Int. J. Adv. Appl. Math. and Mech. 7(2) (2019) 1 – 15 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

On a Conic Through Twelve Notable Points

Research Article

Dasari Naga Vijay Krishna*

Department of Mathematics, Sri chaitanya Educational Institutions Machilipatnam, Bengalore, India

Received 12 July 2019; accepted (in revised version) 20 November 2019

Abstract: In this article we present a conic which passes through a twelve notable points and as a result of the conic we also study few Concurrency, Collinearity and Perspectivity results.

MSC: 97K30 • 68R10

Keywords: Circum Cevian Criangle • Bicevian Conic

1. Introduction

Although Euler was apparently the first person (in 1763) to show that the midpoints of the sides of a triangle and the feet of the altitudes determine a unique circle it was not until 1820 that Brianchon an Poncelet showed that the three midpoints of the segments from the orthocenter to the vertices also lie on the same circle. Hence its name, the nine-point circle [3].

The concept of a nine point circle can be generalized to a nine point ellipse or a nine point hyperbola if we consider a general cevian instead of altitude. Consider three concurrent cevians with cevian point P, locate mid-points E, F and G of the segments from cevian point to the vertices of the triangle. Also locate the feet of the cevian K, N and L. If we draw a conic through any five of the above points, we will get an ellipse and it will also pass through the sixth point [4].

Now locate the mid points of the three sides of the triangle, we will find that these points also fall on the ellipse constructed above. The conic remains an ellipse when the feet of cevians lie on the sides of the triangle and converts to a nine point hyperbola when the feet of the cevians lie on the extensions of the sides.

In our present article we study a special case of the nine point conic where cevian is replaced by internal angular bisector and cevian point as an incenter (I or X(1)). Hence it is clear that we can always construct a conic (ellipse or hyperbola) which passes through the traces of Incenter I((X(1)) and centriod G (X(2)) and it passes through the mid points of AI, BI and CI. Usually this conic is called as bicevian conic, may be this conic is well known in the literature but the way how we are dealing in this article is completely new. Unexpectedly, this nine point conic passes through three more notable points (point of intersections of the line formed by joining the apex of circum cevian triangle of I with the conic) In this short note we discuss how this twelve-point conic is useful in constructing the some notable triangle centers such as X(21), X(56), X(84), X(995), center of 12 point conic X(1125), X(2360), X(3616), X(19861), X(24471). In the journey of this construction, with some modified configurations, unexpectedly we come across a new concurrency of special cevians, as a result we will meet one new triangle center say X(K) whose barycentric coordinates are (8s+a : 8s+b: 8s +c) which is not available in current edition of ETC(1-40000) [2].

* *E-mail address(es):* vijay9290009015@gmail.com

This point Lies on lines $X(i)X(j)$ for these $\{i,j\}$: {1,2}, {3,7988}, {5,7987}, {9,3337}, {12,13462}, {20,10171}, {35,4423}, {36,11108}, {40,3526}, {46,3646}, {55,16863}, {56,16853}, {58,17125}, {63,5506}, {72,3848}, {79,4679}, {140,165}, {182,9587}, {191,3306}, {312,6533}, {377,18514}, {392,10107}, {405,7280}, {442,3847}, {443,3583}, {452,4316}, {474,5010}, {515,5067}, {516,10303}, {547,18481}, {590,19003}, {595,17124}, {615,19004}, {631,1699}, {632,5886}, {944,10172}, {946,3525}, {958,16854}, {993,17536}, {1001,16862}, {1213,16667}, {1265,6018}, {1376,16864}, {1385,5070}, {1420,5726}, {1478,17559}, {1479,17582}, {1573,9336}, {1574,9331}, {1656,3576}, {1743,17398}, {1750,6832}, {1757,6687}, {2093,5443}, {2478,18513}, {2951,6890}, {2975,17546}, {3035,12732}, {3039,9519}, {3068,13942}, {3069,13888}, {3090,5691}, {3097,6683}, {3305,6763}, {3336,5437}, {3338,7308}, {3339,7294}, {3361,5219}, {3467,15297}, {3522,12571}, {3523,3817}, {3524,18483}, {3533,6684}, {3579,15694}, {3585,5084}, {3628,5587},{3653,18357}, {3731,17369}, {3742,4539}, {3746,4413}, {3763,16475}, {3816,17529}, {3825,4197}, {3833,3869}, {3842,17501}, {3844,16491}, {3851,17502}, {3868,4536}, {3876,3894}, {3885,3968}, {3889,4015}, {3890,3918}, {3901,10176}, {3947,5265}, {3973,5257}, {3983,5049}, {4002,10179}, {4297,5056}, {4299,5129}, {4302,17580}, {4324,6904}, {4355,5226}, {4398,17322}, {4440,8715}, {4472,15223},{4653,17551}, {4748,4758}, {4798,25358}, {4873,16673}, {4902,10436}, {4999,5234}, {5044,18398}, {5047,5303}, {5054,9955}, {5055,13624}, {5057,5131},{5204,16857}, {5248,17531}, {5251,16842}, {5253,17534}, {5258,16856}, {5267,16859}, {5284,17535}, {5290,7288}, {5316,13407}, {5326,9819}, {5439,5692}, {5444,10826}, {5563,16855}, {5603,9588}, {5690,16189}, {5715,6878}, {5732,6884}, {5880,7483}, {5901,11224}, {6051,16602}, {6361,15709}, {6532,17155}, {6667,10609}, {6691,25525}, {6701,15671}, {6707,17306}, {6713,15017}, {6824,10857}, {6861,8726}, {6918,15931}, {6972,12565}, {7082,13089}, {7485,9591}, {7486,19925}, {7516,9625}, {7741,8728}, {7746,9592}, {7815,10789}, {7951,17527}, {7982,11231}, {8040,11263}, {8056,24161}, {8185,11284}, {8252,18991}, {8253,18992}, {9581,10543}, {9583,10577}, {9589,10164}, {9612,16845}, {9621,13353}, {9624,11531}, {9779,12512}, {9956,18526}, {9963,15015}, {10124,22791}, {10156,12688}, {10434,19549}, {10980,11374}, {12245,16191}, {13384,17606}, {15668,16468}, {15670,16118}, {15720,22793}, {16291,16678}, {16296,20470}, {16457,17123}, {16469,17245}, {16472,17811}, {16473,17825}, {16844,19749}, {17290,25498}, {17304,24295}, {17400,24342},[1].
In addition to these we see few collinearty perspectivity results.

2. Notation and Background

Let ABC be a non equilateral triangle. We denote its side-lengths by a, b, c, perimeter by 2s, its area by Δ and its circumradius by R, its inradius by r. we will use homogeneous barycentric coordinates with reference to ABC.

Lemma 2.1.
Let $P = (u : v : w)$ be a point not on the side lines of triangle ABC. Then the equation of the conic through the traces of P and the midpoints of the three sides is $\sum_{cyclic} vwx^2 - u(v+w)yz = 0$ having the center as $(2u+v+w : u+2v+w : u+v+2w)$ and also passes through the midpoints of $AP(2u+v+w : v : w)$, $BP(u : u+2v+w : w)$ and $CP(u : v : u+v+2w)$ respectively [5].

