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Simson Identity of Generalized m-step Fibonacci Numbers

Research Article

Yüksel Soykan *

Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

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Abstract: One of the best known and oldest identities for the Fibonacci sequence $\{F_n\}$ is

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Simson or Cassini Identity. In this paper, we generalize this result to generalized m-step Fibonacci numbers and give an attractive formula. Furthermore, we present some Simson's identities of particular generalized m-step Fibonacci sequences. Also we give Simson identity of Gaussian Generalized m-step Fibonacci Numbers.

MSC: 11B39 • 11B83

Keywords: m-step Fibonacci numbers • Simson Identity • Simson Formula • Cassini Identity • Fibonacci numbers • Tribonacci numbers • Tetranacci numbers

1. Introduction

Several generalizations of Fibonacci numbers and identities have been studied by mathematicians over the years. In this paper, we generalize Simson's identity to generalized m-step Fibonacci sequences. Before presenting our main result (Theorem 3.1) we give some background. For $m \geq 2$, the generalized m-step Fibonacci numbers, $\{V_n(V_0, V_1, V_2, ..., V_{m-1}; r_1, r_2, , ..., r_m)\}_{n\geq m}$ (or shortly $\{V_n\}_{n\geq m}$), $(n \geq m)$, is defined by the m-order linear recurrence relation

$$V_n = \sum_{i=1}^{m} r_i V_{n-i} = r_1 V_{n-1} + r_2 V_{n-2} + r_3 V_{n-3} + ... + r_{m-1} V_{n-m-1} + r_m V_{n-m} \tag{1}$$

with m initial terms

$$V_0 = c_0,\ V_1 = c_1,\ V_2 = c_2, ..., V_{m-1} = c_{m-1},$$

where r_i, $1 \leq i \leq m$, are all real numbers and c_i, $0 \leq i \leq m-1$, are all real or complex numbers. Such a sequence is also called the generalized Fibonacci m-sequence, or generalized m-nacci sequence, or the m-generalized Fibonacci sequence.

The sequences $\{V_n\}_{n\geq m}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{r_{m-1}}{r_m} V_{-(n-1)} - \frac{r_{m-2}}{r_m} V_{-(n-2)} - \frac{r_{m-3}}{r_m} V_{-(n-3)} - ... - \frac{r_1}{r_m} V_{-(n-(m-1))} + \frac{1}{r_m} V_{-(n-m)}$$

* *E-mail address(es):* yuksel_soykan@hotmail.com

for $n = m-2, m-1, m, m+1 \ldots$. Therefore, recurrence (1) holds for all integer n.

For $m \geq 2$, the m-step Fibonacci numbers, U_n $(n \geq m)$, is defined by the m-order linear recurrence relation

$$U_n = \sum_{i=1}^{m} U_{n-i} = U_{m-1} + U_{m-2} + U_{m-3} + \ldots + U_{n-m} \tag{2}$$

with m initial terms

$$\begin{cases} U_k = 0 & , \quad -m+2 \leq k \leq 0 \\ U_{-k+1} = 1 & , \quad k = m \end{cases}. \tag{3}$$

Some of the well known members of this m-step Fibonacci numbers include Fibonacci numbers F_n $(m = 2, U = F)$, Tribonacci numbers T_n $(m = 3, U = T)$, Tetranacci numbers M_n $(m = 4, U = M)$ and Pentanacci numbers P_n $(m = 5, U = P)$. Here $r_i = 1$ for all $1 \leq i \leq m$. See Table 1 for some values of these numbers.

Table 1. The first few sequences of m-step Fibonacci numbers.

m	Name	n	−6	−5	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10
2	Fibonacci	F_n	−8	5	−3	2	−1	1	0	1	1	2	3	5	8	13	21	34	55
3	Tribonacci	T_n	−3	2	0	−1	1	0	0	1	1	2	4	7	13	24	44	81	149
4	Tetranacci	M_n	0	0	−1	1	0	0	0	1	1	2	4	8	15	29	56	108	208
5	Pentanacci	P_n	0	−1	1	0	0	0	0	1	1	2	4	8	16	31	61	120	236

Like the m-step Fibonacci numbers, m-step Lucas numbers are defined by the same the m-order recurrence relations (2) but with different initial terms, namely the m-step Lucas numbers, W_n , is defined by the m-order linear recurrence relation

$$W_n = \sum_{i=1}^{m} W_{n-i} = W_{m-1} + W_{m-2} + W_{m-3} + \ldots + W_{n-m} \tag{4}$$

with the m initial terms

$$\begin{cases} W_k = -1 & , \quad -m+1 \leq k \leq -1 \\ W_k = m & , \quad k = 0 \end{cases}. \tag{5}$$

Some of the well known members of this m-step Fibonacci numbers include Lucas numbers L_n $(m = 2, W = L)$, Tribonacci-Lucas numbers K_n $(m = 3, W = K)$, Tetranacci-Lucas numbers R_n $(m = 4, V = n)$ and Pentanacci-Lucas numbers Q_n $(m = 5, W = Q)$. Here $r_i = 1$ for all $1 \leq i \leq m$. See Table 2 for some values of these numbers.

Table 2. The first few sequences of m-step Lucas numbers.

m	Name	n	−4	−3	−2	−1	0	1	2	3	4	5	6	7	8	9	10
2	Lucas	L_n	7	−4	3	−1	2	1	3	4	7	11	18	29	47	76	123
3	Tribonacci-Lucas	K_n	−5	5	−1	−1	3	1	3	7	11	21	39	71	131	241	443
4	Tetranacci-Lucas	R_n	7	−1	−1	−1	4	1	3	7	15	26	51	99	191	367	708
5	Pentanacci-Lucas	Q_n	−1	−1	−1	−1	5	1	3	7	15	31	57	113	223	439	863

Next we consider the case $r_i = 1$ for all $1 \leq i \leq m-1$ and $r_m = 2$. For $m \geq 2$, m-step (order) Jacobsthal numbers, $\{J_n^{(m)}(J_0^{(m)}, J_1^{(m)}, J_2^{(m)}, \ldots, J_{m-1}^{(m)}; 1, 1, , \ldots, 1, 2)\}_{n \geq m}$ (or shortly $\{J_n^{(m)}\}_{n \geq m}$), $(n \geq m)$, is defined by the m-order linear recurrence relation

$$J_n^{(m)} = \sum_{i=1}^{m-1} r_i J_{n-i}^{(m)} + 2J_{n-m}^{(m)} \tag{6}$$

with m initial terms

$$J_0^{(m)} = 0 \text{ and } J_i^{(m)} = 1 \text{ for } i = 1, 2, \ldots, m-1.$$

For the mth order Jacobsthal-Lucas numbers $j_n^{(m)}$ we use the same recursion (6) with initial conditions $j_i^{(m)} = j_i^{(m-1)}$ for $i = 0, 1, 2, \ldots, m-1$ and $j_0^{(2)} = 2$, $j_1^{(2)} = 1$. See Table 3 and Table 4 for mth order Jacobsthal numbers and mth order Jacobsthal-Lucas numbers, respectively.

