another example if we take c_3 arbitrary, $X = x\frac{\partial}{\partial x}$, then Eq.(27) yields (since $\frac{\partial F}{\partial I_2} \neq 0$ and $\frac{\partial F}{\partial I_1} = 0$)

$$\left(I_1 c_1 - c_2 + I_2 c_4 - I_1 I_2 c_6 - I_2{}^2 c_8\right) = 0,$$

this shows that a_5, a_7 arbitrary and a_1, a_2, a_4, a_6, a_8 are zero. Thus we will get the three symmetries

$$\begin{aligned} X_1 &= x\frac{\partial}{\partial x}, \\ X_2 &= x\frac{\partial}{\partial y}, \\ X_3 &= x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}. \end{aligned}$$

We have many cases where only one symmetry occurs . If we take $F = I_1 I_2$ or any function of the product , then the relation (27) gives exactly one symmetry.
Since $\frac{\partial F}{\partial I_1} = I_2$ and $\frac{\partial F}{\partial I_2} = I_1$ by substitution this in Eq.(27) yields

$$\begin{aligned} &\left(-I_1 c_3 + I_1 c_4 + c_5 - {I_1}^2 c_6 - I_2 c_7 - I_1 I_2 c_8\right) I_2 \\ &\quad + \left(I_1 c_1 - c_2 + I_2 c_4 - I_1 I_2 c_6 - {I_2}^2 c_8\right) I_1 = 0. \end{aligned} \tag{31}$$

Further simplification of relationship (31) we get

$$\begin{aligned} &-I_1 I_2 c_3 + I_1 I_2 c_4 + I_2 c_5 - {I_1}^2 I_2 c_6 - {I_2}^2 c_7 - I_1 {I_2}^2 c_8 \\ &\quad + {I_1}^2 c_1 - c_2 I_1 + I_1 I_2 c_4 - {I_1}^2 I_2 c_6 - I_1 {I_2}^2 c_8 = 0. \end{aligned} \tag{32}$$

Finally after simplification of the relationship (32) we get

$$-I_1 I_2 c_3 + 2I_1 I_2 c_4 + I_2 c_5 - 2{I_1}^2 I_2 c_6 - {I_2}^2 c_7 - 2I_1 {I_2}^2 c_8 + {I_1}^2 c_1 - c_2 I_1 = 0.$$

Now put $c_3 = 2c_4$ then Obviously it is c_3, and c_4 are arbitrary and c_1, c_2, c_5, c_6, c_7 and c_8 are zero, since c_3, and c_4 are arbitrary, i.e. $X_3 = x\frac{\partial}{\partial x}$ and $X_4 = y\frac{\partial}{\partial y}$ then the exist exactly one symmetry in this case :

$$X = 2x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

As another example, if we take $F = I_2 e^{-I_1}$, then $\frac{\partial F}{\partial I_1} = -I_2 e^{-I_1}$ and $\frac{\partial F}{\partial I_2} = e^{-I_1}$
Now by using Eq.(27) we get:

$$\begin{aligned} &\left(-I_1 c_3 + I_1 c_4 + c_5 - {I_1}^2 c_6 - I_2 c_7 - I_1 I_2 c_8\right) - I_2 e^{-I_1} \\ &\quad + \left(I_1 c_1 - c_2 + I_2 c_4 - I_1 I_2 c_6 - {I_2}^2 c_8\right) e^{-I_1} = 0, \end{aligned} \tag{33}$$

and by simplifying the relationship (35) yields

$$\begin{aligned} &I_1 I_2 e^{-I_1} c_3 - I_1 I_2 e^{-I_1} - I_2 e^{-I_1} c_5 + {I_1}^2 I_2 e^{-I_1} c_6 + {I_2}^2 e^{-I_1} c_7 \\ &\quad + I_1 {I_2}^2 e^{-I_1} c_8 + I_1 e^{-I_1} c_1 - e^{-I_1 c_2} + I_2 e^{-I_1} c_4 = 0. \end{aligned}$$

Now put $c_4 = 2c_5$ then obviously it is c_4 , and c_5 are arbitrary and c_1, c_2, c_3, c_6, c_7 and c_8 are zero, then there exist exactly one symmetry in this case

$$\begin{aligned} X &= y\frac{\partial}{\partial y} + 2x\frac{\partial}{\partial y} \\ &= X_4 + 2X_5. \end{aligned}$$