Lemma 2.2.
Let P = (u : v : w). The lines AP, BP, CP intersect the circumcircle again at the points $A^{(P)} = \left(\dfrac{-a^2vw}{b^2w+c^2v} : v : w\right)$, $B^{(P)} = \left(u : \dfrac{-b^2wu}{c^2u+a^2w} : w\right)$ and $C^{(P)} = \left(u : v : \dfrac{-c^2uv}{a^2v+b^2u}\right)$. These form the vertices of the Circumcevian triangle of P [5].

2.1. Nine-point conic of Incenter $\left(\mathbb{C}_{I_9}\right)$

Using Lemma 2.1 we can find the barycentric coordinates of the Nine-point conic of incenter $\left(\mathbb{C}_{I_9}\right)$ (Refer Fig. 1)

2.2. Circumcevian triangle of incenter

Using Lemma 2.2 we can find the barycentric coordinates of the circumcevian triangle of incenter (See Fig. 2)
Let us consider three points V_A, V_B and V_C whose barycentric coordinates as in Table 3

Theorem 2.1.
The Nine-point conic of incenter $(\mathbb{C}_{I_9})$ and circumcevian triangle of incenter intersects at the points P, Q, R (midpoints of AI, BI, CI) and V_A, V_B and V_C such that $\mathbb{C}_{I_9} \cap A^{(I)}B^{(I)} = R$ and V_C, $\mathbb{C}_{I_9} \cap B^{(I)}C^{(I)} = P$ and V_A, $\mathbb{C}_{I_9} \cap C^{(I)}A^{(I)} = Q$ and V_B. (Refer Fig. 3)

Table 1.

Points	barycentric coordinates
Traces of Incenter on the sides BC, CA, AB are X, Y, Z respectively	X = (0 : b : c) Y = (a : 0 : c) Z = (a : b : 0)
Traces of centroid G on the sides BC, CA, AB are D, E, F respectively	D = (0 : 1 :1) E = (1 : 0 : 1) F = (1 : 1 : 0)
Midpoints of AI, BI,CI are P, Q, R respectively	P = (2s+a : b :c) Q = (a : 2s+b : c) R = (a : b : 2s+c)
Equation of nine-point conic $\left(\mathbb{C}_{I_9}\right)$ of incenter I	$bcx^2 + cay^2 + abz^2 - a(b+c)yz - b(c+a)zx - c(a+b)xy = 0$
Center of nine-point conic $\left(\mathbb{C}_{I_9}\right)$ of incenter I	center of $\mathbb{C}_{I_9} = \langle \mathbb{C}_{I_9} \rangle = (2s+a : 2s+b : 2s+c)$ $= X(1125)$ = complement of $X(10)$

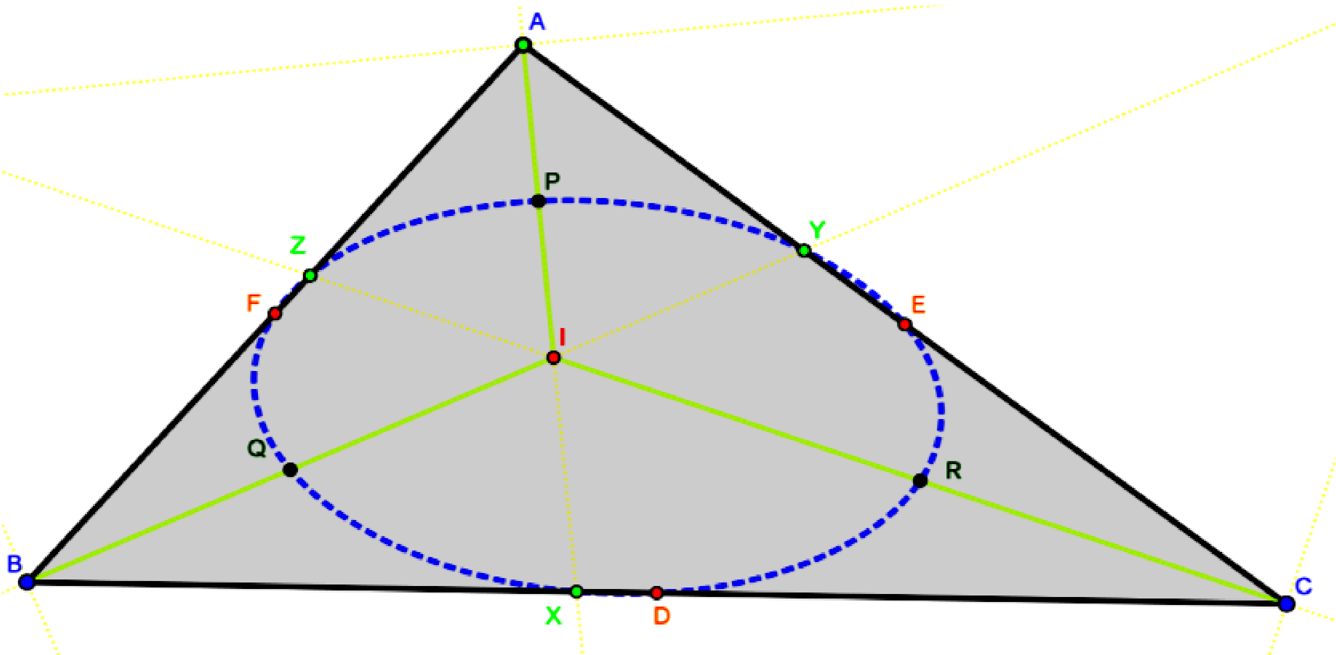

Fig. 1.

Proof. The equation of nine-point conic of incenter $\left(\mathbb{C}_{I_9}\right)$ is given by

$$bcx^2 + cay^2 + abz^2 - a(b+c)yz - b(c+a)zx - c(a+b)xy = 0 \tag{1}$$

Equation of $A^{(I)}B^{(I)}$ is given by

$$b(b+c)x + a(c+a)y - abz = 0 \tag{2}$$

By eliminating abz^2 from the Eqs. (1),(2) and by regrouping, we can factorize the expression as $(bx - ay)\left(cx - cy - (a-b)z\right) = 0$ gives $\frac{x}{y} = \frac{a}{b}$ and $z = \frac{c(x-y)}{(a-b)}$.

By substituting these two in (2), further simplification gives $(x : y : z) = (a : b : a+b+2c)$ and $\left(a^2(s-b) : b^2(s-a) : sc(a+b) - abc\right)$.

Hence

$$\mathbb{C}_{I_9} \cap A^{(I)}B^{(I)} = R \text{ and } V_C = (a : b : 2s+c) \text{ and } \left(a^2(s-b) : b^2(s-a) : sc(a+b) - abc\right)$$

Similarly we can prove the remaining relations.

That is the nine-point conic of incenter passes through other 3 notable points V_A, V_B and V_C. □

Remark:

Table 2.

