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Matrix Sequences of Tetranacci and Tetranacci-Lucas Numbers

Research Article

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Abstract: In this paper, we define Tetranacci and Tetranacci-Lucas matrix sequences and investigate their properties.

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1. Introduction and Preliminaries

Recently, there have been so many studies of the sequences of numbers in the literature, see for example [5, 11]. On the other hand, the matrix sequences have taken so much interest for different type of numbers. For matrix sequences of generalized Horadam type numbers, see for example [2, 3, 6, 17–19, 21, 26], and for matrix sequences of generalized Tribonacci type numbers, see for instance [1, 14, 15, 22, 23].

In this paper, the matrix sequences of Tetranacci and Tetranacci-Lucas numbers will be defined for the first time in the literature. Then, by giving the generating functions, the Binet formulas, and summation formulas over these new matrix sequences, we will obtain some fundamental properties on Tetranacci and Tetranacci-Lucas numbers. Also, we will present the relationship between these matrix sequences.

First, we give some background about Tetranacci and Tetranacci-Lucas numbers.

Tetranacci sequence $\{M_n\}_{n\geq 0}$ (sequence A000078 in [13]) and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ (sequence A073817 in [13]) are defined by the fourth-order recurrence relations

$$M_n = M_{n-1} + M_{n-2} + M_{n-3} + M_{n-4}, \quad M_0 = 0, M_1 = 1, M_2 = 1, M_3 = 2 \tag{1}$$

and

$$R_n = R_{n-1} + R_{n-2} + R_{n-3} + R_{n-4}, \quad R_0 = 4, R_1 = 1, R_2 = 3, R_3 = 7 \tag{2}$$

respectively. The sequences $\{M_n\}_{n\geq 0}$ and $\{R_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$M_{-n} = -M_{-(n-1)} - M_{-(n-2)} - M_{-(n-3)} + M_{-(n-4)}$$

and

$$R_{-n} = -R_{-(n-1)} - R_{-(n-2)} - R_{-(n-3)} - R_{-(n-4)}$$

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Table 1. A few Tetranacci and Tetranacci-Lucas Numbers.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	...
M_n	0	1	1	2	4	8	15	29	56	108	208	401	773	1490	...
M_{-n}	0	0	0	1	−1	0	0	2	−3	1	0	4	−8	5	...
R_n	4	1	3	7	15	26	51	99	191	367	708	1365	2631	5071	...
R_{-n}	4	−1	−1	−1	7	−6	−1	−1	15	−19	4	−1	31	−53	...

for $n = 1,2,3,...$ respectively. Therefore, recurrences (1) and (2) hold for all integer n.This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [7, 9, 10, 12, 16, 24, 25].

Next, we present the first few values of the Tetranacci and Tetranacci-Lucas numbers with positive and negative subscripts in the following Table 1:

We can give some relations between $\{M_n\}$ and $\{R_n\}$ as

$$R_n = -M_{n+3} + 6M_{n+1} - M_n \tag{3}$$

and

$$R_n = -M_{n+2} + 5M_{n+1} - 2M_n - M_{n-1} \tag{4}$$

and also

$$R_n = 4M_{n+1} - 3M_n - 2M_{n-1} - M_{n-2} \tag{5}$$

and

$$R_n = M_n + 2M_{n-1} + 3M_{n-2} + 4M_{n-3} \tag{6}$$

Moreover, we have

$$563M_n = 86R_{n+3} - 61R_{n+2} - 71R_{n+1} - 87R_n \tag{7}$$

and

$$563M_n = 25R_{n+2} + 15R_{n+1} - R_n + 86R_{n-1} \tag{8}$$

and also

$$563M_n = 40R_{n+1} + 24R_n + 111R_{n-1} + 25R_{n-2} \tag{9}$$

Note that the last seven identities hold for all integers n.

It is well known that for all integers n, usual Tetranaci and Tetranacci-Lucas numbers can be expressed using Binet's formulas

$$M_n = \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{10}$$

(see for example [7] or [27])

or

$$M_n = \frac{\alpha-1}{5\alpha-8}\alpha^{n-1} + \frac{\beta-1}{5\beta-8}\beta^{n-1} + \frac{\gamma-1}{5\gamma-8}\gamma^{n-1} + \frac{\delta-1}{5\delta-8}\delta^{n-1} \tag{11}$$

(see for example [4])

and

$$R_n = \alpha^n + \beta^n + \gamma^n + \delta^n \tag{12}$$

respectively, where α, β, γ and δ are the roots of the equation $x^4 - x^3 - x^2 - x - 1 = 0$. Moreover,

$$\begin{aligned}
\alpha &= \frac{1}{4} + \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\
\beta &= \frac{1}{4} + \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 + \frac{13}{4}\omega^{-1}}, \\
\gamma &= \frac{1}{4} - \frac{1}{2}\omega + \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}}, \\
\delta &= \frac{1}{4} - \frac{1}{2}\omega - \frac{1}{2}\sqrt{\frac{11}{4} - \omega^2 - \frac{13}{4}\omega^{-1}},
\end{aligned}$$

where

$$\omega = \sqrt{\frac{11}{12} + \left(\frac{-65}{54} + \sqrt{\frac{563}{108}}\right)^{1/3} + \left(\frac{-65}{54} - \sqrt{\frac{563}{108}}\right)^{1/3}}.$$

Note that we have the following identities:

$$\alpha + \beta + \gamma + \delta = 1, \tag{13}$$

$$\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = -1, \tag{14}$$

$$\alpha\beta\gamma + \alpha\beta\delta + \alpha\gamma\delta + \beta\gamma\delta = 1, \tag{15}$$

$$\alpha\beta\gamma\delta = -1. \tag{16}$$

The generating functions for the Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are

$$\sum_{n=0}^{\infty} M_n x^n = \frac{x}{1-x-x^2-x^3-x^4} \text{ and } \sum_{n=0}^{\infty} R_n x^n = \frac{4-3x-2x^2-x^3}{1-x-x^2-x^3-x^4}. \tag{17}$$

Note that the Binet form of a sequence satisfying (1) and (2) for non-negative integers is valid for all integers n. This result of Howard and Saidak [8] is even true in the case of higher-order recurrence relations as the following theorem shows.