Table 3. The first few sequences of mth order Jacobsthal numbers.

m	Name	n	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
2	second order Jacobsthal	$J_n^{(2)}$	$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	1	1	3	5	11	21	43	85	171	341
3	third order Jacobsthal	$J_n^{(3)}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	1	1	2	5	9	18	37	73	146	293
4	fourth order Jacobsthal	$J_n^{(4)}$	$\frac{5}{8}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	1	1	1	3	7	13	25	51	103	205
5	fifth order Jacobsthal	$J_n^{(5)}$	$\frac{1}{2}$	0	-1	0	1	1	1	1	4	9	17	33	65	132

Table 4. The first few sequences of mth order Jacobsthal-Lucas numbers.

m	Name	n	-3	-2	-1	0	1	2	3	4	5	6	7	8
2	second order Jacobsthal-Lucas	$j_n^{(2)}$	$-\frac{7}{8}$	$\frac{5}{4}$	$-\frac{1}{2}$	2	1	5	7	17	31	65	127	257
3	third order Jacobsthal-Lucas	$j_n^{(3)}$	1	-1	1	2	1	5	10	17	37	74	145	293
4	fourth order Jacobsthal-Lucas	$j_n^{(4)}$	$-\frac{5}{4}$	$\frac{1}{2}$	1	2	1	5	10	20	37	77	154	308
5	fifth order Jacobsthal-Lucas	$j_n^{(5)}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	1	5	10	20	40	77	157	314

For more details about generalized m-step Fibonacci numbers we refer to, for example, the works in [2–4], among others. Now, we consider the cases $m = 2,3,4,5$ of the generalized m-step Fibonacci numbers separately.

Horadam sequence (generalized Fibonacci sequence) $\{V_n(V_0,V_1;r,s)\}_{n\geq 0}$ (or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n = rV_{n-1} + sV_{n-2}, \quad V_0 = c_0, V_1 = c_1, \quad n \geq 2 \tag{7}$$

where V_0, V_1 are arbitrary reel or complex numbers and r, s are real numbers. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{r}{s}V_{-(n-1)} + \frac{1}{s}V_{-(n-2)}$$

for $n = 1,2,3,...$ when $s \neq 0$. Therefore, recurrence (7) holds for all integer n. See Table 5 for a few members of Horadam sequences.

Table 5. A few members of Horadam sequences.

Sequences (Numbers)	Notation
Fibonacci	$\{F_n\} = \{V_n(0,1;1,1)\}$
Lucas	$\{L_n\} = \{V_n(2,1;1,1)\}$
Pell	$\{P_n\} = \{V_n(0,1;2,1)\}$
Pell-Lucas	$\{Q_n\} = \{V_n(2,2;2,1)\}$
second order Jacobsthal	$\{J_n\} = \{V_n(0,1;1,2)\}$
second order Jacobsthal-Lucas	$\{j_n\} = \{V_n(2,1;1,2)\}$

The first few values of the sequences with non-negative indices are shown below (see Table 6).

Table 6. A few values of Horadam sequences with non-negative and negative indices.

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
F_n	-21	13	-8	5	-3	2	-1	1	0	1	1	2	3	5	8	13	21
L_n	47	-29	18	-11	7	-4	3	-1	2	1	3	4	7	11	18	29	47
P_n	-408	169	-70	29	-12	5	-2	1	0	1	2	5	12	29	70	169	408
Q_n	1154	-478	198	-82	34	-14	6	-2	2	2	6	14	34	82	198	478	1154
J_n	$-\frac{85}{256}$	$\frac{43}{128}$	$-\frac{21}{64}$	$\frac{11}{32}$	$-\frac{5}{16}$	$\frac{3}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	1	1	3	5	11	21	43	85
j_n	$\frac{257}{256}$	$-\frac{127}{128}$	$\frac{65}{64}$	$-\frac{31}{32}$	$\frac{17}{16}$	$-\frac{7}{8}$	$\frac{5}{4}$	$-\frac{1}{2}$	2	1	5	7	17	31	65	127	257

The generalized Tribonacci sequence $\{V_n(V_0,V_1,V_2;r,s,t)\}_{n\geq 0}$ (or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}, \quad V_0 = c_0, V_1 = c_1, V_2 = c_2, \quad n \geq 3 \tag{8}$$

where V_0, V_1, V_2 are arbitrary reel or complexs numbers and r, s, t are real numbers. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{s}{t}V_{-(n-1)} - \frac{r}{t}V_{-(n-2)} + \frac{1}{t}V_{-(n-3)}$$

for $n = 1, 2, 3, \ldots$ when $t \neq 0$. Therefore, recurrence (8) holds for all integer n.

In literature, for example, the following names and notations (see Table 7) are used for the special case of r, s, t and initial values.

Table 7. A few members of generalized Tribonacci sequences.

Sequences (Numbers)	Notation
Tribonacci	$\{T_n\} = \{V_n(0,1,1;1,1,1)\}$
Tribonacci-Lucas	$\{K_n\} = \{V_n(3,1,3;1,1,1)\}$
Padovan (Cordonnier)	$\{P_n\} = \{V_n(1,1,1;0,1,1)\}$
Pell-Padovan	$\{R_n\} = \{V_n(1,1,1;0,2,1)\}$
Jacobsthal-Padovan	$\{JP_n\} = \{V_n(1,1,1;0,1,2)\}$
Perrin	$\{Q_n\} = \{V_n(3,0,2;0,1,1)\}$
Pell-Perrin	$\{pQ_n\} = \{V_n(3,0,2;0,2,1)\}$
Jacobsthal-Perrin	$\{JQ_n\} = \{V_n(3,0,2;0,1,2)\}$
Padovan-Perrin	$\{S_n\} = \{V_n(0,0,1;0,1,1)\}$
Narayana	$\{N_n\} = \{V_n(0,1,1;1,0,1)\}$
third order Jacobsthal	$\{J_n\} = \{V_n(0,1,1;1,1,2)\}$
third order Jacobsthal-Lucas	$\{j_n\} = \{V_n(2,1,5;1,1,2)\}$

The first few values of the sequences with non-negative and negative indices are shown below (see Table 8).