Circum cevian triangle of incenter $I(a:b:c)$	barycentric coordinates
Vertices $A^{(I)}$, $B^{(I)}$ and $C^{(I)}$	$A^{(I)}=\big(-a^2\ :\ b(b+c)\ :\ c(b+c)\big)$, $B^{(I)}=\big(a(c+a)\ :\ -b^2\ :\ c(c+a)\big)$ $C^{(I)}=\big(a(a+b)\ :\ b(a+b)\ :\ -c^2\big)$
Sides $A^{(I)}B^{(I)}$, $B^{(I)}C^{(I)}$and $C^{(I)}A^{(I)}$	$A^{(I)}B^{(I)}: b(b+c)x+a(c+a)y-abz=0$, $B^{(I)}C^{(I)}: -bcx+c(c+a)y+b(a+b)z=0$ $C^{(I)}A^{(I)}: c(b+c)x-cay+a(a+b)z=0$

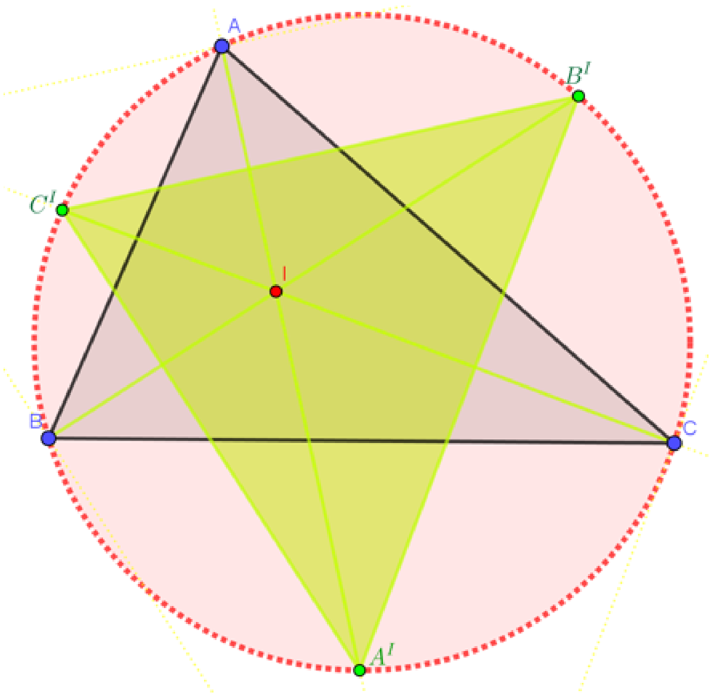

Fig. 2.

1.

Point of intersection of Nine-point conic of incenter $(\mathbb{C}_{I_9})$ with the circumcevian triangle of incenter	Name of the Point of intersections
$\mathbb{C}_{I_9}\cap A^{(I)}B^{(I)}$	R and V_C
$\mathbb{C}_{I_9}\cap B^{(I)}C^{(I)}$	P and V_A
$\mathbb{C}_{I_9}\cap C^{(I)}A^{(I)}$	Q and V_B

2. It is easy to verify that the set of lines $\{A^{(I)}B^{(I)}, CI\}$, $\{B^{(I)}C^{(I)}, AI\}$ and $\{C^{(I)}A^{(I)}, BI\}$ are perpendicular at their point of intersections R, P and Q respectively.
3. It is clear that the circumcevian triangle of I is similar to the excentral triangle of ABC.
4. Theorem 2.1 also gives the construction of the points V_A, $V_B$and V_C.
5. The twelve points $D, E, F, X, Y, Z, P, Q, R$ and V_A, V_B, V_C lie on a conic.
6. This twelve point bicevian conic contains the Feuerbach point $X(11)$.
7. The twelve point conic $\mathbb{C}_{I_{12}}$ which we discussed in Theorem 2.1 has center $\langle\mathbb{C}_{I_{12}}\rangle$ and the coordinates of center= $(2s+a:2s+b:2s+c)$ = complement of $X(10) = X(1125)$

Table 3.

Points	barycentric coordinates
V_A	$\left(sa(b+c)-abc\ :\ b^2(s-c)\ :\ c^2(s-b)\right)$
V_B	$\left(a^2(s-c)\ :\ sb(c+a)-abc\ :\ c^2(s-a)\right)$
V_C	$\left(a^2(s-b)\ :\ b^2(s-a)\ :\ sc(a+b)-abc\right)$

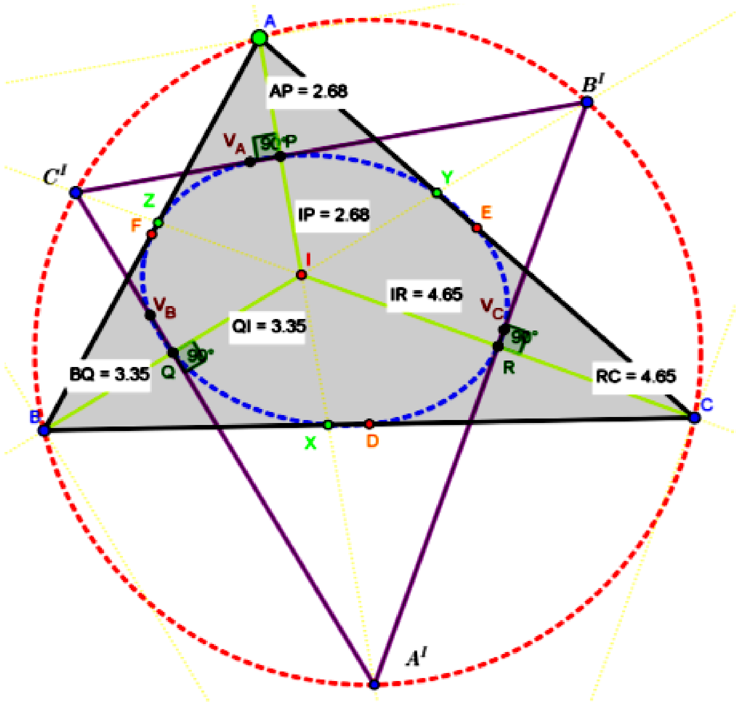

Fig. 3.

3. Some Collinearity and Perspectivity Results

Proposition 3.1.
The lines AV_A, BV_B and CV_C are concur at ***EXSIMILICENTER*** $X(56)$. *That is the triangles ABC and $V_AV_BV_C$ are perspective and the perspector is* $X(56)$.

Proof. The lines AV_A, BV_B and CV_C have barycentric equations

$$(0)x + c^2(s-b)y - b^2(s-c)z = 0 \tag{3}$$

$$c^2(s-a)x + (0)y - a^2(s-c)z = 0 \tag{4}$$

$$b^2(s-a)x - a^2(s-b)y - (0)z = 0 \tag{5}$$

It is clear that, $\begin{vmatrix} 0 & c^2(s-b) & -b^2(s-c) \\ c^2(s-a) & 0 & -a^2(s-c) \\ b^2(s-a) & -a^2(s-b) & 0 \end{vmatrix} = 0$

Hence (3), (4) and (4) are concur. And they intersect at $\left(\dfrac{a^2}{s-a} : \dfrac{b^2}{s-b} : \dfrac{c^2}{s-c}\right) = \boldsymbol{EXSIMILICENTER} = X(56)$. □

Remark: This article gives a new way of constructing the point *X(56)*. (Refer Fig. 5)

Proposition 3.2.
The lines $A^{(I)}V_A$, $B^{(I)}V_B$ and $C^{(I)}V_C$ are concur at X(995).That is the triangles $A^{(I)}B^{(I)}C^{(I)}$ and $V_AV_BV_C$ are perspective and the perspector is X(995).