Theorem 1.1 ([8]).
Let $\{w_n\}$ be a sequence such that

$$\{w_n\} = a_1 w_{n-1} + a_2 w_{n-2} + \ldots + a_k w_{n-k}$$

for all integers n, with arbitrary initial conditions $w_0, w_1, \ldots, w_{k-1}$. Assume that each a_i and the initial conditions are complex numbers. Write

$$\begin{aligned} f(x) &= x^k - a_1 x^{k-1} - a_2 x^{k-2} - \ldots - a_{k-1} x - a_k \\ &= (x-\alpha_1)^{d_1}(x-\alpha_2)^{d_2}\ldots(x-\alpha_h)^{d_h} \end{aligned} \tag{18}$$

with $d_1 + d_2 + \ldots + d_h = k$, and $\alpha_1, \alpha_2, \ldots, \alpha_k$ distinct. Then

(a) *For all n,*

$$w_n = \sum_{m=1}^{k} N(n,m)(\alpha_m)^n \tag{19}$$

where

$$N(n,m) = A_1^{(m)} + A_2^{(m)} n + \ldots + A_{r_m}^{(m)} n^{r_m - 1} = \sum_{u=0}^{r_m - 1} A_{u+1}^{(m)} n^u$$

with each $A_i^{(m)}$ a constant determined by the initial conditions for $\{w_n\}$. Here, equation (19) is called the Binet form (or Binet formula) for $\{w_n\}$. We assume that $f(0) \neq 0$ so that $\{w_n\}$ can be extended to negative integers n.

If the zeros of (18) are distinct, as they are in our examples, then

$$w_n = A_1(\alpha_1)^n + A_2(\alpha_2)^n + \ldots + A_k(\alpha_k)^n.$$

(b) *The Binet form for $\{w_n\}$ is valid for all integers n.*

2. The Matrix Sequences of Tetranacci and Tetranacci-Lucas Numbers

In this section we define Tetranacci and Tetranacci-Lucas matrix sequences and investgate their properties.

Definition 2.1.
For any integer $n \geq 0$, the Tetranacci matrix $(\mathcal{T}_n)$ and Tetranacci-Lucas matrix $(\mathcal{R}_n)$ are defined by

$$\mathcal{M}_n = \mathcal{M}_{n-1} + \mathcal{M}_{n-2} + \mathcal{M}_{n-3} + \mathcal{M}_{n-4}, \quad (20)$$
$$\mathcal{R}_n = \mathcal{R}_{n-1} + \mathcal{R}_{n-2} + \mathcal{R}_{n-3} + \mathcal{R}_{n-4}, \quad (21)$$

respectively, with initial conditions

$$\mathcal{M}_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \mathcal{M}_1 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \mathcal{M}_2 = \begin{pmatrix} 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \mathcal{M}_3 = \begin{pmatrix} 4 & 4 & 3 & 2 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

and

$$\mathcal{R}_0 = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & -3 & -2 & -1 \\ -1 & 5 & -2 & -1 \\ -1 & 0 & 6 & -1 \end{pmatrix}, \mathcal{R}_1 = \begin{pmatrix} 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & -3 & -2 & -1 \\ -1 & 5 & -2 & -1 \end{pmatrix},$$
$$\mathcal{R}_2 = \begin{pmatrix} 7 & 8 & 4 & 3 \\ 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 \\ 4 & -3 & -2 & -1 \end{pmatrix}, \mathcal{R}_3 = \begin{pmatrix} 15 & 11 & 10 & 7 \\ 7 & 8 & 4 & 3 \\ 3 & 4 & 5 & 1 \\ 1 & 2 & 3 & 4 \end{pmatrix}.$$

The sequences $\{\mathcal{M}_n\}_{n\geq 0}$ and $\{\mathcal{R}_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{M}_{-n} = -\mathcal{M}_{-(n-1)} - \mathcal{M}_{-(n-2)} - \mathcal{M}_{-(n-3)} + \mathcal{M}_{-(n-4)}$$

and

$$\mathcal{R}_{-n} = -\mathcal{R}_{-(n-1)} - \mathcal{R}_{-(n-2)} - \mathcal{R}_{-(n-3)} + \mathcal{R}_{-(n-4)}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (20) and (21) hold for all integers n.

The following theorem gives the nth general terms of the Tetranacci and Tetranacci-Lucas matrix sequences.

Theorem 2.1.
For any integer $n \geq 0$, we have the following formulas of the matrix sequences:

$$\mathcal{M}_n = \begin{pmatrix} M_{n+1} & M_n + M_{n-1} + M_{n-2} & M_n + M_{n-1} & M_n \\ M_n & M_{n-1} + M_{n-2} + M_{n-3} & M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-2} + M_{n-3} + M_{n-4} & M_{n-2} + M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-3} + M_{n-4} + M_{n-5} & M_{n-3} + M_{n-4} & M_{n-3} \end{pmatrix} \quad (22)$$

$$\mathcal{R}_n = \begin{pmatrix} R_{n+1} & R_n + R_{n-1} + R_{n-2} & R_n + R_{n-1} & R_n \\ R_n & R_{n-1} + R_{n-2} + R_{n-3} & R_{n-1} + R_{n-2} & R_{n-1} \\ R_{n-1} & R_{n-2} + R_{n-3} + R_{n-4} & R_{n-2} + R_{n-3} & R_{n-2} \\ R_{n-2} & R_{n-3} + R_{n-4} + R_{n-5} & R_{n-3} + R_{n-4} & R_{n-3} \end{pmatrix}. \quad (23)$$

Proof. We prove (22) by strong mathematical induction on n. (23) can be proved similarly.