Table 8. A few values of generalized Tribonacci sequences.

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8
T_n	4	1	-3	2	0	-1	1	0	0	1	1	2	4	7	13	24	44
K_n	3	-15	11	-1	-5	5	-1	-1	3	1	3	7	11	21	39	71	131
P_n	0	1	-1	1	0	0	1	0	1	1	1	2	2	3	4	5	7
R_n	67	-41	25	-15	9	-5	3	-1	1	1	1	3	3	7	9	17	25
JP_n	$\frac{23}{128}$	$-\frac{3}{64}$	$-\frac{1}{32}$	$\frac{5}{16}$	$-\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	0	1	1	1	3	3	5	9	11	19
Q_n	5	-1	-2	4	-3	2	1	-1	3	0	2	3	2	5	5	7	10
pQ_n	156	-96	59	-36	22	-13	8	-4	3	0	2	3	4	8	11	20	30
JQ_n	$\frac{161}{256}$	$-\frac{85}{128}$	$\frac{25}{64}$	$\frac{19}{32}$	$-\frac{15}{16}$	$\frac{11}{8}$	$\frac{1}{4}$	$-\frac{1}{2}$	3	0	2	6	2	10	14	14	34
S_n	1	-2	2	-1	0	1	-1	1	0	0	1	0	1	1	1	2	2
N_n	0	-2	1	1	-1	0	1	0	0	1	1	1	2	3	4	6	9
J_n	$\frac{55}{128}$	$-\frac{9}{64}$	$-\frac{9}{32}$	$\frac{7}{16}$	$-\frac{1}{8}$	$-\frac{1}{4}$	$\frac{1}{2}$	0	0	1	1	2	5	9	18	37	73
j_n	$-\frac{41}{32}$	$\frac{7}{16}$	$\frac{7}{8}$	$-\frac{5}{4}$	$\frac{1}{2}$	1	-1	1	2	1	5	10	17	37	74	145	293

The generalized Tetranacci sequence $\{V_n(V_0, V_1, V_2, V_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3} + uV_{n-4}, \qquad V_0 = c_0, V_1 = c_1, V_2 = c_2, \, , V_3 = c_3, \; n \geq 4 \tag{9}$$

where V_0, V_1, V_2, V_3 are arbitrary reel or complex numbers and r, s, t, u are real numbers. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{t}{u}V_{-(n-1)} - \frac{s}{u}V_{-(n-2)} - \frac{r}{u}V_{-(n-3)} + \frac{1}{u}V_{-(n-4)}$$

for $n = 1, 2, 3, \ldots$ when $u \neq 0$. Therefore, recurrence (9) holds for all integer n.

In literature, for example, the following names and notations (see Table 9) are used for the special case of r, s, t, u and initial values.

Table 9. A few members of generalized Tetranacci sequences.

Sequences (Numbers)	Notation
Tetranacci	$\{M_n\} = \{V_n(0,1,1,2;1,1,1,1)\}$
Tetranacci-Lucas	$\{R_n\} = \{V_n(4,1,3,7;1,1,1,1)\}$
fourth order Jacobsthal	$\{J_n\} = \{V_n(0,1,1,1;1,1,1,2)\}$
fourth order Jacobsthal-Lucas	$\{j_n\} = \{V_n(2,1,5,10;1,1,1,2)\}$

The first few values of the sequences with non-negative and negative indices are shown below (see Table 10).

Table 10. A few values of generalized Tetranacci sequences.

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
M_n	-3	2	0	0	-1	1	0	0	0	1	1	2	4	8	15	29	56	108
R_n	15	-1	-1	-6	7	-1	-1	-1	4	1	3	7	15	26	51	99	191	367
J_n	$-\frac{51}{256}$	$\frac{77}{128}$	$\frac{13}{64}$	$-\frac{19}{32}$	$-\frac{3}{16}$	$\frac{5}{8}$	$\frac{1}{4}$	$-\frac{1}{2}$	0	1	1	1	3	7	13	25	51	103
j_n	$\frac{103}{128}$	$-\frac{89}{64}$	$\frac{7}{32}$	$\frac{7}{16}$	$\frac{7}{8}$	$-\frac{5}{4}$	$\frac{1}{2}$	1	2	1	5	10	20	37	77	154	308	613

The generalized Pentanacci sequence $\{V_n(V_0, V_1, V_2, V_3, V_4; r, s, t, u, v)\}_{n\geq 0}$ (or shortly $\{V_n\}_{n\geq 0}$) is defined as follows:

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3} + uV_{n-4} + vV_{n-5},\ V_0 = c_0, V_1 = c_1, V_2 = c_2,\ V_3 = c_3, V_4 = c_4,\ n \geq 5 \tag{10}$$

where V_0, V_1, V_2, V_3, V_4 are arbitrary reel or complex numbers and r, s, t, u are real numbers. The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{u}{v}V_{-n+1} - \frac{t}{v}V_{-n+2} - \frac{s}{v}V_{-n+3} - \frac{r}{v}V_{-n+4} + \frac{1}{v}V_{-n+5}$$

for $n = 1, 2, 3, ...$ when $u \neq 0$. Therefore, recurrence (10) holds for all integer n.

In literature, for example, the following names and notations (see Table 11) are used for the special case of r, s, t, u, v and initial values.

Table 11. A few members of generalized Pentanacci sequences.

Sequences (Numbers)	Notation
Pentanacci	$\{P_n\} = \{V_n(0,1,1,2,4;1,1,1,1,1)\}$
Pentanacci-Lucas	$\{Q_n\} = \{V_n(5,1,3,7,15;1,1,1,1,1)\}$
fifth order Jacobsthal	$\{J_n\} = \{V_n(0,1,1,1,1;1,1,1,1,2)\}$
fifth order Jacobsthal-Lucas	$\{j_n\} = \{V_n(2,1,5,10,20;1,1,1,1,2)\}$

The first few values of the sequences with non-negative and negative indices are shown below (see Table 12).