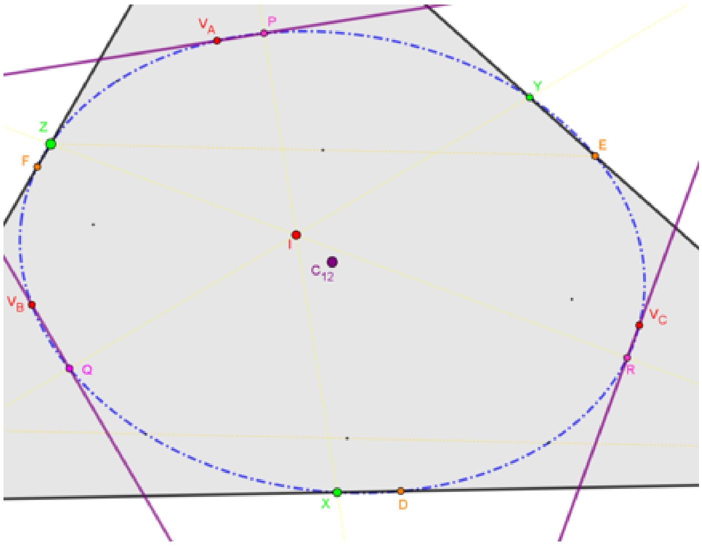

Fig. 4.

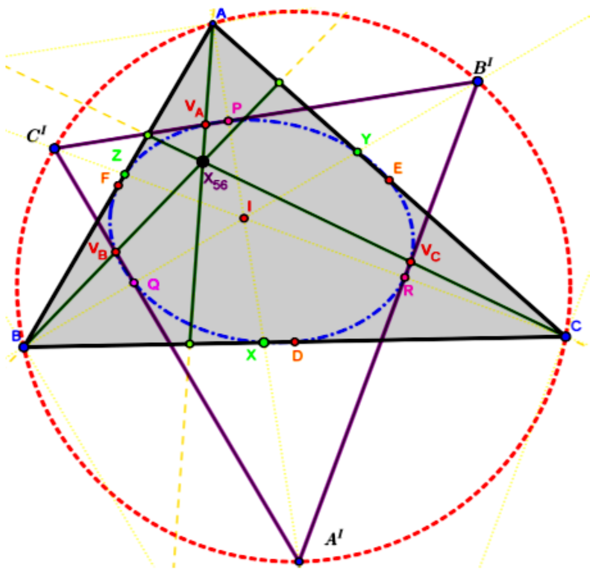

Fig. 5.

Proof. The lines $A^{(I)}V_A$, $B^{(I)}V_B$ and $C^{(I)}V_C$ have barycentric equations

$$bc(b^2-c^2)x - ca(b^2+c^2+ca)y + ab(b^2+c^2+ab)z = 0 \tag{6}$$

$$bc(c^2+a^2+bc)x + ca(c^2-a^2)y - ab(c^2+a^2+ab)z = 0 \tag{7}$$

$$-bc(a^2+b^2+bc)x + ca(a^2+b^2+ca)y + ab(a^2-b^2)z = 0 \tag{8}$$

It is clear that, $\begin{vmatrix} bc(b^2-c^2) & -ca(b^2+c^2+ca) & ab(b^2+c^2+ab) \\ bc(c^2+a^2+bc) & ca(c^2-a^2) & -ab(c^2+a^2+ab) \\ -bc(a^2+b^2+bc) & ca(a^2+b^2+ca) & ab(a^2-b^2) \end{vmatrix} = 0.$

Hence (6), (7) and (8) are concur. And they concur at $\left(a^2(4s^2-2sa-3bc)\ :\ b^2(4s^2-2sb-3ca)\ :\ c^2(4s^2-2sc-3ab)\right) = X(995)$. □

Remark:

1. This article gives a new way of constructing the point *X(995)*. (Refer Fig. 6).
2. The four points incenter *X(1)*, centroid *X(2)*, *X(995)*, *X(K)* are collinear. (Refer Fig. 6).

Proof. Consider a line in barycentric coordinates which contains the points *X(1)* and *X(2)* is

$$(b-c)x+(c-a)y+(a-b)z=0 \tag{9}$$

Now it is easy to verify that center $X(K)=(8s+a:8s+b:8s+c)$ lies on (9) since $(8s+a)(b-c)+(8s+b)(c-a)+(8s+c)(a-b)=0$.
Similarly we can verify that

$$\sum_{a,b,c} a^2(4s^2-2sa-3bc)(b-c)=4s^2\sum_{a,b,c} a^2(b-c)-2s\sum_{a,b,c} a^3(b-c)-3abc\sum_{a,b,c} a(b-c)$$
$$=-4s^2(a-b)(b-c)(c-a)+2s(a-b)(b-c)(c-a)(a+b+c)-0=0$$

So the point X(995) lies on (9). Hence proved. □

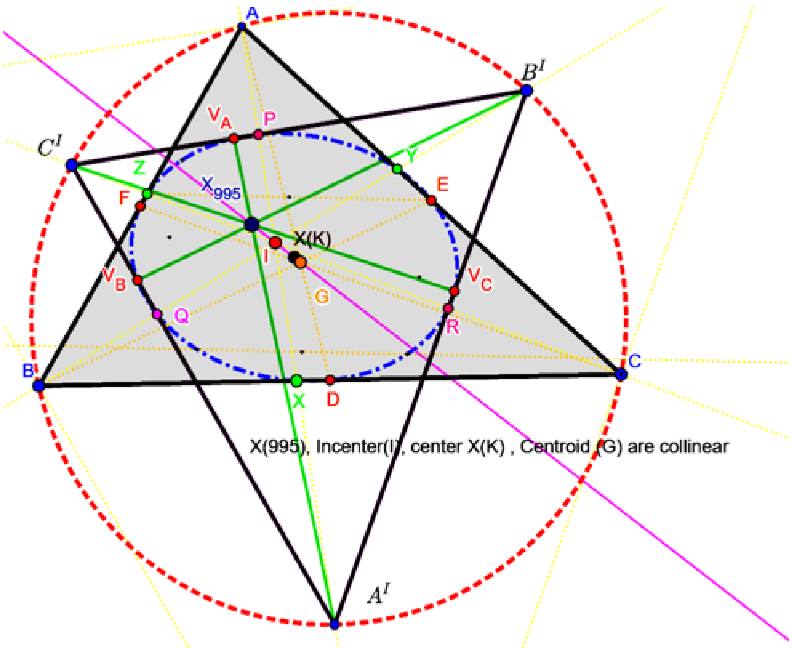

Fig. 6.