If $n = 0$ then we have

$$\mathcal{M}_0 = \begin{pmatrix} M_1 & M_0 + M_{-1} + M_{-2} & M_0 + M_{-1} & M_0 \\ M_0 & M_{-1} + M_{-2} + M_{-3} & M_{-1} + M_{-2} & M_{-1} \\ M_{-1} & M_{-2} + M_{-3} + M_{-4} & M_{-2} + M_{-3} & M_{-2} \\ M_{-2} & M_{-3} + M_{-4} + M_{-5} & M_{-3} + M_{-4} & M_{-3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

which is true and

$$\mathcal{M}_1 = \begin{pmatrix} M_2 & M_1 + M_0 + M_{-1} & M_1 + M_0 & M_1 \\ M_1 & M_0 + M_{-1} + M_{-2} & M_0 + M_{-1} & M_0 \\ M_0 & M_{-1} + M_{-2} + M_{-3} & M_{-1} + M_{-2} & M_{-1} \\ M_{-1} & M_{-2} + M_{-3} + M_{-4} & M_{-2} + M_{-3} & M_{-2} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

which is true. Assume that the equality holds for $n \leq k$. For $n = k+1$, using the fourth-order recurrence relations (1), we have

$$\begin{aligned}\mathcal{M}_{k+1} &= \mathcal{M}_k + \mathcal{M}_{k-1} + \mathcal{M}_{k-2} + \mathcal{M}_{k-3} \\ &= \begin{pmatrix} M_{k+2} & M_{k-1}+M_{k+1}+M_k & M_{k+1}+M_k & M_{k+1} \\ M_{k+1} & M_{k-1}+M_{k-2}+M_k & M_{k-1}+M_k & M_k \\ M_k & M_{k-1}+M_{k-2}+M_{k-3} & M_{k-1}+M_{k-2} & M_{k-1} \\ M_{k-1} & M_{k-2}+M_{k-3}+M_{k-4} & M_{k-2}+M_{k-3} & M_{k-2} \end{pmatrix}\end{aligned}$$

Thus, by strong induction on n, this proves (22).

We now give the Binet formulas for the Tetranacci and Tetranacci-Lucas matrix sequences.

Theorem 2.2.
For every integer n, the Binet formulas of the Tetranacci and Tetranacci-Lucas matrix sequences are given by

$$\mathcal{M}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n + D_1\delta^n, \tag{24}$$
$$\mathcal{R}_n = A_2\alpha^n + B_2\beta^n + C_2\gamma^n + D_2\delta^n. \tag{25}$$

where

$$\begin{aligned}
A_1 &= \frac{\mathcal{M}_0 + \alpha^{-1}(\alpha+1)\mathcal{M}_1 + \alpha(\alpha-1)\mathcal{M}_2 + \alpha\mathcal{M}_3}{\alpha\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}, B_1 = \frac{\mathcal{M}_0 + \beta^{-1}(\beta+1)\mathcal{M}_1 + \beta(\beta-1)\mathcal{M}_2 + \beta\mathcal{M}_3}{\beta\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}, \\
C_1 &= \frac{\mathcal{M}_0 + \gamma^{-1}(\gamma+1)\mathcal{M}_1 + \gamma(\gamma-1)\mathcal{M}_2 + \gamma\mathcal{M}_3}{\gamma\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}, D_1 = \frac{\mathcal{M}_0 + \delta^{-1}(\delta+1)\mathcal{M}_1 + \delta(\delta-1)\mathcal{M}_2 + \delta\mathcal{M}_3}{\delta\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)} \\
A_2 &= \frac{\mathcal{R}_0 + \alpha^{-1}(\alpha+1)\mathcal{R}_1 + \alpha(\alpha-1)\mathcal{R}_2 + \alpha\mathcal{R}_3}{\alpha\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}, B_2 = \frac{\mathcal{R}_0 + \beta^{-1}(\beta+1)\mathcal{R}_1 + \beta(\beta-1)\mathcal{R}_2 + \beta\mathcal{R}_3}{\beta\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}, \\
C_2 &= \frac{\mathcal{R}_0 + \gamma^{-1}(\gamma+1)\mathcal{R}_1 + \gamma(\gamma-1)\mathcal{R}_2 + \gamma\mathcal{R}_3}{\gamma\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}, D_2 = \frac{\mathcal{R}_0 + \delta^{-1}(\delta+1)\mathcal{R}_1 + \delta(\delta-1)\mathcal{R}_2 + \delta\mathcal{R}_3}{\delta\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}.
\end{aligned}$$

Proof. We prove the theorem only for $n \geq 0$ because of Theorem 1.1. We prove (24). By the assumption, the characteristic equation of (20) is $x^4 - x^3 - x^2 - x - 1 = 0$ and the roots of it are α, β, γ and δ. So it's general solution is given by

$$\mathcal{M}_n = A_1\alpha^n + B_1\beta^n + C_1\gamma^n + D_1\delta^n.$$

Using initial condition which is given in Definition 2.1, and also applying lineer algebra operations, we obtain the matrices A_1, B_1, C_1, D_1 as desired. This gives the formula for $\mathcal{M}_n$.

Similarly we have the formula (25).

The well known Binet formulas for Tetranacci and Tetranacci-Lucas numbers are given in (10) and (12) respectively. But, we will obtain these functions in terms of Tetranacci and Tetranacci-Lucas matrix sequences as a consequence of Theorem 2.1 and Theorem 2.2. To do this, we will give the formulas for these numbers by means of the related matrix sequences. In fact, in the proof of next corollary, we will just compare the linear combination of the 2nd row and 1st column entries of the matrices.

Corollary 2.1.
For every integers n, the Binet's formulas for Tetranacci and Tetranacci-Lucas numbers are given as

$$\begin{aligned}
M_n &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\
R_n &= \alpha^n + \beta^n + \gamma^n + \delta^n.
\end{aligned}$$

Proof. From Theorem 2.2, we have

$$\begin{aligned}
\mathcal{M}_n &= A_1\alpha^n + B_1\beta^n + C_1\gamma^n + D_1\delta^n \\
&= \frac{\mathcal{M}_0 + \alpha^{-1}(\alpha+1)\mathcal{M}_1 + \alpha(\alpha-1)\mathcal{M}_2 + \alpha\mathcal{M}_3}{\alpha\left(\alpha-\beta\right)\left(\alpha-\gamma\right)(\alpha-\delta)}\alpha^n + \frac{\mathcal{M}_0 + \beta^{-1}(\beta+1)\mathcal{M}_1 + \beta(\beta-1)\mathcal{M}_2 + \beta\mathcal{M}_3}{\beta(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^n \\
&\quad + \frac{\mathcal{M}_0 + \gamma^{-1}(\gamma+1)\mathcal{M}_1 + \gamma(\gamma-1)\mathcal{M}_2 + \gamma\mathcal{M}_3}{\gamma(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^n + \frac{\mathcal{M}_0 + \delta^{-1}(\delta+1)\mathcal{M}_1 + \delta(\delta-1)\mathcal{M}_2 + \delta\mathcal{M}_3}{\delta(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^n
\end{aligned}$$