Table 12. A few values of generalized Pentanacci sequences.

n	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
P_n	0	0	0	-1	1	0	0	0	0	1	1	2	4	8	16	31	61	120
Q_n	-1	-1	-7	9	-1	-1	-1	-1	5	1	3	7	15	31	57	113	223	439
J_n	$\frac{31}{64}$	$-\frac{1}{32}$	$-\frac{17}{16}$	$-\frac{1}{8}$	$\frac{3}{4}$	$\frac{1}{2}$	0	-1	0	1	1	1	1	4	9	17	33	65
j_n	$\frac{13}{128}$	$\frac{13}{64}$	$\frac{13}{32}$	$\frac{13}{16}$	$-\frac{11}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	1	5	10	20	40	77	157	314	628

2. Particular Cases of Main Result

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

A search of the literature turns up that there are many identities including Simson (Cassini), Catalan, d'Ocagne, Melham, Tagiuri, Gelin-Cesaro, Gould identities, see for example, [5–12]. See also [13–15] for the work of some generalizations of Fibonacci numbers.

Next, we consider generalized Horadam numbers $V_n = rV_{n-1} + sV_{n-2}$ with 2 initial terms $V_0 = c_0$, $V_1 = c_1$ and present a formula for those numbers.

Theorem 2.1 (Simson Formula of Horadam Numbers).
For all integers n we have

$$\begin{vmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{vmatrix} = (-1)^n s^n \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}. \tag{11}$$

Proof. We proof by induction on n. Firstly, we prove the formula (11) for $n \geq 0$. For $n = 0$, it is obvious that the formula is true. Now, we assume that the formula (11) is true for $n = k$, that is

$$\begin{vmatrix} V_{k+1} & V_k \\ V_k & V_{k-1} \end{vmatrix} = (-1)^k s^k \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}.$$

Then by induction hypothesis, we obtain

$$\begin{aligned} \begin{vmatrix} V_{k+2} & V_{k+1} \\ V_{k+1} & V_k \end{vmatrix} &= \begin{vmatrix} rV_{k+1}+sV_k & V_{k+1} \\ rV_k+sV_{k-1} & V_k \end{vmatrix} = r\begin{vmatrix} V_{k+1} & V_{k+1} \\ V_k & V_k \end{vmatrix} + s\begin{vmatrix} V_k & V_{k+1} \\ V_{k-1} & V_k \end{vmatrix} \\ &= -s\begin{vmatrix} V_{k+1} & V_k \\ V_k & V_{k-1} \end{vmatrix} = -s\left((-1)^k s^k \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}\right) \\ &= (-1)^{k+1} s^{k+1} \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}. \end{aligned}$$

i.e., the formula (11) is true for $n = k+1$. Thus, (11) hold for all integers $n \geq 1$.

Now we consider the formula (11) for $n \leq -1$. Take $h = -n$ so that $h \geq 1$. So we need to prove by induction that for $h \geq 1$ we have

$$\begin{vmatrix} V_{-h+1} & V_{-h} \\ V_{-h} & V_{-h-1} \end{vmatrix} = (-1)^{-h} s^{-h} \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}. \tag{12}$$

For $h = 1$, the formula is true because

$$\begin{aligned} \begin{vmatrix} V_0 & V_{-1} \\ V_{-1} & V_{-2} \end{vmatrix} &= -\begin{vmatrix} V_{-1} & V_0 \\ V_{-2} & V_{-1} \end{vmatrix} = -\begin{vmatrix} -\frac{r}{s}V_0+\frac{1}{s}V_1 & V_0 \\ -\frac{r}{s}V_{-1}+\frac{1}{s}V_0 & V_{-1} \end{vmatrix} \\ &= -\begin{vmatrix} -\frac{r}{s}V_0 & V_0 \\ -\frac{r}{s}V_{-1} & V_{-1} \end{vmatrix} - \begin{vmatrix} \frac{1}{s}V_1 & V_0 \\ \frac{1}{s}V_0 & V_{-1} \end{vmatrix} = -\frac{1}{s}\begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}. \end{aligned}$$

Now, we assume that the formula (12) is true for $h = k$, that is

$$\begin{vmatrix} V_{-k+1} & V_{-k} \\ V_{-k} & V_{-k-1} \end{vmatrix} = (-1)^{-k} s^{-k} \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}. \tag{13}$$

Then by induction hypothesis (13), we obtain

$$\begin{aligned} \begin{vmatrix} V_{-(k+1)+1} & V_{-(k+1)} \\ V_{-(k+1)} & V_{-(k+1)-1} \end{vmatrix} &= \begin{vmatrix} V_{-k} & V_{-k-1} \\ V_{-k-1} & V_{-k-2} \end{vmatrix} = -\begin{vmatrix} V_{-k-1} & V_{-k} \\ V_{-k-2} & V_{-k-1} \end{vmatrix} = -\begin{vmatrix} -\frac{r}{s}V_{-k}+\frac{1}{s}V_{-k+1} & V_{-k} \\ -\frac{r}{s}V_{-k-1}+\frac{1}{s}V_{-k} & V_{-k-1} \end{vmatrix} \\ &= -\begin{vmatrix} -\frac{r}{s}V_{-k} & V_{-k} \\ -\frac{r}{s}V_{-k-1} & V_{-k-1} \end{vmatrix} - \begin{vmatrix} \frac{1}{s}V_{-k+1} & V_{-k} \\ \frac{1}{s}V_{-k} & V_{-k-1} \end{vmatrix} \\ &= -\frac{1}{s}\begin{vmatrix} V_{-k+1} & V_{-k} \\ V_{-k} & V_{-k-1} \end{vmatrix} = -\frac{1}{s}\left((-1)^{-k} s^{-k} \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}\right) \\ &= (-1)^{-k+1} s^{-(k+1)} \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix} = (-1)^{-(k+1)} s^{-(k+1)} \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix} \end{aligned}$$

so that the formula (12) is true for $h = k+1$. Thus, (12) holds for all integers $h \geq 1$ and so (11) holds for all integers $n \leq -1$. This completes the proof. □

Remark 2.1.

Theorem 2.1 is given in Horadam [10, 11]). In fact, in [10], Horadam gave a beautiful formula more general case, namely Catalan Identity for Horadam numbers. We provide the proof of Theorem 2.1 here because it pave the way the method to prove the general case.