Well if we take $F = e^{{I_1}^2} I_2$, then by using the relation (27) we get the one symmetry

$$X = 3x\frac{x}{\partial x} + 2y\frac{\partial}{\partial y}.$$

So there is an infinite number of one symmetry in cases. In order to show that consider the first integral

$$F = \frac{1}{2}{I_1}^2 - bI_2, b \neq 0.$$

By using (27) get

$$\begin{aligned} &\left(-I_1 c_3 + I_1 c_4 + c_5 - {I_1}^2 c_6 - I_2 c_7 - I_1 I_2 c_8\right) I_1 \\ &+ \left(I_1 c_1 - c_2 + I_2 c_4 - I_1 I_2 c_6 - {I_2}^2 c_8\right)(-b) = 0, \end{aligned}$$

separation with respect to powers of I_1 and I_2 yields

$$-{I_1}^2c_3 + {I_1}^2c_4 + c_5I_1 - {I_1}^3c_6 - I_1I_2c_7 - {I_1}^2I_2c_8$$
$$-I_1bc_1 + bc_2 - I_2bc_4 + I_1I_2bc_6 + {I_2}^2BC_8 = 0,$$

obviously it is c_1, and c_5 are arbitrary and c_2, c_3, c_4, c_6, c_7 and c_8 are zero then there exist exactly one symmetry in this case $c_5 = cc_1$ .i.e.

$$X = x\frac{\partial}{\partial y} + b\frac{\partial}{\partial x}$$
$$= X_5 + cX_1.$$

Therefore the one symmetry case is not unique.

Case 2. Two Symmetry.

If we have the symmetries $X = \frac{\partial}{\partial y}$ and $X = x\frac{\partial}{\partial y}$ which form the two dimensional Abelian algebra, then a_2 and a_5 are arbitrary in Eq. (27) this mean that $\frac{\partial F}{\partial I_1} = \frac{\partial F}{\partial I_2} = 0$. In this case there is no symmetry for any first integral of the simplest second-ordr ODE.
The same if we consider $X = \frac{\partial}{\partial y}$ and $Y = x\frac{\partial}{\partial x}$ which forms a two-dimensional non-Abelian algebra.
Here again there is no symmetry for any first integral of the simplest second-ordr ODE. In previous cases, if the symmetry is of a simple type, it can not be obtaine two symmetries of an integral.
Thus there have to be combinations of the symmetries. Example of that combination, if we take $F = I_2 - I_1$ have independent integral and satisfy the one condition

$$\frac{\partial F}{\partial I_1} + \frac{\partial F}{\partial I_2} = 0,$$

hence this F admits two symmetries. Since $\frac{\partial F}{\partial I_1} = -1$ and $\frac{\partial F}{\partial I_2} = 1$ substituting this in the relation (27) we get

$$\left(-I_1c_3 + I_1c_4 + c_5 - {I_1}^2c_6 - I_2c_7 - I_1I_2c_8\right)(-1)$$
$$+\left(I_1c_1 - c_2 + I_2c_4 - I_1I_2c_6 - {I_2}^2c_8\right)(1) = 0, \quad (34)$$

simplify the relationship (34)

$$I_1c_3 - I_1c_4 - c_5 + {I_1}^2c_6 + I_2c_7 + I_1I_2c_8 + I_1c_1 - c_2 + I_2c_4 - I_1I_2c_6 - {I_2}^2c_8 = 0,$$

obviously it is c_1, c_2, c_3 and c_5 are arbitrary and c_4, c_6 and c_7 are zero thats mean $a_3 = -a_1$ and $a_5 = -a_2$ then

$$X = \frac{\partial}{\partial x} - x\frac{\partial}{\partial x}, \quad (35)$$
$$Y = \frac{\partial}{\partial y} - x\frac{\partial}{\partial y}. \quad (36)$$

The Lie algebra component of Lie bracket $[X, Y] = -Y$.[9]

Remark 4.1.
Lie Bracket of two vector field

$$[X, Y] = [a\frac{\partial}{\partial x_i}, b\frac{\partial}{\partial x_j}]$$
$$= a\frac{\partial b}{\partial x_i}\frac{\partial}{\partial x_j} - b\frac{\partial a}{\partial x_j}\frac{\partial}{\partial x_i},$$

if $a = b = 1$ then $[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}] = [X, Y] = 0$
Now we find the Lie bracket of (35)

$$[X, Y] = [\frac{\partial}{\partial x} - x\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial y}]$$
$$= [\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] + [\frac{\partial}{\partial x}, -x\frac{\partial}{\partial y}] + [-x\frac{\partial}{\partial x}, \frac{\partial}{\partial y}] + [-x\frac{\partial}{\partial x}, -x\frac{\partial}{\partial y}].$$

since $[\frac{\partial}{\partial x},\frac{\partial}{\partial y}]=0$, because $a=b=1$ then

$$\begin{aligned}[X,Y] &= \left[1\frac{\partial(-x)}{\partial x}\frac{\partial}{\partial y}-(-x)\frac{\partial(1)}{\partial y}\frac{\partial}{\partial x}\right]+\left[-x\frac{\partial(1)}{\partial x}-(1)\frac{\partial(-x)}{\partial y}\frac{\partial}{\partial x}\right]\\ &+\left[-x\frac{\partial(-x)}{\partial x}\frac{\partial}{\partial y}-(-x)\frac{\partial(-x)}{\partial y}\frac{\partial}{\partial x}\right]\\ &=[-\frac{\partial}{\partial y}+0]+[0-0]+[x\frac{\partial}{\partial y}]\\ &=-\left(\frac{\partial}{\partial y}-x\frac{\partial}{\partial y}\right)\\ &=-Y.\end{aligned}$$