By Theorem 2.1, we know that

$$\mathcal{M}_n=\begin{pmatrix} M_{n+1} & M_n+M_{n-1}+M_{n-2} & M_n+M_{n-1} & M_n \\ M_n & M_{n-1}+M_{n-2}+M_{n-3} & M_{n-1}+M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-2}+M_{n-3}+M_{n-4} & M_{n-2}+M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-3}+M_{n-4}+M_{n-5} & M_{n-3}+M_{n-4} & M_{n-3} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above two equations, then we obtain

$$\begin{aligned}\mathcal{M}_n &= \frac{(2\alpha+\frac{1}{\alpha}(\alpha+1)+\alpha(\alpha-1))}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^{n-1}+\frac{(2\beta+\frac{1}{\beta}(\beta+1)+\beta(\beta-1))}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}\beta^{n-1} \\ &\quad +\frac{(2\gamma+\frac{1}{\gamma}(\gamma+1)+\gamma(\gamma-1))}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}\gamma^{n-1}+\frac{(2\delta+\frac{1}{\delta}(\delta+1)+\delta(\delta-1))}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}\delta^{n-1} \\ &= \frac{\alpha^{n+2}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^{n+2}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} \\ &\quad +\frac{\gamma^{n+2}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^{n+2}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}.\end{aligned}$$

From Theorem 2.2, we obtain

$$\begin{aligned}\mathcal{R}_n &= A_2\alpha^n+B_2\beta^n+C_2\gamma^n++D_2\delta^n \\ &= \frac{\mathcal{R}_0+\alpha^{-1}(\alpha+1)\mathcal{R}_1+\alpha(\alpha-1)\mathcal{R}_2+\alpha\mathcal{R}_3}{\alpha(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n+\frac{\mathcal{R}_0+\beta^{-1}(\beta+1)\mathcal{R}_1+\beta(\beta-1)\mathcal{R}_2+\beta\mathcal{R}_3}{\beta(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\beta^n \\ &\quad +\frac{\mathcal{R}_0+\gamma^{-1}(\gamma+1)\mathcal{R}_1+\gamma(\gamma-1)\mathcal{R}_2+\gamma\mathcal{R}_3}{\gamma(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\gamma^n+\frac{\mathcal{R}_0+\delta^{-1}(\delta+1)\mathcal{R}_1+\delta(\delta-1)\mathcal{R}_2+\delta\mathcal{R}_3}{\delta(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\delta^n\end{aligned}$$

By Theorem 2.1, we know that

$$\mathcal{R}_n=\begin{pmatrix} R_{n+1} & R_n+R_{n-1}+R_{n-2} & R_n+R_{n-1} & R_n \\ R_n & R_{n-1}+R_{n-2}+R_{n-3} & R_{n-1}+R_{n-2} & R_{n-1} \\ R_{n-1} & R_{n-2}+R_{n-3}+R_{n-4} & R_{n-2}+R_{n-3} & R_{n-2} \\ R_{n-2} & R_{n-3}+R_{n-4}+R_{n-5} & R_{n-3}+R_{n-4} & R_{n-3} \end{pmatrix}.$$

Now, if we compare the 2nd row and 1st column entries with the matrices in the above last two equations, then we obtain

$$\begin{aligned}R_n &= \frac{(7\alpha+\frac{1}{\alpha}(\alpha+1)+3\alpha(\alpha-1)+4)}{\alpha(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}\alpha^n+\frac{(7\beta+\frac{1}{\beta}(\beta+1)+3\beta(\beta-1)+4)}{\beta(\beta-\beta)(\beta-\gamma)(\beta-\delta)}\beta^n \\ &\quad +\frac{(7\gamma+\frac{1}{\gamma}(\gamma+1)+3\gamma(\gamma-1)+4)}{\gamma(\gamma-\beta)(\gamma-\gamma)(\gamma-\delta)}\gamma^n+\frac{(7\delta+\frac{1}{\delta}(\delta+1)+3\delta(\delta-1)+4)}{\delta(\delta-\beta)(\delta-\gamma)(\delta-\delta)}\delta^n.\end{aligned}$$

Using the relations (13)-(16) and considering α,β,γ and δ are the roots the equation $x^4-x^3-x^2-x-1=0$, we obtain

$$\begin{aligned}\frac{(7\alpha+\frac{1}{\alpha}(\alpha+1)+3\alpha(\alpha-1)+4)}{\alpha(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} &= 1 \\ \frac{(7\beta+\frac{1}{\beta}(\beta+1)+3\beta(\beta-1)+4)}{\beta(\beta-\beta)(\beta-\gamma)(\beta-\delta)} &= 1 \\ \frac{(7\gamma+\frac{1}{\gamma}(\gamma+1)+3\gamma(\gamma-1)+4)}{\gamma(\gamma-\beta)(\gamma-\gamma)(\gamma-\delta)} &= 1 \\ \frac{(7\delta+\frac{1}{\delta}(\delta+1)+3\delta(\delta-1)+4)}{\delta(\delta-\beta)(\delta-\gamma)(\delta-\delta)} &= 1.\end{aligned}$$

So finally we conclude that

$$R_n=\alpha^n+\beta^n+\gamma^n+\delta^n$$

as required.

Now, we present summation formulas for Tetranacci and Tetranacci-Lucas matrix sequences.