We can write Theorem 2.1 as

$$f(n) = (-1)^n s^n f(0)$$

where $f(n) = \begin{vmatrix} V_{n+1} & V_n \\ V_n & V_{n-1} \end{vmatrix}$ and $f(0) = \begin{vmatrix} V_1 & V_0 \\ V_0 & V_{-1} \end{vmatrix}$. In the following Table 13, we present Simsons's formula of particular Horadam sequences.

Table 13. Simsons's formula of some Horadam sequences.

Sequence: V_n	Simson Formula	Sequence: V_n	Simson Formula
F_n	$f(n)=(-1)^n$	L_n	$f(n)=5(-1)^{n-1}$
P_n	$f(n)=(-1)^n$	Q_n	$f(n)=8(-1)^{n-1}$
J_n	$f(n)=(-1)^n 2^{n-1}$	j_n	$f(n)=9(-1)^{n-1}2^{n-1}$

Next we consider generalized Tribonacci numbers $V_n = rV_{n-1} + sV_{n-2} + tV_{n-3}$ with 3 initial terms $V_0 = c_0$, $V_1 = c_1$, $V_2 = c_2$.

Theorem 2.2 (Simson Formula of Generalized Tribonacci Numbers).
For all integers n we have

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = t^n \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{14}$$

Proof. We prove by induction on n. Firstly, we prove the formula (14) for $n \geq 0$. For $n = 0$, it is obvious that the formula is true. Now, we assume that the formula (14) is true for $n = k$, that is

$$\begin{vmatrix} V_{k+2} & V_{k+1} & V_k \\ V_{k+1} & V_k & V_{k-1} \\ V_k & V_{k-1} & V_{k-2} \end{vmatrix} = t^k \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}.$$

Then by induction hypothesis, we obtain

$$\begin{aligned} & \begin{vmatrix} V_{k+3} & V_{k+2} & V_{k+1} \\ V_{k+2} & V_{k+1} & V_k \\ V_{k+1} & V_k & V_{k-1} \end{vmatrix} \\ = & \begin{vmatrix} rV_{k+2}+sV_{k+1}+tV_k & V_{k+2} & V_{k+1} \\ rV_{k+1}+sV_k+tV_{k-1} & V_{k+1} & V_k \\ rV_k+sV_{k-1}+tV_{k-2} & V_k & V_{k-1} \end{vmatrix} \\ = & \begin{vmatrix} rV_{k+2} & V_{k+2} & V_{k+1} \\ rV_{k+1} & V_{k+1} & V_k \\ rV_k & V_k & V_{k-1} \end{vmatrix} + \begin{vmatrix} sV_{k+1} & V_{k+2} & V_{k+1} \\ sV_k & V_{k+1} & V_k \\ sV_{k-1} & V_k & V_{k-1} \end{vmatrix} + \begin{vmatrix} tV_k & V_{k+2} & V_{k+1} \\ tV_{k-1} & V_{k+1} & V_k \\ tV_{k-2} & V_k & V_{k-1} \end{vmatrix} \\ = & t \begin{vmatrix} V_k & V_{k+2} & V_{k+1} \\ V_{k-1} & V_{k+1} & V_k \\ V_{k-2} & V_k & V_{k-1} \end{vmatrix} = t \begin{vmatrix} V_{k+2} & V_{k+1} & V_k \\ V_{k+1} & V_k & V_{k-1} \\ V_k & V_{k-1} & V_{k-2} \end{vmatrix} \\ = & t \left(t^k \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix} \right) = t^{k+1} \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix} \end{aligned}$$

i.e., the formula (14) is true for $n = k+1$. Thus, (14) holds for all integers $n \geq 1$.

Now we consider the formula (14) for $n \leq -1$. Take $h = -n$ so that $h \geq 1$. So we need to prove by induction that for $h \geq 1$

$$\begin{vmatrix} V_{-h+2} & V_{-h+1} & V_{-h} \\ V_{-h+1} & V_{-h} & V_{-h-1} \\ V_{-h} & V_{-h-1} & V_{-h-2} \end{vmatrix} = t^{-h} \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{15}$$

For $h = 1$, the formula is true because

$$\begin{aligned} \begin{vmatrix} V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \\ V_{-1} & V_{-2} & V_{-3} \end{vmatrix} & = \begin{vmatrix} V_{-1} & V_1 & V_0 \\ V_{-2} & V_0 & V_{-1} \\ V_{-3} & V_{-1} & V_{-2} \end{vmatrix} = \begin{vmatrix} -\frac{s}{t}V_0 - \frac{r}{t}V_1 + \frac{1}{t}V_2 & V_1 & V_0 \\ -\frac{s}{t}V_{-1} - \frac{r}{t}V_0 + \frac{1}{t}V_1 & V_0 & V_{-1} \\ -\frac{s}{t}V_{-2} - \frac{r}{t}V_{-1} + \frac{1}{t}V_0 & V_{-1} & V_{-2} \end{vmatrix} \\ & = \begin{vmatrix} -\frac{s}{t}V_0 & V_1 & V_0 \\ -\frac{s}{t}V_{-1} & V_0 & V_{-1} \\ -\frac{s}{t}V_{-2} & V_{-1} & V_{-2} \end{vmatrix} + \begin{vmatrix} -\frac{r}{t}V_1 & V_1 & V_0 \\ -\frac{r}{t}V_0 & V_0 & V_{-1} \\ -\frac{r}{t}V_{-1} & V_{-1} & V_{-2} \end{vmatrix} + \begin{vmatrix} \frac{1}{t}V_2 & V_1 & V_0 \\ \frac{1}{t}V_1 & V_0 & V_{-1} \\ \frac{1}{t}V_0 & V_{-1} & V_{-2} \end{vmatrix} \\ & = \frac{1}{t} \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \end{aligned}$$

Now, we assume that the formula (15) is true for $h = k$, that is

$$\begin{vmatrix} V_{-k+2} & V_{-k+1} & V_{-k} \\ V_{-k+1} & V_{-k} & V_{-k-1} \\ V_{-k} & V_{-k-1} & V_{-k-2} \end{vmatrix} = t^{-k} \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{16}$$