Proposition 3.3.
The points $V_1=BV_C\cap CV_B$, $V_2=CV_A\cap AV_C$ and $V_3=AV_B\cap BV_A$ and the triangles ABC and $V_1V_2V_3$ are perspective.(Refer Fig. 7)

Proof. The lines BV_C and CV_B have barycentric equations

$$(sc(a+b)-abc)x+(0)y-a^2(s-b)z=0 \tag{10}$$

$$(sb(c+a)-abc)x-a^2(s-c)y+(0)z=0 \tag{11}$$

Lines (6), (7) intersect at the point $V_1=\left(a^2 : \dfrac{sb(a+c)-abc}{s-c} : \dfrac{sc(a+b)-abc}{s-b}\right)$
Similarly

$$V_2=CV_A\cap AV_C=\left(\frac{sa(b+c)-abc}{s-c} : b^2 : \frac{sc(a+b)-abc}{s-a}\right)$$

and

$$V_3=AV_B\cap BV_A=\left(\frac{sa(b+c)-abc}{s-b} : \frac{sb(a+b)-abc}{s-a} : c^2\right)$$

From these coordinates, it is clear that triangles $V_1V_2V_3$ is perspective with ABC at

$$\Lambda=AV_1\cap BV_2\cap CV_3=\left(\frac{sa(b+c)-abc}{s-a} : \frac{sb(c+a)-abc}{s-b} : \frac{sc(a+b)-abc}{s-c}\right) \tag{12}$$

□

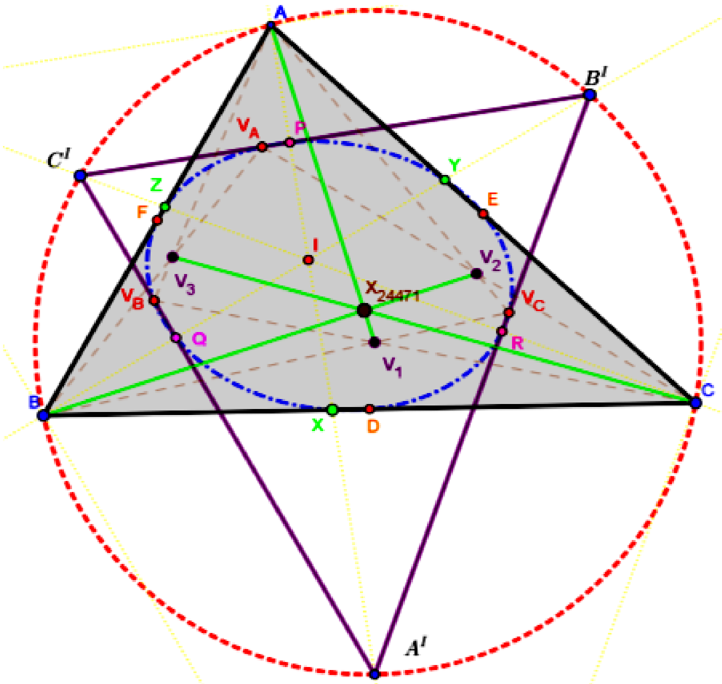

Fig. 7.

Remark: The perspector $\Lambda = \left(\frac{sa(b+c)-abc}{s-a} : \frac{sb(c+a)-abc}{s-b} : \frac{sc(a+b)-abc}{s-c}\right)$ given in (12) is the triangle center $X(24471) = \Lambda = X(6)X(57) \cap X(7)X(8)$.

Proposition 3.4.
The lines $A^{(I)}V_1$, $B^{(I)}V_2$ and $C^{(I)}V_3$ are concur at X(2360) That is the triangles $A^{(I)}B^{(I)}C^{(I)}$ and $V_1V_2V_3$ are perspective and the perspector is X(2360).

Proof. The lines $A^{(I)}V_1$, $B^{(I)}V_2$ and $C^{(I)}V_3$ have barycentric equations

$$sbc(b^2-c^2)x + a^2c(c+a)(s-c)y - a^2b(a+b)(s-b)z = 0 \tag{13}$$

$$-b^2c(b+c)(s-c)x + sca(c^2-a^2)y + b^2a(a+b)(s-a)z = 0 \tag{14}$$

$$c^2b(b+c)(s-b)x - c^2a(c+a)(s-a)y + sab(a^2-b^2)z = 0 \tag{15}$$

It is clear that $\begin{vmatrix} sbc(b^2-c^2) & a^2c(c+a)(s-c) & -a^2b(a+b)(s-b) \\ -b^2c(b+c)(s-c) & sca(c^2-a^2) & b^2a(a+b)(s-a) \\ c^2b(b+c)(s-b) & -c^2a(c+a)(s-a) & sab(a^2-b^2) \end{vmatrix} = 0.$
Hence (13), (14) and (15) are concur. And they concur at

$$\left(\frac{a^2}{b+c}(2s^2(s-a)-bc(b+c)) : \frac{b^2}{c+a}(2s^2(s-b)-ca(c+a)) : \frac{c^2}{a+b}(2s^2(s-c)-ab(a+b))\right) = X(2360)$$

Hence the triangles $A^{(I)}B^{(I)}C^{(I)}$ and $V_1V_2V_3$ are perspective and the perspector is *X(2360).* □

Remark:

1. This article gives a new way of constructing the point *X(2360)*(Refer Fig. 8).
2. The points $X(56), X(995), X(2360)$ are collinear. (Refer Fig. 9).

Proposition 3.5.
The lines V_AV_1, V_BV_2 and V_CV_3 are concur at X(19861). That is the triangles $V_AV_BV_C$ and $V_1V_2V_3$ are perspective and the perspector is X(19861).

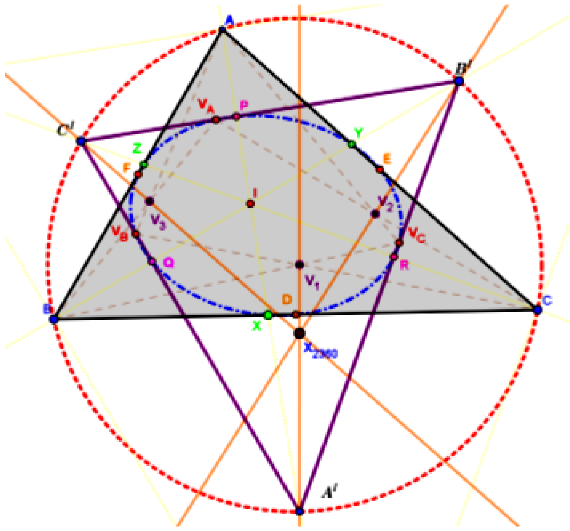

Fig. 8.

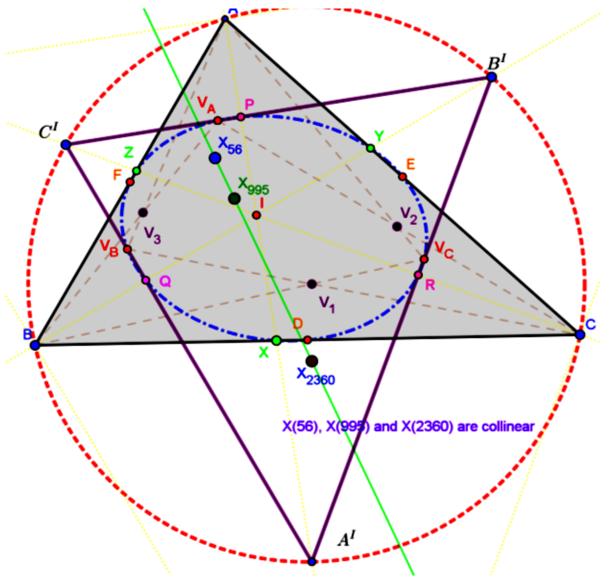

Fig. 9.