Theorem 2.3.
For $m \geq j \geq 0$, we have

$$\sum_{i=0}^{n-1} \mathscr{M}_{mi+j} = \frac{\begin{array}{c} -2(\mathscr{M}_{mn+2m+j} + \mathscr{M}_{mn+m+j} - \mathscr{M}_{2m+j} - \mathscr{M}_{m+j}) + 2R_m(\mathscr{M}_{mn+m+j} - \mathscr{M}_{m+j}) \\ +2(-1)^m(\mathscr{M}_{mn-m+j} - \mathscr{M}_{-m+j}) + (R_{2m} + 2R_m - R_m^2 - 2)(\mathscr{M}_{mn+j} - \mathscr{M}_j) \end{array}}{(R_m^2 - R_{2m} - 2R_m + 2 + 2(-1)^m(1 - R_{-m}))} \tag{26}$$

and

$$\sum_{i=0}^{n-1} \mathscr{R}_{mi+j} = \frac{\begin{array}{c} -2(\mathscr{R}_{mn+2m+j} + \mathscr{R}_{mn+m+j} - \mathscr{R}_{2m+j} - \mathscr{R}_{m+j}) + 2R_m(\mathscr{R}_{mn+m+j} - \mathscr{R}_{m+j}) \\ +2(-1)^m(\mathscr{R}_{mn-m+j} - \mathscr{R}_{-m+j}) + (R_{2m} + 2R_m - R_m^2 - 2)(\mathscr{R}_{mn+j} - \mathscr{R}_j) \end{array}}{(R_m^2 - R_{2m} - 2R_m + 2 + 2(-1)^m(1 - R_{-m}))} \tag{27}$$

Proof. Note that,

$$\begin{aligned} \sum_{i=0}^{n-1} \mathscr{M}_{mi+j} &= \sum_{i=0}^{n-1} (A_1\alpha^{mi+j} + B_1\beta^{mi+j} + C_1\gamma^{mi+j} + D_1\delta^{mi+j}) \\ &= A_1\alpha^j(\frac{\alpha^{mn}-1}{\alpha^m-1}) + B_1\beta^j(\frac{\beta^{mn}-1}{\beta^m-1}) + C_1\gamma^j(\frac{\gamma^{mn}-1}{\gamma^m-1}) + D_1\delta^j(\frac{\delta^{mn}-1}{\delta^m-1}) \end{aligned}$$

and

$$\begin{aligned} \sum_{i=0}^{n-1} \mathscr{R}_{mi+j} &= \sum_{i=0}^{n-1} (A_2\alpha^{mi+j} + B_2\beta^{mi+j} + C_2\gamma^{mi+j} + D_2\delta^{mi+j}) \\ &= A_2\alpha^j(\frac{\alpha^{mn}-1}{\alpha^m-1}) + B_2\beta^j(\frac{\beta^{mn}-1}{\beta^m-1}) + C_{21}\gamma^j(\frac{\gamma^{mn}-1}{\gamma^m-1}) + D_2\delta^j(\frac{\delta^{mn}-1}{\delta^m-1}) \end{aligned}$$

Simplifying the last equalities in the last two expression imply (26) and (27) as required.

As in Corollary 2.1, in the proof of next Corollary, we just compare the linear combination of the 2nd row and 1st column entries of the relevant matrices.

Corollary 2.2.
For $m \geq j \geq 0$, we have

$$\sum_{i=0}^{n-1} M_{mi+j} = \frac{\begin{array}{c} -2(M_{mn+2m+j} + M_{mn+m+j} - M_{2m+j} - M_{m+j}) + 2R_m(M_{mn+m+j} - M_{m+j}) \\ +2(-1)^m(M_{mn-m+j} - M_{-m+j}) + (R_{2m} + 2R_m - R_m^2 - 2)(M_{mn+j} - M_j) \end{array}}{(R_m^2 - R_{2m} - 2R_m + 2 + 2(-1)^m(1 - R_{-m}))} \tag{28}$$

and

$$\sum_{i=0}^{n-1} R_{mi+j} = \frac{\begin{array}{c} -2(R_{mn+2m+j} + R_{mn+m+j} - R_{2m+j} - R_{m+j}) + 2R_m(R_{mn+m+j} - R_{m+j}) \\ +2(-1)^m(R_{mn-m+j} - R_{-m+j}) + (R_{2m} + 2R_m - R_m^2 - 2)(R_{mn+j} - R_j) \end{array}}{(R_m^2 - R_{2m} - 2R_m + 2 + 2(-1)^m(1 - R_{-m}))} \tag{29}$$

Note that using the above Corollary we obtain the following well known formulas (see, for example, [25]]) (taking $m = 1, j = 0$):

$$\sum_{i=0}^{n-1} M_i = \frac{M_{n+2} - M_n + M_{n-1} - 1}{3} \quad \text{and} \quad \sum_{i=0}^{n-1} R_i = \frac{R_{n+2} - R_n + R_{n-1} + 2}{3}.$$

We can write last two formulas as

$$\sum_{i=0}^{n} M_i = \frac{M_{n+2} + 2M_n + M_{n-1} - 1}{3} \quad \text{and} \quad \sum_{i=0}^{n} R_i = \frac{R_{n+2} + 2R_n + R_{n-1} + 2}{3}.$$

We now give generating functions of $\mathscr{M}$ and $\mathscr{R}$.

Theorem 2.4.
The generating function for the Tetranacci and Tetranacci-Lucas matrix sequences are given as

$$\sum_{n=0}^{\infty}\mathcal{M}_n x^n = \frac{\begin{pmatrix} 1 & x^3+x^2+x & x^2+x & x \\ x & 1-x & x^3+x^2 & x^2 \\ x^2 & x-x^2 & -x^2-x+1 & x^3 \\ x^3 & x^2-x^3 & -x^3-x^2+x & -x^3-x^2-x+1 \end{pmatrix}}{1-x-x^2-x^3-x^4}$$

and

$$\sum_{n=0}^{\infty}\mathcal{R}_n x^n = \frac{\begin{pmatrix} 4x^3+3x^2+2x+1 & -3x^3+2x^2+2x+2 & -2x^3-4x^2+2x+3 & -x^3-2x^2-3x+4 \\ -x^3-2x^2-3x+4 & 5x^3+5x^2+5x-3 & -2x^3+4x^2+5x-2 & -x^3-2x^2+5x-1 \\ -x^3-2x^2+5x-1 & 5-8x & 6x^3+7x^2-2 & -x^3+6x^2-1 \\ -x^3+6x^2-1 & 5x-8x^2 & x^3-6x^2-8x+6 & 7x^3+x^2-1 \end{pmatrix}}{1-x-x^2-x^3-x^4}$$

respectively.