Then by induction hypothesis, we obtain

$$\begin{aligned}
&\begin{vmatrix} V_{-(k+1)+2} & V_{-(k+1)+1} & V_{-(k+1)} \\ V_{-(k+1)+1} & V_{-(k+1)} & V_{-(k+1)-1} \\ V_{-(k+1)} & V_{-(k+1)-1} & V_{-(k+1)-2} \end{vmatrix} = \begin{vmatrix} V_{-k+1} & V_{-k} & V_{-k-1} \\ V_{-k} & V_{-k-1} & V_{-k-2} \\ V_{-k-1} & V_{-k-2} & V_{-k-3} \end{vmatrix} \\
&= \begin{vmatrix} V_{-k-1} & V_{-k+1} & V_{-k} \\ V_{-k-2} & V_{-k} & V_{-k-1} \\ V_{-k-3} & V_{-k-1} & V_{-k-2} \end{vmatrix} = \begin{vmatrix} -\frac{s}{t}V_{-k} - \frac{r}{t}V_{-k+1} + \frac{1}{t}V_{-k+2} & V_{-k+1} & V_{-k} \\ -\frac{s}{t}V_{-k-1} - \frac{r}{t}V_{-k} + \frac{1}{t}V_{-k+1} & V_{-k} & V_{-k-1} \\ -\frac{s}{t}V_{-k-2} - \frac{r}{t}V_{-k-1} + \frac{1}{t}V_{-k} & V_{-k-1} & V_{-k-2} \end{vmatrix} \\
&= \begin{vmatrix} -\frac{s}{t}V_{-k} & V_{-k+1} & V_{-k} \\ -\frac{s}{t}V_{-k-1} & V_{-k} & V_{-k-1} \\ -\frac{s}{t}V_{-k-2} & V_{-k-1} & V_{-k-2} \end{vmatrix} + \begin{vmatrix} -\frac{r}{t}V_{-k+1} & V_{-k+1} & V_{-k} \\ -\frac{r}{t}V_{-k} & V_{-k} & V_{-k-1} \\ -\frac{r}{t}V_{-k-1} & V_{-k-1} & V_{-k-2} \end{vmatrix} + \begin{vmatrix} \frac{1}{t}V_{-k+2} & V_{-k+1} & V_{-k} \\ \frac{1}{t}V_{-k+1} & V_{-k} & V_{-k-1} \\ \frac{1}{t}V_{-k} & V_{-k-1} & V_{-k-2} \end{vmatrix} \\
&= \frac{1}{t}\begin{vmatrix} V_{-k+2} & V_{-k+1} & V_{-k} \\ V_{-k+1} & V_{-k} & V_{-k-1} \\ V_{-k} & V_{-k-1} & V_{-k-2} \end{vmatrix} = \frac{1}{t}t^{-k}\begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix} = t^{-(k+1)}\begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}
\end{aligned}$$

i.e., the formula (15) is true for $h = k+1$. Thus, (15) holds for all integers $h \geq 1$ and so (14) holds for all integers $n \leq -1$. This completes the proof. □

We can write Theorem 2.2 as

$$f(n) = t^n f(0)$$

where $f(n) = \begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix}$ and $f(0) = \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}$.

In the following Table 14, we present Simsons's formula of particular generalized Tribonacci sequences.

Table 14. Simsons's formula of some generalized Tribonacci sequences.

Sequence: V_n	Simson Formula	Sequence: V_n	Simson Formula
T_n	$f(n) = -1$	K_n	$f(n) = -44$
P_n	$f(n) = -1$	R_n	$f(n) = -4$
JP_n	$f(n) = -2^n$	Q_n	$f(n) = -23$
pQ_n	$f(n) = -11$	JQ_n	$f(n) = -13 \times 2^{n+1}$
S_n	$f(n) = -1$	N_n	$f(n) = -1$
J_n	$f(n) = -2^{n-1}$	j_n	$f(n) = -9 \times 2^{n+1}$

Next we consider generalized Tetranacci numbers $V_n = rV_{n-1} + sV_{n-2} + tV_{n-3} + uV_{n-4}$ with 4 initial terms

$$V_0 = c_0,\ V_1 = c_1,\ V_2 = c_2, V_3 = c_3.$$

Theorem 2.3 (Simson Formula of Generalized Tetranacci Numbers).
For all integers n we have

$$\begin{vmatrix} V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} \end{vmatrix} = (-1)^n u^n \begin{vmatrix} V_3 & V_2 & V_1 & V_0 \\ V_2 & V_1 & V_0 & V_{-1} \\ V_1 & V_0 & V_{-1} & V_{-2} \\ V_0 & V_{-1} & V_{-2} & V_{-3} \end{vmatrix}. \tag{17}$$

The proof can be given exactly as the proof of Theorem 2.1, so we omit it.
We can write Theorem 2.3 as

$$f(n) = (-1)^n u^n f(0)$$

where $f(n)=\begin{vmatrix} V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} \end{vmatrix}$ and $f(0)=\begin{vmatrix} V_3 & V_2 & V_1 & V_0 \\ V_2 & V_1 & V_0 & V_{-1} \\ V_1 & V_0 & V_{-1} & V_{-2} \\ V_0 & V_{-1} & V_{-2} & V_{-3} \end{vmatrix}$.

In the following Table 15, we present Simsons's formula of particular generalized Tetranacci sequences.

Table 15. Simsons's formula of some generalized Tetranacci sequences.

Sequence: V_n	Simson Formula
M_n	$f(n)=(-1)^{n-1}$
R_n	$f(n)=563(-1)^n$
J_n	$f(n)=0$
j_n	$f(n)=(-1)^n2^{n-2}3^5$

Next we consider generalized Pentanacci numbers $V_n=rV_{n-1}+sV_{n-2}+tV_{n-3}+uV_{n-4}+vV_{n-5}$ with 5 initial terms $V_0=c_0, V_1=c_1, V_2=c_2, V_3=c_3, V_4=c_4$.

Theorem 2.4 (Simson Formula of Generalized Pentanacci Numbers).
For all integers n we have

$$\begin{vmatrix} V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \end{vmatrix} = v^n \begin{vmatrix} V_4 & V_3 & V_2 & V_1 & V_0 \\ V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \end{vmatrix}. \tag{18}$$

The proof can be given exactly as the proof of Theorem 2.2, so we omit it.
We can write Theorem 2.4 as

$f(n)=v^n f(0)$

where $f(n)=\begin{vmatrix} V_{n+4} & V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+3} & V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} & V_{n-3} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} & V_{n-4} \end{vmatrix}$ and $f(0)=\begin{vmatrix} V_4 & V_3 & V_2 & V_1 & V_0 \\ V_3 & V_2 & V_1 & V_0 & V_{-1} \\ V_2 & V_1 & V_0 & V_{-1} & V_{-2} \\ V_1 & V_0 & V_{-1} & V_{-2} & V_{-3} \\ V_0 & V_{-1} & V_{-2} & V_{-3} & V_{-4} \end{vmatrix}$.