Proof. The lines V_AV_1, V_BV_2 and V_CV_3 have barycentric equations

$$(b-c)(2s^3-2s^2a+s(a^2-2bc)+abc)x-a(s-c)(2s^2-2sb+b^2)y+a(s-b)(2s^2-2sc+c^2)z=0$$

$$b(s-c)(2s^2-2sa+a^2)x+(c-a)(2s^3-2s^2b+s(b^2-2ca)+abc)y-b(s-a)(2s^2-2sc+c^2)z=0$$

$$-c(s-b)(2s^2-2sa+a^2)x+c(s-a)(2s^2-2sb+b^2)y+(a-b)(2s^3-2s^2c+s(c^2-2ab)+abc)z=0$$

After transformations using

$$k_a=2s^2-2sa+a^2=s^2+(s-a)^2$$

$$k_b=2s^2-2sb+b^2=s^2+(s-b)^2$$

$$k_c=2s^2-2sc+c^2=s^2+(s-c)^2$$

$$-\left(\frac{k_a-k_b}{c}\right)=a-b,-\left(\frac{k_b-k_c}{a}\right)=b-c \text{ and } -\left(\frac{k_c-k_a}{b}\right)=c-a$$

and

$$(2s^3-2s^2a+s(a^2-2bc)+abc)=s(2s^2-2sa+a^2)-bc(b+c)=s(k_a)-bc(b+c)$$

$$(2s^3-2s^2b+s(b^2-2ca)+abc)=s(2s^2-2sb+b^2)-ca(c+a)=s(k_b)-ca(c+a)$$

$$(2s^3-2s^2c+s(c^2-2ab)+abc)=s(2s^2-2sc+c^2)-ab(a+b)=s(k_c)-ab(a+b)$$

And let

$$abcT = a^2k_a + b^2k_b + c^2k_c - 4\Delta(k_bk_c\cot A + k_ck_a\cot B + k_ak_b\cot C)$$
$$= a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b)$$

We get the equations of the lines V_AV_1 , V_BV_2 and V_CV_3 as

$$(k_b-k_c)(sk_a - bc(b+c))x + a^2(s-c)(k_b)y - a^2(s-b)(k_c)z = 0 \tag{16}$$
$$-b^2(s-c)(k_a)x + (k_c-k_a)(sk_b - ca(c+a))y + b^2(s-a)(k_c)z = 0 \tag{17}$$
$$c^2(s-b)(k_a)x - c^2(s-a)(k_b)y + (k_a-k_b)(sk_c - ab(a+b))z = 0 \tag{18}$$

It is clear that,

$$\begin{vmatrix} (k_b-k_c)(sk_a-bc(b+c)) & a^2(s-c)(k_b) & -a^2(s-b)(k_c) \\ -b^2(s-c)(k_a) & (k_c-k_a)(sk_b-ca(c+a)) & b^2(s-a)(k_c) \\ c^2(s-b)(k_a) & -c^2(s-a)(k_b) & (k_a-k_b)(sk_c-ab(a+b)) \end{vmatrix} = 0$$

Hence (16), (17) and (18) are concur, and they concur at

$$(aT + abc(4s-a) \ : \ bT + abc(4s-b) \ : \ cT + abc(4s-c))$$
$$= \left(\left(a \left(\sum_{a,b,c} a^2(a-b-c) + 2bc(b+c) + 4abc \right) \right) : \left(b \left(\sum_{a,b,c} b^2(b-c-a) + 2ca(c+a) + 4abc \right) \right) : \left(c \left(\sum_{a,b,c} c^2(c-a-b) + 2ab(a+b) + 4abc \right) \right) \right)$$
$$= \left(\left(a \left(\sum_{a,b,c} a^2(a-b-c) + 2bc(b+c) + 4abc \right) \right) \ : \ \left(b \left(\sum_{a,b,c} b^2(b-c-a) + 2ca(c+a) + 4abc \right) \right) \ : \ \left(c \left(\sum_{a,b,c} c^2(c-a-b) + 2ab(a+b) + 4abc \right) \right) \right)$$
$$= X(19861)$$

Hence the triangles $V_AV_BV_C$ and $V_1V_2V_3$ are perspective and the perspector is $X(19861)$. (Refer Fig. 10) □

Remark: $X(56)$, $X(19861)$ and $X(24471)$ are collinear. (Refer Fig. 11)

Proposition 3.6.
The points $P_1 = BR \cap CQ$, $Q_1 = CP \cap AR$ and $R_1 = AQ \cap BP$ and the triangles ABC and $P_1Q_1R_1$ are perspective.(refer Fig. 12)

Proof. The lines BR and CQ have barycentric equations

$$(2s+c)x + (0)y - az = 0 \tag{19}$$
$$(2s+b)x - ay + (0)z = 0 \tag{20}$$

Lines (19), (20) intersect at the point $P_1 = (a \ : \ 2s+b \ : \ 2s+c)$.
Similarly $Q_1 = CP \cap AR = (2s+a \ : \ b \ : \ 2s+c)$ and $R_1 = AQ \cap BP = (2s+a \ : \ 2s+b \ : \ c)$

From these coordinates, it is clear that triangles $P_1Q_1R_1$ is perspective with ABC at

$$\lambda = AP_1 \cap BQ_1 \cap CR_1 = (2s+a \ : \ 2s+b \ : \ 2s+c) \tag{21}$$

□

Remark:

1. The perspector $\lambda = (2s+a \ : \ 2s+b \ : \ 2s+c)$ given in (21) is the triangle center X(1125)[1].
2. It is clear that P_1, Q_1 and R_1 are the centroids of the triangles BCI, CAI and ABI . The triangle $P_1Q_1R_1$ is homothetic to ABC, and the center of homothety is $\lambda = AP_1 \cap BQ_1 \cap CR_1 = (2s+a \ : \ 2s+b \ : \ 2s+c)$ =X(1125)=complement of X(10).

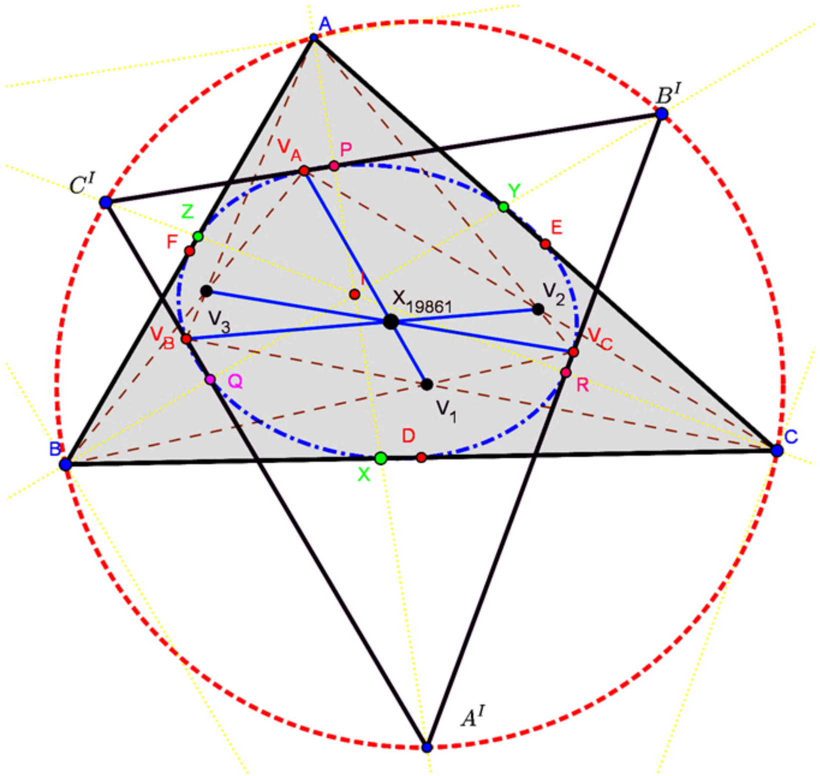

Fig. 10.