Now if we take $F=c\frac{I_2}{I_1}-\frac{1}{I_1}$, then $\frac{\partial F}{\partial I_1}=\frac{1}{{I_1}^2}-c\frac{I_2}{{I_1}^2}$ and $\frac{\partial F}{\partial I_2}=\frac{c}{I_1}$, by using the relation (27) subtituting this, we get

$$-\frac{c_3}{I_1}+cc_3\frac{I_2}{I_1}+\frac{c_4}{I_1}+\frac{c_5}{{I_1}^2}-cc_5\frac{I_2}{{I_1}^2}-c_6-c_7\frac{I_2}{{I_1}^2}+cc_7\frac{{I_2}^2}{{I_1}^2}-\frac{I_2}{I_1}c_8+c_1c-\frac{c_2C}{I_1}=0,$$

clear it $c_4=c_2c$ and $c_1c=c_6$ thats means

$$X=cX_1+X_6,$$
$$Y=cX_2+X_4,$$

two symmetry span a two-dimensional algebra with

$$[X,Y]=-cX.$$

We conclude from the above examples that the two symmetry case is not unique.

Now if we take the simplest first-order ODE

$$y'=0, \tag{37}$$

by using Lie symmetry condition $\zeta^{(1)}=0$ of Eq.(37) yields

$$\eta_x+(\eta_y-\xi_x)y'-\xi_y(y')^2=0,$$

we set the coefficient y' to zero

$$\eta_x=0, \tag{38}$$

$$\eta_y-\xi_x=0, \tag{39}$$

$$\xi_y=0, \tag{40}$$

from Eq.(40)

$$\xi=A(x)\quad\text{then}\quad\xi_x=A'(x),$$

from Eq.(39)

$$\eta_y=\xi_x\quad\text{then}\quad\eta_y=A'(x), \tag{41}$$

by integrate both side of Eq.(41) yields

$$\eta=A'(x)y+B(x),$$

then

$$\eta_x=A''(x)y+B'(x).$$

Finally we get the tangent vector.
Since $\eta_x = 0$ and $\eta_x = A''(x)y + B'(x)$, so

$$A''(x)y + B''(x) = 0,$$

this lead to $A''(x) = 0$ and $B'(x) = 0$, by twice integrate $A''(x) = 0$ both side yields

$$A(x) = c_1 x + c_2,$$

and by integrate $B'(x) = 0$ both side yields

$$B(x) = c_3,$$

where c_1, c_2 and c_3 are constant,
now by substituting $B(x)$ and $A'(x)$ in value of η yields

$$\eta = c_1 y + c_3.$$

Now the infinitesimal operator

$$\begin{aligned} X &= \sum_{i=1}^{3} c_i X_i \\ &= c_1 X_1 + c_2 X_2 + c_3 X_3, \end{aligned}$$

since

$$X = \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \zeta^{(1)} \frac{\partial}{\partial y'},$$

and

$$\zeta^{(1)} = 0,$$

then

$$\begin{aligned} X &= \xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} \\ &= (c_1 X + c_2) \frac{\partial}{\partial x} + (c_1 y + c_3) \frac{\partial}{\partial y} \\ &= c_1 (x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}) + c_2 \frac{\partial}{\partial x} + c_3 \frac{\partial}{\partial y}, \end{aligned}$$

so the Eq.(37) which has the symmetries

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y}, \\ X_2 = X \frac{\partial}{\partial x} &+ y \frac{\partial}{\partial y}, \\ X_3 &= \frac{\partial}{\partial y}. \end{aligned}$$

If we take

$$X = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$$

it easy to see that $X = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is a point symmetry generator of Eq.(37) if $X^{[1]} y' \mid_{y'=0} = 0$ with

$$\zeta^{(1)} = D_x(\eta) - y' D_x(\xi),$$

in which D_x is the totaly differential operator and $X^{[1]}$ is the first extend operator X
i.e

$$\begin{aligned} \zeta^{(1)} &= D_x(\eta) - y' D_x(\xi) \\ &= y \frac{\partial y}{\partial x} + y' \frac{\partial y}{\partial y} - y' \left(\frac{\partial x}{\partial x} + y' \frac{\partial x}{\partial y} \right) \\ &= y' - y' \\ &= 0. \end{aligned}$$

$$X^{[1]}y' = \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y}\right)y'$$
$$= \left(\xi\frac{\partial y'}{\partial x} + \eta\frac{\partial y'}{\partial y}\right)$$
$$= X\frac{\partial y'}{\partial x} + y\frac{\partial y'}{\partial y}$$
$$= 0,$$

then $X = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ point symmetry of y' if

$$X^{[1]}y' \mid_y' = 0.$$

Since $\eta = y$ then $\eta_x = 0$
$\eta = \eta(y)$ where η arbitrary function of y. Therefore

$$X = \xi(x,y)\frac{\partial}{\partial x} + \eta(y)\frac{\partial}{\partial y}.$$