Proof. We prove the Tetranacci case. Suppose that $g(x)=\sum_{n=0}^{\infty}\mathcal{M}_n x^n$ is the generating function for the sequence $\{\mathcal{M}_n\}_{n\geq 0}$. Then, using Definition 2.1, and substracting $xf(x)$, $x^2f(x)$, $x^3f(x)$ and $x^4f(x)$ from $f(x)$ we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
&(1-x-x^2-x^3-x^4)g(x)\\
=\;& \sum_{n=0}^{\infty}\mathcal{M}_n x^n - x\sum_{n=0}^{\infty}\mathcal{M}_n x^n - x^2\sum_{n=0}^{\infty}\mathcal{M}_n x^n - x^3\sum_{n=0}^{\infty}\mathcal{M}_n x^n - x^4\sum_{n=0}^{\infty}\mathcal{M}_n x^n\\
=\;& \sum_{n=0}^{\infty}\mathcal{M}_n x^n - \sum_{n=0}^{\infty}\mathcal{M}_n x^{n+1} - \sum_{n=0}^{\infty}\mathcal{M}_n x^{n+2} - \sum_{n=0}^{\infty}\mathcal{M}_n x^{n+3} - \sum_{n=0}^{\infty}\mathcal{M}_n x^{n+4}\\
=\;& \sum_{n=0}^{\infty}\mathcal{M}_n x^n - \sum_{n=1}^{\infty}\mathcal{M}_{n-1} x^{n} - \sum_{n=2}^{\infty}\mathcal{M}_{n-2} x^{n} - \sum_{n=3}^{\infty}\mathcal{M}_{n-3} x^{n} - \sum_{n=4}^{\infty}\mathcal{M}_{n-4} x^{n}\\
=\;& (\mathcal{M}_0+\mathcal{M}_1x+\mathcal{M}_2x^2+\mathcal{M}_3x^3)-(\mathcal{M}_0x+\mathcal{M}_1x^2+\mathcal{M}_2x^3)-(\mathcal{M}_0x^2+\mathcal{M}_1x^3)-\mathcal{M}_0x^3\\
&+\sum_{n=4}^{\infty}(\mathcal{M}_n-\mathcal{M}_{n-1}-\mathcal{M}_{n-2}-\mathcal{M}_{n-3}-\mathcal{M}_{n-4})x^n\\
=\;& \mathcal{M}_0+(\mathcal{M}_1-\mathcal{M}_0)x+(\mathcal{M}_2-\mathcal{M}_1-\mathcal{M}_0)x^2+(\mathcal{M}_3-\mathcal{M}_2-\mathcal{M}_1-\mathcal{M}_0)x^3
\end{aligned}$$

Rearranging above equation, we get

$$g(x)=\frac{\mathcal{M}_0+(\mathcal{M}_1-\mathcal{M}_0)x+(\mathcal{M}_2-\mathcal{M}_1-\mathcal{M}_0)x^2+(\mathcal{M}_3-\mathcal{M}_2-\mathcal{M}_1-\mathcal{M}_0)x^3}{1-x-x^2-x^3-x^4}.$$

This completes the proof.

Tetranacci-Lucas case can be proved similarly.

The well known generating functions for Tetranacci and Tetranacci-Lucas numbers are as in (17). However, we will obtain these functions in terms of Tetranacci and Tetranacci-Lucas matrix sequences as a consequence of Theorem 2.4. To do this, we will again compare the the 2nd row and 1st column entries with the matrices in Theorem 2.4. Thus we have the following corollary.

Corollary 2.3.
The generating functions for the Tetranacci sequence $\{M_n\}_{n\geq 0}$ and Tetranacci-Lucas sequence $\{R_n\}_{n\geq 0}$ are given as

$$\sum_{n=0}^{\infty}M_n x^n = \frac{x}{1-x-x^2-x^3-x^4} \quad \text{and} \quad \sum_{n=0}^{\infty}R_n x^n = \frac{4-3x-2x^2-x^3}{1-x-x^2-x^3-x^4}. \tag{30}$$

respectively.

3. Relation Between Tetranacci and Tetranacci-Lucas Matrix Sequences

The following theorem shows that there always exist interrelation between Tetranacci and Tetranacci-Lucas matrix sequences.

Theorem 3.1.
For the matrix sequences $\{\mathscr{M}_n\}$ and $\{\mathscr{R}_n\}$, we have the following identities.

(a) $\mathscr{R}_n = -\mathscr{M}_{n+3} + 6\mathscr{M}_{n+1} - \mathscr{M}_n$,

(b) $\mathscr{R}_n = -\mathscr{M}_{n+2} + 5\mathscr{M}_{n+1} - 2\mathscr{M}_n - \mathscr{M}_{n-1}$,

(c) $\mathscr{R}_n = 4\mathscr{M}_{n+1} - 3\mathscr{M}_n - 2\mathscr{M}_{n-1} - \mathscr{M}_{n-2}$,

(d) $\mathscr{R}_n = \mathscr{M}_n + 2\mathscr{M}_{n-1} + 3\mathscr{M}_{n-2} + 4\mathscr{M}_{n-3}$

(e) $563\mathscr{M}_n = 86\mathscr{R}_{n+3} - 61\mathscr{R}_{n+2} - 71\mathscr{R}_{n+1} - 87\mathscr{R}_n$,

(f) $563\mathscr{M}_n = 25\mathscr{R}_{n+2} + 15\mathscr{R}_{n+1} - R_n + 86\mathscr{R}_{n-1}$,

(g) $563\mathscr{M}_n = 40\mathscr{R}_{n+1} + 24\mathscr{R}_n + 111\mathscr{R}_{n-1} + 25\mathscr{R}_{n-2}$.

Proof. From (3), (4), (5) and (6), (a), (b), (c) and (d) follow. From (7), (8) and (9), (e), (f) and (g) follow.

Lemma 3.1.
For all non-negative integers m and n, we have the following identities.

(a) $\mathscr{R}_0\mathscr{M}_n = \mathscr{M}_n\mathscr{R}_0 = \mathscr{R}_n$,

(b) $\mathscr{M}_0\mathscr{R}_n = \mathscr{R}_n\mathscr{M}_0 = \mathscr{R}_n$.