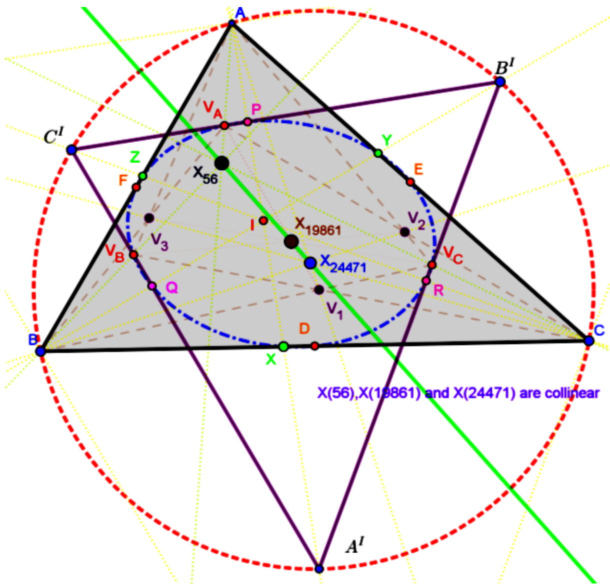

Fig. 11.

3. The point X(1125) is also the center of twelve point conic of incenter $\mathbb{C}_{I12}$. This article gives a new way of constructing the point X(1125).

Proposition 3.7.
The lines $A^{(I)}P_1$, $B^{(I)}Q_1$ and $C^{(I)}R_1$ are concur at Schiffler point. That is the triangles $A^{(I)}B^{(I)}C^{(I)}$ and $P_1Q_1R_1$ are perspective and the perspector is Schiffler point.

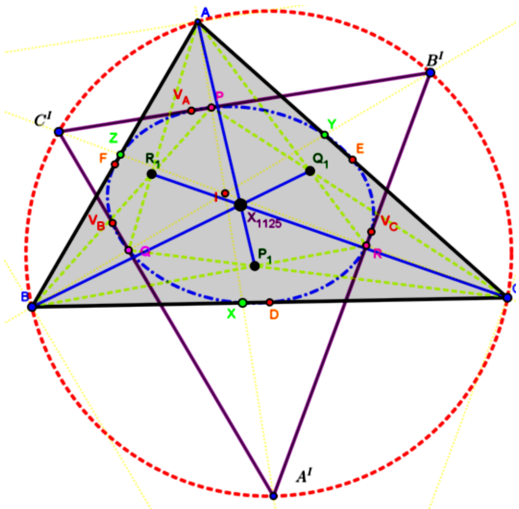

Fig. 12.

Proof. The lines $A^{(I)}P_1$, $B^{(I)}Q_1$ and $C^{(I)}R_1$ have barycentric equations

$$(b^2-c^2)x+a(c+a)y-a(a+b)z=0 \tag{22}$$

$$b(b+c)x+(c^2-a^2)y-a(a+b)z=0 \tag{23}$$

$$c(b+c)x-a(a+b)y+(a^2-b^2)z=0 \tag{24}$$

It is clear that

$$\begin{vmatrix} (b^2-c^2) & a(c+a) & -a(a+b) \\ b(b+c) & (c^2-a^2) & -a(a+b) \\ c(b+c) & -a(a+b) & (a^2-b^2) \end{vmatrix} = 0$$

Hence (22), (23) and (24) are concur. And they concur at

$$\left(\frac{a}{b+c}(s-a)\ :\ \frac{b}{c+a}(s-b)\ :\ \frac{c^2}{a+b}(s-c)\right) = Schiffler\ point = X(21)$$

Hence the triangles $A^{(I)}B^{(I)}C^{(I)}$ and $P_1Q_1R_1$ are perspective and the perspector is Schiffler point. □

Remark:

1. This article gives a new way of constructing the point X(21).(Refer Fig. 13).
2. The points X(19861), X(21), X(2360) are collinear (Refer Fig. 14).

Proposition 3.8.
The lines PP_1 , QQ_1 and RR_1 are concur at X(3616). That is the triangles PQR and $P_1Q_1R_1$ are perspective and the perspector is X(3616).

Proof. The lines PP_1 , QQ_1 and RR_1 have barycentric equations

$$(b-c)x-(4s-b)y+(4s-c)z=0 \tag{25}$$

$$(4s-a)x+(c-a)y-(4s-c)z=0 \tag{26}$$

$$-(4s-a)x+(4s-b)y+(a-b)z=0 \tag{27}$$

It is clear that

$$\begin{vmatrix} (b-c) & -(4s-b) & (4s-c) \\ (4s-a) & (c-a) & -(4s-c) \\ -(4s-a) & (4s-b) & (a-b) \end{vmatrix} = 0$$

Hence (25), (26) and (27) are concur. And they concur at

$$(s+a\ :\ s+b\ :\ s+c) = X(3616).$$

Hence the triangles PQR and $P_1Q_1R_1$ are perspective and the perspector is X(3616). (Refer Fig. 15) □

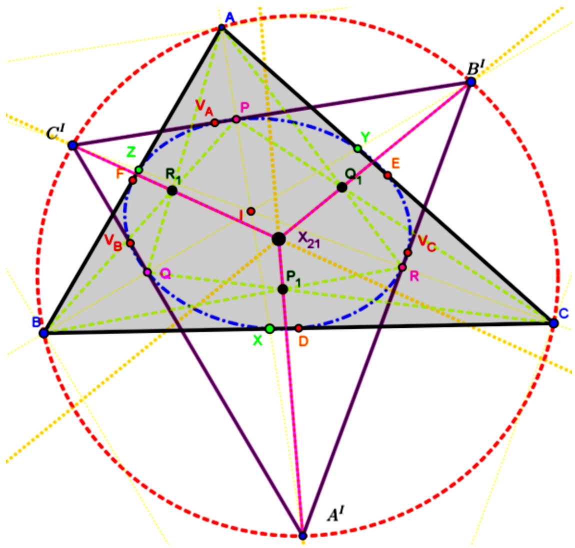

Fig. 13.