Thus there is an infinite number of point symmetries. If we choose

$$X = \xi(x,y)\frac{\partial}{\partial x}$$

Now $I = y$ first integral of Eq.(37).
It has point symmetry X if $XI = 0$ then $\xi(x,y)\frac{\partial y}{\partial x} = 0$ there is an infinite number of symmetries of the first integral $I = y$ of (37)
Let $F = F(I) = XI\frac{\partial F}{\partial I}$, therefor X as in $X = \xi(x,y)\frac{\partial}{\partial x}$ is asymmetry of $I = y$ and also any function of $F(y)$

5. Scalar linear nth-order differential equations

The general, homogeneous, form is

$$y^{(p)} + \sum_{i=0}^{p-1} a_i(x)y^{(i)} = 0,\ p \geq 1, \tag{42}$$

we can reduces Eq.(42) for $p \geq 2$ to

$$y^{(p)} + \sum_{i=0}^{p-2} a_i(x)y^{(i)} = 0,\ p \geq 2. \tag{43}$$

Theorem 5.1.
The Lie point symmetry generator

$$X = \xi(x)\frac{\partial}{\partial x} + \left[\left(\frac{p-1}{2}\xi' + c_0\right)y + \eta(x)\right]\frac{\partial}{\partial y}, \tag{44}$$

is admitted by Equation (43) for $p \geq 3$ and is the most general, where c_0 is a constant, $\eta(x)$ satisfies Eq.(43) and $\xi(x)$ is determined by the relations

$$\frac{(n+1)!(i-1)}{(n-i)!(i+1)!}\xi^{(i+1)} + 2i\xi' a_{n-i} + 2\xi a'_{n-i}$$
$$+ \sum_{j=2}^{i-1} a_{n-j}\frac{(n-j)![n(i-j-1)+i+j-1]}{(n-i)!(i-j+1)!}\xi^{(i-j+1)} = 0,\ i = 1,\cdots,n. \tag{45}$$

Definition 5.1 (Principal Lie algebra[2]).
For arbitrary coefficients $a_i(x)$ Eq.(43) admits the Lie algebra spanned by the $p+1$ homogeneous and superposition operators

$$X_1 = y\frac{\partial}{\partial y}, \tag{46}$$

$$X_i + 1 = \eta_i(x)\frac{\partial}{\partial y},\ i = 1,\cdots,n, \tag{47}$$

where $\eta_i(x)$ are p linearly independent solution of Eq.(43). This algebra is referred to as the principal Lie algebra of Eq.(43).

consider the simplest pth-order equation

$$y^{(p)} = 0, \; p \geq 3. \tag{48}$$

We have $(n+1)$ symmetry generators giving by Eqs.(46) and (47) with $\eta_i(x) = c_i x^{i-1}, \; i = 1, \cdots, n$ where $c_i^{'}s$ are arbitrary constants. Moreover, the use of Eq.(45), since $a_i = 0$, gives
$\xi = A_0 + A_1 x + A_2 x^2, \; A_i$ are constants.
The extension is maximum, i.e. three dimensional. Therefore, the maximum symmetry algebra of Eq.(48) is spanned by Eqs.(46) and (47) with η_i given above and

$$X_p + 2 = \frac{\partial}{\partial x}, \; X_p + 3 = x\frac{\partial}{\partial x}, \; X_p + 4 = x^2\frac{\partial}{x} + (p-1)xy\frac{\partial}{\partial y}. \tag{49}$$

6. The algebra structure of the first integral of third-order linear equation

consider the simplest third-order ODE

$$y''' = 0, \tag{50}$$

which as is well-know has the $n+4$ symmetries. [2, 7]
since $p = 3$ and $\eta_i(x) = c_i x^{i-1}$, c_i constant, $i = 1, 2, 3$

$$\eta_1(x) = c_1,$$
$$\eta_2(x) = c_2 x,$$
$$\eta_3(x) = c_3 x^2,$$

if $c_i = 1$, for $i = 1, 2, 3$ then

$$\eta_1(x) = 1,$$
$$\eta_2(x) = x,$$
$$\eta_3(x) = x^2.$$

From Eqs.(46) and (47) we get

$$X_1 = \frac{\partial}{\partial y},$$
$$X_2 = x\frac{\partial}{\partial y},$$
$$X_3 = x^2\frac{\partial}{\partial y},$$
$$X_4 = y\frac{\partial}{\partial y},$$

since $X_n + 2 = \frac{\partial}{\partial x}$, $X_n + 3 = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$ and $X_n + 4 = x^2\frac{\partial}{\partial x} + (n-1)xy\frac{\partial}{\partial x}$.
If $n = 3$ then

$$X_5 = \frac{\partial}{\partial x},$$
$$X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$
$$X_7 = x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}.$$