Proof. Identities can be established easily. Note that to show (a) we need to use all the relations (3), (4), (5) and (6). In the next Corollary, we obtain (9) using Lemma 3.1 and then (b) follows from (a).

Corollary 3.1.
We have the following identities.

(a) $M_n = \frac{1}{563}(40R_{n+1} + 24R_n + 111R_{n-1} + 25R_{n-2})$,

(b) $\mathscr{M}_n = \frac{1}{563}(40\mathscr{R}_{n+1} + 24\mathscr{R}_n + 111\mathscr{R}_{n-1} + 25\mathscr{R}_{n-2})$.

Proof. From Lemma 3.1 (a), we know that $\mathscr{R}_0\mathscr{M}_n = \mathscr{M}_n\mathscr{R}_0$. To show (a), use Theorem 2.1 for the matrix $\mathscr{M}_n$ and calculate the matrix operation $\mathscr{R}_0^{-1}\mathscr{R}_n$ and then compare the 2nd row and 1st column entries with the matrices $\mathscr{M}_n$ and $\mathscr{R}_0^{-1}\mathscr{R}_n$. Now (b) follows from (a).

To prove the following Theorem we need the next Lemma.

Lemma 3.2.
Let $A_1, B_1, C_1, D_1; A_2, B_2, C_2, D_2$ as in Theorem 2.2. Then the following relations hold:

$$\begin{aligned} A_1^2 &= A_1,\ B_1^2 = B_1,\ C_1^2 = C_1,\ D_1^2 = D_1, \\ A_1B_1 &= B_1A_1 = A_1C_1 = C_1A_1 = A_1D_1 = D_1A_1 = C_1B_1 = B_1C_1 = D_1B_1 = B_1D_1 = C_1D_1 = D_1C_1 = (0), \\ A_2B_2 &= B_2A_2 = A_2C_2 = C_2A_2 = A_2D_2 = D_2A_2 = C_2B_2 = B_2C_2 = D_2B_2 = B_2D_2 = C_2D_2 = D_2C_2 = (0). \end{aligned}$$

Proof. Using the relations (13)-(16), required equalities can be established by matrix calculations.

Theorem 3.2.
For all non-negative integers m and n, we have the following identities.

(a) $\mathscr{M}_m\mathscr{M}_n = \mathscr{M}_{m+n} = \mathscr{M}_n\mathscr{M}_m$,

(b) $\mathscr{M}_m\mathscr{R}_n = \mathscr{R}_n\mathscr{M}_m = \mathscr{R}_{m+n}$,

(c) $\mathscr{R}_m\mathscr{R}_n = \mathscr{R}_n\mathscr{R}_m = \mathscr{M}_{m+n+6} - 12\mathscr{M}_{m+n+4} + 2\mathscr{M}_{m+n+3} + 36\mathscr{M}_{m+n+2} - 12\mathscr{M}_{m+n+1} + \mathscr{M}_{m+n}$,

(d) $\mathscr{R}_m\mathscr{R}_n = \mathscr{R}_n\mathscr{R}_m = \mathscr{M}_{m+n+4} - 10\mathscr{M}_{m+n+3} + 29\mathscr{M}_{m+n+2} - 18\mathscr{M}_{m+n+1} - 6\mathscr{M}_{m+n} + 4\mathscr{M}_{m+n-1} + \mathscr{M}_{m+n-2}$,

(e) $\mathscr{R}_m\mathscr{R}_n = \mathscr{R}_n\mathscr{R}_m = 16\mathscr{M}_{m+n+2} - 24\mathscr{M}_{m+n+1} - 7\mathscr{M}_{m+n} + 4\mathscr{M}_{m+n-1} + 10\mathscr{M}_{m+n-2} + 4\mathscr{M}_{m+n-3} + \mathscr{M}_{m+n-4}$

(f) $\mathscr{R}_m\mathscr{R}_n = \mathscr{R}_n\mathscr{R}_m = \mathscr{M}_{m+n} + 4\mathscr{M}_{m+n-1} + 10\mathscr{M}_{m+n-2} + 20\mathscr{M}_{m+n-3} + 25\mathscr{M}_{m+n-4} + 24\mathscr{M}_{m+n-5} + 16\mathscr{M}_{m+n-6}$.

Proof.

(a) Using Lemma 3.2 we obtain

$$\begin{aligned}
\mathscr{M}_m\mathscr{M}_n &= (A_1\alpha^m + B_1\beta^m + C_1\gamma^m + D_1\delta^m)(A_1\alpha^n + B_1\beta^n + C_1\gamma^n + D_1\delta^n) \\
&= A_1^2\alpha^{m+n} + B_1^2\beta^{m+n} + C_1^2\gamma^{m+n} + D_1^2\delta^{m+n} \\
&\quad + A_1B_1\alpha^m\beta^n + A_1C_1\alpha^m\gamma^n + A_1D_1\alpha^m\delta^n \\
&\quad + B_1A_1\beta^m\alpha^n + B_1C_1\beta^m\gamma^n + B_1D_1\beta^m\delta^n \\
&\quad + C_1A_1\gamma^m\alpha^n + C_1B_1\gamma^m\beta^n + C_1D_1\gamma^m\delta^n \\
&\quad + D_1A_1\delta^m\alpha^n + D_1B_1\delta^m\beta^n + D_1C_1\delta^m\gamma^n \\
&= A_1\alpha^{m+n} + B_1\beta^{m+n} + C_1\gamma^{m+n} + D_1\delta^{m+n} \\
&= \mathscr{M}_{m+n}.
\end{aligned}$$

(b) By Lemma 3.1, we have

$$\mathscr{M}_m\mathscr{R}_n = \mathscr{M}_m\mathscr{M}_n\mathscr{R}_0.$$

Now from (a) and again by Lemma 3.1 we obtain $\mathscr{M}_m\mathscr{R}_n = \mathscr{M}_{m+n}\mathscr{R}_0 = \mathscr{R}_{m+n}$.
It can be shown similarly that $\mathscr{R}_n\mathscr{M}_m = \mathscr{R}_{m+n}$.