In the following Table 16, we present Simsons's formula of particular generalized Pentanacci sequences.

Table 16. Simsons's formula of some generalized Pentanacci sequences.

Sequence: V_n	Simson Formula
P_n	$f(n)=1$
Q_n	$f(n)=9584$
J_n	$f(n)=2^{n-2}\times 11$
j_n	$f(n)=2^{n-3}\times 3^4\times 19$

3. Main Result: Simson Formula of Generalized m-step Fibonacci Numbers

Now we consider the m-order linear recurrence relation

$$V_n=\sum_{i=1}^{m} r_iV_{n-i}=r_1V_{n-1}+r_2V_{n-2}+r_3V_{n-3}+...+r_mV_{n-m}.$$

For $m\geq 2$, we define f by

$$f(n)=\begin{vmatrix} V_{n+m-1} & V_{n+m-2} & V_{n+m-3} & \cdots & V_{n+2} & V_{n+1} & V_n \\ V_{n+m-2} & V_{n+m-3} & V_{n+m-4} & \cdots & V_{n+1} & V_n & V_{n-1} \\ V_{n+m-3} & V_{n+m-4} & V_{n+m-5} & \cdots & V_n & V_{n-1} & V_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{n+2} & V_{n+1} & V_n & \cdots & V_{n-m+5} & V_{n-m+4} & V_{n-m+3} \\ V_{n+1} & V_n & V_{n-1} & \cdots & V_{n-m+4} & V_{n-m+3} & V_{n-m+2} \\ V_n & V_{n-1} & V_{n-2} & \cdots & V_{n-m+3} & V_{n-m+2} & V_{n-m+1} \end{vmatrix}.$$

Note that

$$f(0)=\begin{vmatrix} V_{m-1} & V_{m-2} & V_{m-3} & \cdots & V_2 & V_1 & V_0 \\ V_{m-2} & V_{m-3} & V_{m-4} & \cdots & V_1 & V_0 & V_{-1} \\ V_{m-3} & V_{m-4} & V_{m-5} & \cdots & V_0 & V_{-1} & V_{-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_2 & V_1 & V_0 & \cdots & V_{m+5} & V_{-m+4} & V_{-m+3} \\ V_1 & V_0 & V_{-1} & \cdots & V_{m+4} & V_{-m+3} & V_{-m+2} \\ V_0 & V_{-1} & V_{-2} & \cdots & V_{m+3} & V_{-m+2} & V_{-m+1} \end{vmatrix}.$$

Motivated by the cases $m=2,3,4,5$, we are ready to present our main result for the arbitrary m.

Theorem 3.1 (Simson Formula of Generalized m-step Fibonacci Numbers).
Let $m\geq 2$. Then for all integers n we have

$$f(n)=y(n)r_m^n f(0) \tag{19}$$

where

$$y(n)=\begin{cases} 1 & , \ m \text{ odd} \\ (-1)^n & , \ m \text{ even} \end{cases}.$$

Proof. We prove the theorem by induction for $n\geq 0$, the proof of the case $n\leq -1$ being similar. As in the proof of the cases $m=2,3,4,5$ we need to consider m separately as odd and even. We provide the proof of the even cases. For $n=0$, it is obvious that the formula is true. Now, we assume that the formula (19) is true for $n=k$. Then we will complete the inductive step $n=k+1$ as follows: Note that

$$f(k+1)=\begin{vmatrix} V_{k+m} & V_{k+m-1} & V_{k+m-2} & \cdots & V_{k+3} & V_{k+2} & V_{k+1} \\ V_{k+m-1} & V_{k+m-2} & V_{k+m-3} & \cdots & V_{k+2} & V_{k+1} & V_k \\ V_{k+m-2} & V_{k+m-3} & V_{k+m-4} & \cdots & V_{k+1} & V_k & V_k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{k+3} & V_{k+2} & V_{k+1} & \cdots & V_{k-m+6} & V_{k-m+5} & V_{k-m+4} \\ V_{k+2} & V_{k+1} & V_k & \cdots & V_{k-m+5} & V_{k-m+4} & V_{k-m+3} \\ V_{k+1} & V_k & V_{k-1} & \cdots & V_{k-m+4} & V_{k-m+3} & V_{k-m+2} \end{vmatrix}.$$

Using the recurrence relations

$$\begin{aligned} V_{k+m} &= r_1V_{k+m-1}+r_2V_{k+m-2}+r_3V_{k+m-3}+...+r_mV_k \\ V_{k+m-1} &= r_1V_{k+m-2}+r_2V_{k+m-3}+r_3V_{k+m-4}+...+r_mV_{k-1} \\ V_{k+m-2} &= r_1V_{k+m-3}+r_2V_{k+m-4}+r_3V_{k+m-5}+...+r_mV_{k-2} \\ &\vdots \\ V_{k+3} &= r_1V_{k+2}+r_2V_{k+1}+r_3V_k+...+r_mV_{k-m+3} \\ V_{k+2} &= r_1V_{k+1}+r_2V_k+r_3V_{k-1}+...+r_mV_{k-m+2} \\ V_{k+1} &= r_1V_k+r_2V_{k-1}+r_3V_{k+1-3}+...+r_mV_{k-m+1} \end{aligned}$$

in the 1^{st} column of the determinant $f(k+1)$ and expanding 1^{st} column as $m-1$ additions and then after rearranging the determinant, we obtain