So the simplest Eq.(50) has seven symmetris

$$X_1 = \frac{\partial}{\partial y},$$
$$X_2 = x\frac{\partial}{\partial y},$$
$$X_3 = x^2\frac{\partial}{\partial y},$$
$$X_5 = \frac{\partial}{\partial x},$$
$$X_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$
$$X_7 = x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}.$$

It is obvious that Eq.(50) has three functionally independent first integral

$$I_1 = y'', \tag{51}$$

$$I_2 = xy'' - y', \tag{52}$$

$$I_3 = \frac{1}{2}x^2 y'' - xy' + y, \tag{53}$$

the first integral (51) has four symmetries.
since $X^{[2]} I_1 = 0$ then

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= x\frac{\partial}{\partial y}, \\ X_4 &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}, \end{aligned} \tag{54}$$

Lie point symmetry of I_1.
i.e
Since

$$X^{[2]} I_2 = 0,$$

because

$$X^{[2]} I_2 = \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta^{(1)}\frac{\partial}{\partial y'} + \zeta^{(2)}\frac{\partial}{\partial y''}\right) y'' = \zeta^{(2)} = 0,$$

$$\zeta^{(2)} = \eta_{xx} + y'\left(2\eta_{xy} - \xi_{xx}\right) + (y')^2\left(\eta_{yy} - 2\xi_{xy}\right) - \xi_{yy}(y')^3 + \left(\eta_y - 2\xi_x - 3\xi_y y'\right) y'' = 0.$$

If $X_1 = \frac{\partial}{\partial x}$ then $\eta = 1$ and $\xi = 0$
means $X_1 = 1\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y}$ then we get $\eta_{xx} = 0$, $\eta_{xy} = \xi_{xx} = 0$ and $\xi_y = \eta_y = 0$.
So $X^{[2]} I_1 = \zeta^{(2)} = 0$ then $X_1 = \frac{\partial}{\partial x}$ Lie point symmetry of I_1
In the same way we can show it X_2, X_3 and X_4 are Lie point symmetry of I_2.
Now the second first integral I_2 has three symmetries

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial y}, \\ Y_2 &= x^2\frac{\partial}{\partial y}, \\ Y_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \end{aligned} \tag{55}$$

by using the condition $X^{[2]} I_2 = 0$ we get the results above in the same way as before.
Now the third first integral I_3 also has four symmetries

$$\begin{aligned} H_1 &= x\frac{\partial}{\partial x}, \\ H_2 &= x\frac{\partial}{\partial y}, \\ H_3 &= x^2\frac{\partial}{\partial y}, \\ H_4 &= x^2\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}. \end{aligned} \tag{56}$$

Note further that the symmetries in Eq.(56) are found by multiplying those of Eq.(54) by the factor x.

Classifying relation for the symmetries

let F be an arbitrary function of the integrals I_1, I_2 and I_3, namely

$$F = F(I_1, I_2, I_3).$$

The symmetry of this general function of the first integrals is

$$X^{[2]}F = X^{[2]}I_1\frac{\partial}{\partial I_1} + X^{[2]}I_2\frac{\partial}{\partial I_2} + X^{[2]}I_3\frac{\partial}{\partial I_3} = 0, \tag{57}$$

where

$$\begin{aligned}
X^{[2]}I_1 &= \left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta^{(1)}\frac{\partial}{\partial y'} + \zeta^{(2)}\frac{\partial}{\partial y''}\right] y'' \\
&= \zeta^{(2)}, \\
X^{[2]}I_2 &= \left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta^{(1)}\frac{\partial}{\partial y'} + \zeta^{(2)}\frac{\partial}{\partial y''}\right] (xy'' - y') \\
&= \xi y'' - \zeta^{(1)} + x\zeta^{(2)}, \\
X^{[2]}I_3 &= \left[\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta^{(1)}\frac{\partial}{\partial y'} + \zeta^{(2)}\frac{\partial}{\partial y''}\right] (\frac{1}{2}x^2y'' - xy' + y) \\
&= \xi(xy'' - y') + \eta - x\zeta^{(1)} + \frac{1}{2}x^2\zeta^{(2)}.
\end{aligned}$$