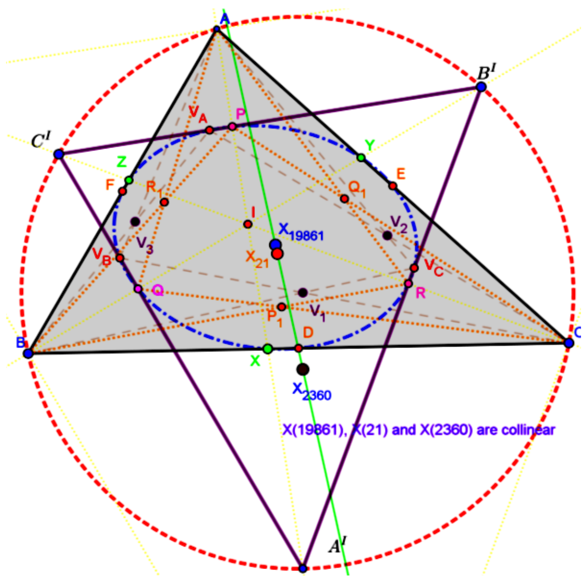

Fig. 14.

Remark:

1. This article gives a new way of constructing the point X(3616).
2. X(1), X(2), X(995), center of the conic $\mathbb{C}_{I_{12}}$ X(1125) , X(3616), X(19861) and X(K) .

Proof. Consider a line in barycentric coordinates which contains the points X(1) and X(2) is [see Eq. (9)]

$$(b-c)x+(c-a)y+(a-b)z=0$$

In remark-2 of Proposition 3.2 we proved that X(995) , X(K) lies on (9) and now it is clear that the points center of the conic $\mathbb{C}_{I_{12}}$ X(1125), X(3616) also lies on (9). That is all the six points X(1), X(2), X(995), center of the conic X(1125) , X(3616) , X(19861) and X(K) are collinear. □

Proposition 3.9.
The diagonal points of the quadrangle BCV_BV_C are X(56), V_1 and W_1. The points W_2, W_3 are similarly defined. Then The lines $A^{(I)}W_1$, $B^{(I)}W_2$ and $C^{(I)}W_3$ form a perspective triangle $A_WB_WC_W$ with ABC with perspector X(84)= Isogonal conjugate of Bevan point. (Refer **??***)*

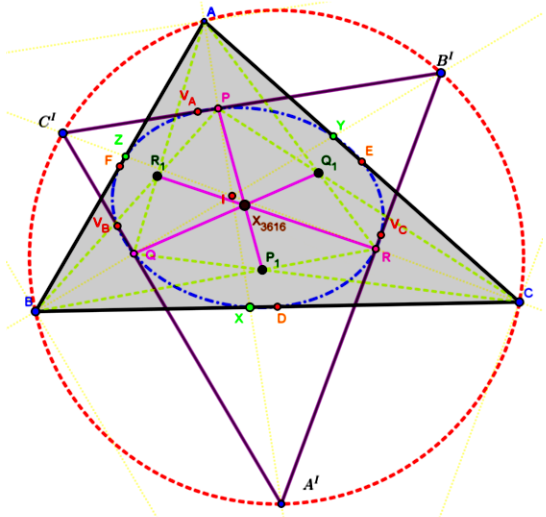

Fig. 15.

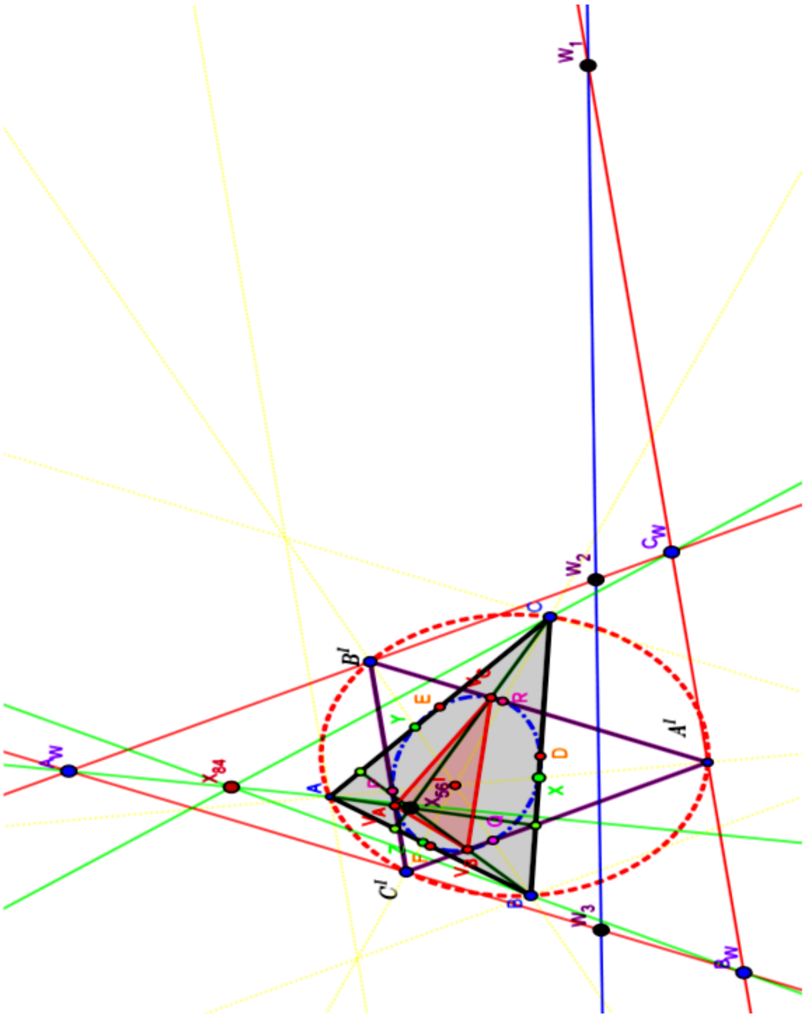

Fig. 16.

4. Conclusion

This article has shown new and more elegant approach to construct some notable triangle centers such as $X(21), X(56), X(84), X(995)$, center of the conic $\mathbb{C}_{I_{12}}$ $X(1125), X(2360), X(3616), X(19861), X(24471)$ and $X(K)$ using a bicevian conic (12 point conic).

Acknowledgement

The author would like to thank to Angel Montesdeoca, with a lot of patience he helped the author in identifying the certain concurrencies as prescribed triangle centers and also the author want to thank for his constant assistance to bring the article to the present form. The author is would like to thank an anonymous referee for his/her kind comments and suggestions, which lead to a better presentation of this paper.

References

[1] Angel Montesdeoca, Hyacinthos message 5003, 4998, october, 2018.
[2] C. Kimberling, Encyclopedia of Triangle Centers, available at http://faculty.evansville.edu/ck6/encyclopedia/ETC.html
[3] Dasari Naga Vijay Krishna, The New Proof of Ptolemy's Theorem & Nine Point Circle Theorem, Mathematics and Computer Science 1(4) (2016) 93-100. doi: 10.11648/j.mcs.20160104.14
[4] Michael De Villiers, The nine-point conic: a rediscovery and proof by computer, International Journal of Mathematical Education in Science and Technology 37(1) (2006) 7–14.
[5] P. Yiu, Introduction to the Geometry of the Triangle, Florida Atlantic University Lecture Notes, 2001; revised, 2013, available at http://math.fau.edu/Yiu/Geometry.html