$$\begin{aligned} f(k+1) &= r_m\begin{vmatrix} V_k & V_{k+m-1} & V_{k+m-2} & \cdots & V_{k+3} & V_{k+2} & V_{k+1} \\ V_{k-1} & V_{k+m-2} & V_{k+m-3} & \cdots & V_{k+2} & V_{k+1} & V_k \\ V_{k-2} & V_{k+m-3} & V_{k+m-4} & \cdots & V_{k+1} & V_k & V_k \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{k-m+3} & V_{k+2} & V_{k+1} & \cdots & V_{k-m+6} & V_{k-m+5} & V_{k-m+4} \\ V_{k-m+2} & V_{k+1} & V_k & \cdots & V_{k-m+5} & V_{k-m+4} & V_{k-m+3} \\ V_{k+1-m} & V_k & V_{k-1} & \cdots & V_{k-m+4} & V_{k-m+3} & V_{k-m+2} \end{vmatrix} \\ &= -r_m\begin{vmatrix} V_{k+m-1} & V_{k+m-2} & V_{k+m-3} & \cdots & V_{k+2} & V_{k+1} & V_k \\ V_{k+m-2} & V_{k+m-3} & V_{k+m-4} & \cdots & V_{k+1} & V_k & V_{k-1} \\ V_{k+m-3} & V_{k+m-4} & V_{k+m-5} & \cdots & V_k & V_{k-1} & V_{k-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{k+2} & V_{k+1} & V_k & \cdots & V_{k-m+5} & V_{k-m+4} & V_{k-m+3} \\ V_{k+1} & V_k & V_{k-1} & \cdots & V_{k-m+4} & V_{k-m+3} & V_{k-m+2} \\ V_k & V_{k-1} & V_{k-2} & \cdots & V_{k-m+3} & V_{k-m+2} & V_{k-m+1} \end{vmatrix} \\ &= -r_m((-1)^k r_m^k f(0))=(-1)^{k+1}r_m^{k+1}f(0). \end{aligned}$$

This completes the inductive step and the proof of the theorem.

□

Remark 3.1.
Of course, this paper could be shortened. To calculate Simson Identity we needed sequences and the values of the elements of those sequences. But a search of the literature shows that it is not easy to find sequences of altogether the case $m = 2,3,4,5$ of the generalized m-step Fibonacci numbers in a single reference. So, as much as presenting new results, we wanted to fill this gap as well by giving the sequences and the values of their elements as tables.

4. Simson Formula of Gaussian Generalized m-step Fibonacci Numbers

For $m \geq 2$, the Gaussian generalized m-step Fibonacci numbers, $\{GV_n(GV_0, GV_1, GV_2, ..., GV_{m-1}; r_1, r_2, , ..., r_m)\}_{n\geq m}$ (or shortly $\{GV_n\}_{n\geq m}$), $(n \geq m)$, is defined by the m-order linear recurrence relation

$$GV_n = \sum_{i=1}^{m} r_i GV_{n-i} = r_1 GV_{n-1} + r_2 GV_{n-2} + r_3 GV_{n-3} + ... + r_{m-1} GV_{n-m-1} + r_m GV_{n-m} \tag{20}$$

with m initial terms

$$GV_0 = V_0 + iV_{-1},\ GV_1 = V_1 + iV_0,\ GV_2 = V_2 + iV_1, ..., GV_{m-1} = V_{m-1} + iV_{m-2},$$

where r_i, $1 \leq i \leq m$, are all real numbers and V_i, $0 \leq i \leq m-1$, are all real or complex numbers. Such a sequence is also called the Gaussian generalized Fibonacci m-sequence, or Gaussian generalized m-nacci sequence, or the Gaussian m-generalized Fibonacci sequence. For more information on Gaussian numbers, see [8].

The sequences $\{GV_n\}_{n\geq m}$ can be extended to negative subscripts by defining

$$GV_{-n} = -\frac{r_{m-1}}{r_m} GV_{-(n-1)} - \frac{r_{m-2}}{r_m} GV_{-(n-2)} - \frac{r_{m-3}}{r_m} GV_{-(n-3)} - ... - \frac{r_1}{r_m} GV_{-(n-(m-1))} + \frac{1}{r_m} GV_{-(n-m)}$$

for $n = m-2, m-1, m, m+1....$ Therefore, recurrence (20) holds for all integer n.

Note that for $n \geq 0$

$$GV_n = V_n + iV_{n-1} \tag{21}$$

and

$$GV_{-n} = V_{-n} + iV_{-n-1}.$$

For $m \geq 2$, we define h by

$$h(n) = \begin{vmatrix} GV_{n+m-1} & GV_{n+m-2} & GV_{n+m-3} & \cdots & GV_{n+2} & GV_{n+1} & GV_n \\ GV_{n+m-2} & GV_{n+m-3} & GV_{n+m-4} & \cdots & GV_{n+1} & GV_n & GV_{n-1} \\ GV_{n+m-3} & GV_{n+m-4} & GV_{n+m-5} & \cdots & GV_n & GV_{n-1} & GV_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ GV_{n+2} & GV_{n+1} & GV_n & \cdots & GV_{n-m+5} & GV_{n-m+4} & GV_{n-m+3} \\ GV_{n+1} & GV_n & GV_{n-1} & \cdots & GV_{n-m+4} & GV_{n-m+3} & GV_{n-m+2} \\ GV_n & GV_{n-1} & GV_{n-2} & \cdots & GV_{n-m+3} & GV_{n-m+2} & GV_{n-m+1} \end{vmatrix}.$$

Note that

$$h(0) = \begin{vmatrix} GV_{m-1} & GV_{m-2} & GV_{m-3} & \cdots & GV_2 & GV_1 & GV_0 \\ GV_{m-2} & GV_{m-3} & GV_{m-4} & \cdots & GV_1 & GV_0 & GV_{-1} \\ GV_{m-3} & GV_{m-4} & GV_{m-5} & \cdots & GV_0 & GV_{-1} & GV_{-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ GV_2 & GV_1 & GV_0 & \cdots & GV_{m+5} & GV_{-m+4} & GV_{-m+3} \\ GV_1 & GV_0 & GV_{-1} & \cdots & GV_{m+4} & GV_{-m+3} & GV_{-m+2} \\ GV_0 & GV_{-1} & GV_{-2} & \cdots & GV_{m+3} & GV_{-m+2} & GV_{-m+1} \end{vmatrix}.$$

Now we present Simson Formula for Gaussian generalized m-step Fibonacci numbers.

Theorem 4.1 (Simson Formula of Gaussian Generalized m-step Fibonacci Numbers).
Let $m \geq 2$. Then for all integers n we have

$$h(n) = z(n) r_m^n h(0) \tag{22}$$

where

$$z(n) = \begin{cases} 1 & , \ m \text{ odd} \\ (-1)^n & , \ m \text{ even} \end{cases}.$$

The proof is exactly as in the proof of Theorem 3.1.

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