Now we will find coefficient functions ξ,η,$\zeta^{(1)}$ and $\zeta^{(2)}$
since η is linearly independent solution of Eq.(43) then

$$\eta = c_1 + xc_2 + x^2c_3 + yc_4 + yc_6 + 2xyc_7.$$

Since $\xi = A_0 + A_1x + A_2x^2$, A_i constant
then

$$\xi = c_5 + xc_6 + x^2c_7,$$

$$\begin{aligned}
\zeta^{(1)} &= D_x(\eta) - y'D_x(\xi) \\
&= \frac{\partial}{\partial x}(\eta) + y'\frac{\partial}{\partial y}(\eta) - y'\left(\frac{\partial}{\partial x}(\xi) + y'\frac{\partial}{\partial y}(\xi)\right) \\
&= c_2 + 2c_3x + 2yc_7 + y'(c_4 + c_6 + 2xc_7) - y'(c_6 + 2c_7x + y'(0)) \\
&= c_2 + 2xc_3 + y'c_4 + 2yc_7
\end{aligned}$$

$$\zeta^{(2)} = 2c_3 + y''c_4 - y''c_6 + (2y' - 2xy'')c_7,$$

these are obtained by setting

$$X^{[2]} = \sum_{i=1}^{7} a_iX_i^{[2]},$$

where the X_i are the symmetry generator and the a_i are constant.
After substitution of the value of $X^{[2]}I_1$,$X^{[2]}I_2$ and $X^{[2]}I_3$ with ξ , η , $\zeta^{(1)}$ and $\zeta^{(2)}$ in Eq.(57).
Now we arrive at the classifying relation

$$\begin{aligned}
&[2c_3 + (c_4 - c_6)I_1 - 2c_7I_2]\frac{\partial F}{\partial I_1} \\
&+ (-c_2 + c_4I_2 + c_5I_2 - 2c_7I_3)\frac{\partial F}{\partial I_2} \\
&+ [c_1 + (c_4 + c_6)I_3 + c_5I_2]\frac{\partial F}{\partial I_3} = 0,
\end{aligned} \tag{58}$$

the relation (58) explicitly provides the relationship between the symmetries and the first integrals of the simple third-order Eq.(50). We use the classifying relation (58) to establish the number and property of symmetries possessed by the first integrals of the simplest third-order Eq.(50).
In this research we study three cases

Case 1. No Symmetry.

If F is any arbitrary function of I_1, I_2 and I_3 then $\frac{\partial F}{\partial I_1}$, $\frac{\partial F}{\partial I_2}$ and $\frac{\partial F}{\partial I_3}$ are not related to each other. In this case we have from relation (58) and

$$\frac{\partial F}{\partial I_1}=\frac{\partial F}{\partial I_2}=\frac{\partial F}{\partial I_3}=0$$

that

$$2c_3+(c_4-c_6)I_1-2c_7I_2=0, \tag{59}$$

$$-c_2+c_4I_2+c_5I_1-2c_7I_3=0, \tag{60}$$

$$c_1+(c_4+c_6)I_3+c_5I_2=0, \tag{61}$$

we see that from Eqs.(59), (60) and (61) that all the $a's$ are zero. So no symmetry exists in this case. As an example if we take $F=I_1I_2\ln I_3$, then the relation (58) becomes

$$\begin{aligned}&[2c_3+(c_4-c_6)I_1-2c_7I_2]I_2I_3\ln I_3\\&+(-c_2+c_4I_2+c_5I_2-2c_7I_3)I_1I_2\ln I_3\\&+[c_1+(c_4+c_6)I_3+c_5I_2]I_1I_2=0,\end{aligned}$$

its clear $c's=0$, so no symmetry exists in this case

Case 2. One Symmetry.

If F satisfies the relation (58), then there exists one symmetry. For the simple symmetries of Eq.(50) one obtains further symmetries except for X_6 which we consider below if we take $F=I_1I_2I_3$ or any function of this product, then the relation (58) becomes

$$\begin{aligned}&[2c_3+(c_4-c_6)I_2-2c_7I_2]I_2I_3\\&+(-c_2+c_4I_2+c_5I_2-2c_7I_3)I_1I_3\\&+[c_1+(c_4+c_6)I_3+c_5I_2]I_1I_2=0,\end{aligned} \tag{62}$$

Simplify the Eq.(62) yields

$$\begin{aligned}&2c_3I_2I_3+c_4I_1I_2I_3-c_6I_1I_2I_3-2c_7I_2^2I_3-c_2I_1I_3+c_4I_1I_2I_3\\&+c_5I_1^2I_3-2c_7I_1I_3^2+c_1I_1I_2+c_4I_1I_2I_3+c_6I_1I_2I_3+c_5I_1I_2^2=0,\end{aligned} \tag{63}$$

in Eq.(63), c_1 to c_7 are zero expect c_6 which gives one symmetry

$$X_6=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial x}.$$

Case 3. Two Symmetry.

Here there are many cases as well. We begin by using the Lie Table 1 for the classification of the two-dimensional algebras.[5]
such that for example $L_{2,1}$ denotes the second realizations of the 1 Lie algebra of dimension 2.
i.e.the notation $L^{\alpha}_{2,i}$ where 2 refers to the dimension of the algebra, i to the number of the algebra in some given ordinary and α is the realizations as an algebra many have more than on realizations for example $L^{II}_{2,1}$. Now from Table 1. [2] They are

$$\begin{aligned}&Y_1=\frac{\partial}{\partial y},\quad Y_2=\frac{\partial}{\partial x},\\&Y_1=\frac{\partial}{\partial y},\quad Y_2=x\frac{\partial}{\partial y},\\&Y_1=\frac{\partial}{\partial y},\quad Y_1=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y},\\&Y_1=\frac{\partial}{\partial y},\quad Y_2=y\frac{\partial}{\partial y}.\end{aligned} \tag{64}$$

Table 1. Realizations of two dimensional algebras in the real plane

p=$\frac{\partial}{\partial x}$ and $q=\frac{\partial}{\partial y}$	
Algebra	Realizations in (x,y) plane
$L^{I}_{2,1}$	$X_1 = p, X_2 = q,$
$L^{II}_{2,1}$	$X_1 = q, X_2 = xq,$
$L^{I}_{2,2}$	$X_1 = q, X_2 = xp + yq,$
$L^{II}_{2,2}$	$X_1 = q, X_2 = yq.$

These form subalgebra of the Lie algebra of symmetries of Eq.(50) as can clearly be observed.
Now we take the first realization listed above (64)

$$Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = \frac{\partial}{\partial x}$$

If c_1 is arbitrary in relation (58) That means it F is independent of I_3 Further $X_5 = \frac{\partial}{\partial x}$ yields that F does not depend on I_2 as well.
Since we require that $\frac{\partial F}{\partial I_1} \neq 0$ then we have

$$2c_3 + I_1(c_4 - c_6) - 2c_7 I_2 = 0,$$

from which it follows that $c_3 = c_7 = 0$ and $c_4 = c_6$, in the end we possing with two more symmetries.

$$\begin{aligned} X &= y\frac{\partial}{\partial y} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \\ &= x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} \\ &= X_4 + X_6. \end{aligned}$$

Now we take the second realizations listed Eq.(64)

$$Y_1 = \frac{\partial}{\partial y}, \quad Y_2 = x\frac{\partial}{\partial y}$$

we get if c_1 arbitrary in relation (58) then $X_1 = \frac{\partial}{\partial y}$ this means F is independent of I_3 and if $X_2 = x\frac{\partial}{\partial y}$ yield that F does not dependent on I_2 we require that $\frac{\partial F}{\partial I_1} \neq 0$ so

$$2c_3 + I_1(c_4 - c_6) - 2c_7 I_2 = 0,$$

then $c_3 = c_7 = 0$ and $c_4 = c_6$. So in the end we have more than two symmetries X_2 and $X_4 + X_6$ then we get the symmetry

$$X_1 = \frac{\partial}{\partial y}, \quad X_2 = x\frac{\partial}{\partial y},$$

and since c_5 arbitrary then $X_5 = \frac{\partial}{\partial x}$.
Then the symmetry of first integral I_1 are

$$\begin{aligned} X_1 &= \frac{\partial}{\partial x}, \\ X_2 &= \frac{\partial}{\partial y}, \\ X_3 &= x\frac{\partial}{\partial y}, \\ X_4 &= \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y}. \end{aligned}$$

Now if we take the third realization listed above Eq.(64).
$Y_1 = \frac{\partial}{\partial y}$, $Y_2 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$, we find the symmetries of I_2

$$\begin{aligned} Y_1 &= \frac{\partial}{\partial y}, \\ Y_2 &= x^2\frac{\partial}{\partial y}, \\ Y_3 &= x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \end{aligned}$$

The fourth realization results in a two symmetry case as c_1 and c_4 arbitrary thats means

$$\frac{\partial F}{\partial I_3}=0, \quad I_1\frac{\partial F}{\partial I_1}+I_2\frac{\partial F}{\partial I_2}=0$$

which as solution $F=H(I_2/I_1)$.
The further substitution of this form into the relation (58) constrains all the $c's$ equal zero except for c_1 and c_4. This result prompts the following simple products and quotients that do give two symmetries.
If $F=I_1I_2$ then relation (58) get

$$[2c_3+(c_4-c_6)I_1-2c_7I_2]I_2+(-c_2+c_4I_2+c_5I_1-2c_7I_3)I_1=0,$$

we observe that c_2,c_3,c_5 and c_7 are zero whereas c_1 is arbitrary and $c_6=2c_4$
i.e.

$$2c_3I_2+c_4I_1I_2-c_6I_1I_2-2c_7I_2^2-c_2I_1+c_4I_1I_2+c_5I_1^2-2c_7I_1I_3=0,$$

more simplification we get

$$2c_3I_2+2c_4I_1I_2-c_6I_1I_2-2c_7I_2^2-c_2I_1+c_5I_1^2-2c_7I_1I_3=0$$

Therefore we obtain the two symmetries

$$X_1=\frac{\partial}{\partial Y},$$
$$Y=x\frac{\partial}{\partial x}+3y\frac{\partial}{\partial Y},$$

which form a two-dimensional algebra with

$$[X,Y]=3X_1$$

If we set $F=I_1I_3$ then we in the end getting c_1, c_3, c_4, c_5 and c_7 are zero,since c_2 and c_6 are arbitrary so they result in two symmetry

$$X_2=x\frac{\partial}{\partial Y},$$
$$X_6=x\frac{\partial}{\partial x}+y\frac{\partial}{\partial Y},$$

and

$$[X_2,X_6]=0$$

If we take $F=I_3/I_1$ this shows that c_2, c_4 are arbitrary and the resulting two symmetries.Since $\frac{\partial F}{\partial I_2}=0$, $\frac{\partial F}{\partial I_1}=-I_3/I_1^2$, $\frac{\partial F}{\partial I_3}=1/I_1$
Now by using (58) we get

$$-2c_3I_3/I_1^2-c_4I_3/I_1+c_6I_3/I_1+2c_7I_2I_3/I_1^2+c_1/I_1+c_4I_3/I_1+c_6I_3/I_1+c_5I_2/I_1=0$$

Clear it c_4 arbitrary and c_3, c_6, c_7 are zero. Since $\frac{\partial F}{\partial I_2}=0$ then from (58) we arrive

$$-c_2+c_4I_2+c_5I_1-2c_7I_3=0$$

c_2 arbitrary then $X_2=x\frac{\partial}{\partial Y}$ with

$$[X_2,X_4]=X_2.$$

If we take $F=I_3/I_2$,
since $\frac{\partial F}{\partial I_1}=0$, $\frac{\partial F}{\partial I_2}=-I_3/I_2^2$, $\frac{\partial F}{\partial I_3}=1/I_2$
Now by using Eq.(58) we arrive

$$c_2I_3/I_2^2-c_4I_3/I_2-c_5I_1I_3/I_2^2+2c_7I_3^2/I_2^2+c_1/I_2+c_4I_3/I_2+c_6I_3/I_2+c_5I_2/I_2=0,$$

its clear $c_2=0$,$c_5=0$,$c_6=0$, $c_7=0$ and c_3, c_4 arbitrary,
So we obtain the two symmetries

$$X_3=x^2\frac{\partial}{\partial Y},$$
$$X_4=y\frac{\partial}{\partial Y},$$

with Lie bracket $[X_3,X_4]$.

7. Symmetry properties of first integrals of higher order ODEs

Consider the p_{th}-order ODE of maximal symmetry

$$y^{(p)} = 0, \quad p \geq 3, \tag{65}$$

this ODE (65) has $p+4$ symmetries as it's known. Our attention will be about the first integral and p first integrals which are

$$I_1 = y^{(p-1)}, \tag{66}$$

and

$$I_p = \sum_{i=1}^{p} \frac{(-1)^{i-1}}{(p-1)!} x^{p-i} y^{(p-i)}. \tag{67}$$

The first integral (66) has $p+1$ symmetries which are

$$\begin{aligned} X_i &= x^{i-1}\frac{\partial}{\partial y}, \quad i = 1,\cdots,p-1, \\ X_p &= \frac{\partial}{\partial x}, \\ X_p + 1 &= x\frac{\partial}{\partial x} + (p-1)y\frac{\partial}{\partial y}, \end{aligned} \tag{68}$$

these formulas an $p+1$-dimensional subalgebra of the Eq.(65). the first integral Eq.(67) has symmetries

$$\begin{aligned} Y_i &= x^{i}\frac{\partial}{\partial y}, \quad i = 1,\cdots,p-1, \\ Y_p &= x\frac{\partial}{\partial x}, \\ Y_p + 1 &= x^2\frac{\partial}{\partial x} + (p-1)xy\frac{\partial}{\partial y}, \end{aligned} \tag{69}$$

note that it is the Eq.(69) comes from multiplying the symmetries of Eq.(68) by x.

8. Conclusion

In this research we have provided the algebraic structure of first integrals of simplest second-, third and higher-order ordinary differential equations or any scalar linearizable, by point transformation, ODE. Firstly, we derived the relationship between the symmetries and the first integrals of the simplest ordinary differential equation. By analyzing this classifying relation, we were able to establish the number of symmetries possessed by any first integral of the simplest equation. We obtained the important result that the symmetries admitted by a first integral can be 0, 1, 2 . It was observed that the zero symmetry case was rather surprising or unexpected as one does not have a route to integration of the equation due to the lack of any symmetry and this too for the simplest equation. The one and two symmetry cases were not unique - there were many first integrals with differing one and two symmetry structures.Finally, we studied completely the situation when a first integral has three symmetries. We used the classification of realizations in the plane adapted as simplest equation. We showed that the only three-dimensional algebra admitted by a first integral of the simplest equation is $L^{I}_{3;5}$ which is admitted by the functionally independent integrals I_1 and I_2 as well as their quotient I_2/I_1. We discussed this research study the point symmetry properties of the first integrals of $y''' = 0$ which also represents all linearizable by point transformations third-order ODEs that reduce to this class. Finally we study symmetry properties of first integrals of higher ODE.

Acknowledgements

The authors are very grateful to the referees for useful comments and suggestions towards the improvement of this paper.

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