(c) Using (a) and Theorem 3.1 (a) we obtain

$$\begin{aligned}
\mathscr{R}_m\mathscr{R}_n &= (-\mathscr{M}_{m+3} + 6\mathscr{M}_{m+1} - \mathscr{M}_m)(-\mathscr{M}_{n+3} + 6\mathscr{M}_{n+1} - \mathscr{M}_n) \\
&= \mathscr{M}_{m+3}\mathscr{M}_{n+3} - 6\mathscr{M}_{m+3}\mathscr{M}_{n+1} + \mathscr{M}_{m+3}\mathscr{M}_n - 6\mathscr{M}_{m+1}\mathscr{M}_{n+3} \\
&\quad + 36\mathscr{M}_{m+1}\mathscr{M}_{n+1} - 6\mathscr{M}_{m+1}\mathscr{M}_n + \mathscr{M}_m\mathscr{M}_{n+3} - 6\mathscr{M}_m\mathscr{M}_{n+1} + \mathscr{M}_m\mathscr{M}_n \\
&= \mathscr{M}_{m+n+6} - 6\mathscr{M}_{m+n+4} + \mathscr{M}_{m+n+3} - 6\mathscr{M}_{m+n+4} + 36\mathscr{M}_{m+n+2} - 6\mathscr{M}_{m+n+1} \\
&\quad + \mathscr{M}_{m+n+3} - 6\mathscr{M}_{m+n+1} + \mathscr{M}_{m+n} \\
&= \mathscr{M}_{m+n+6} - 12\mathscr{M}_{m+n+4} + 2\mathscr{M}_{m+n+3} + 36\mathscr{M}_{m+n+2} - 12\mathscr{M}_{m+n+1} + \mathscr{M}_{m+n}.
\end{aligned}$$

It can be shown similarly that $\mathscr{R}_n\mathscr{R}_m = \mathscr{M}_{m+n+6} - 12\mathscr{M}_{m+n+4} + 2\mathscr{M}_{m+n+3} + 36\mathscr{M}_{m+n+2} - 12\mathscr{M}_{m+n+1} + \mathscr{M}_{m+n}$.
The remaining of identities can be proved by considering again (a) and Theorem 3.1.

Comparing matrix entries and using Theorem 2.1 we have next result.

Corollary 3.2.
For Tetranacci and Tetranacci-Lucas numbers, we have the following identities:

(a) $M_{m+n} = M_{n+1}M_m + M_n(M_{m-1} + M_{m-2} + M_{m-3}) + M_{n-1}(M_{m-1} + M_{m-2}) + M_{n-2}M_{m-1}$

(b) $R_{m+n} = R_{n+1}M_m + R_n(M_{m-1} + M_{m-2} + M_{m-3}) + R_{n-1}(M_{m-1} + M_{m-2}) + R_{n-2}M_{m-1}$

(c) $R_mR_{n+1} + R_n(R_{m-1} + R_{m-2} + R_{m-3}) + R_{m-1}R_{n-2} + R_{n-1}(R_{m-1} + R_{m-2}) = M_{m+n+6} - 12M_{m+n+4} + 2M_{m+n+3} + 36M_{m+n+2} - 12M_{m+n+1} + M_{m+n}$

(d) $R_mR_{n+1} + R_n(R_{m-1} + R_{m-2} + R_{m-3}) + R_{m-1}R_{n-2} + R_{n-1}(R_{m-1} + R_{m-2}) = M_{m+n+4} - 10M_{m+n+3} + 29M_{m+n+2} - 18M_{m+n+1} - 6M_{m+n} + 4M_{m+n-1} + M_{m+n-2}$

(e) $R_mR_{n+1} + R_n(R_{m-1} + R_{m-2} + R_{m-3}) + R_{m-1}R_{n-2} + R_{n-1}(R_{m-1} + R_{m-2}) = 16M_{m+n+2} - 24M_{m+n+1} - 7M_{m+n} + 4M_{m+n-1} + 10M_{m+n-2} + 4M_{m+n-3} + M_{m+n-4}$

(f) $R_mR_{n+1} + R_n(R_{m-1} + R_{m-2} + R_{m-3}) + R_{m-1}R_{n-2} + R_{n-1}(R_{m-1} + R_{m-2}) = M_{m+n} + 4M_{m+n-1} + 10M_{m+n-2} + 20M_{m+n-3} + 25M_{m+n-4} + 24M_{m+n-5} + 16M_{m+n-6}$.

Proof.

(a) From Theorem 3.2 we know that $\mathcal{M}_m\mathcal{M}_n = \mathcal{M}_{m+n}$. Using Theorem 2.1, we can write this result as

$$\begin{pmatrix} M_{m+1} & M_m + M_{m-1} + M_{m-2} & M_m + M_{m-1} & M_m \\ M_m & M_{m-1} + M_{m-2} + M_{m-3} & M_{m-1} + M_{m-2} & M_{m-1} \\ M_{m-1} & M_{m-2} + M_{m-3} + M_{m-4} & M_{m-2} + M_{m-3} & M_{m-2} \\ M_{m-2} & M_{m-3} + M_{m-4} + M_{m-5} & M_{m-3} + M_{m-4} & M_{m-3} \end{pmatrix}$$

$$\times \begin{pmatrix} M_{n+1} & M_n + M_{n-1} + M_{n-2} & M_n + M_{n-1} & M_n \\ M_n & M_{n-1} + M_{n-2} + M_{n-3} & M_{n-1} + M_{n-2} & M_{n-1} \\ M_{n-1} & M_{n-2} + M_{n-3} + M_{n-4} & M_{n-2} + M_{n-3} & M_{n-2} \\ M_{n-2} & M_{n-3} + M_{n-4} + M_{n-5} & M_{n-3} + M_{n-4} & M_{n-3} \end{pmatrix}$$

$$= \begin{pmatrix} M_{m+n+1} & M_{m+n} + M_{m+n-1} + M_{m+n-2} & M_{m+n} + M_{m+n-1} & M_{m+n} \\ M_{m+n} & M_{m+n-1} + M_{m+n-2} + M_{m+n-3} & M_{m+n-1} + M_{m+n-2} & M_{m+n-1} \\ M_{m+n-1} & M_{m+n-2} + M_{m+n-3} + M_{m+n-4} & M_{m+n-2} + M_{m+n-3} & M_{m+n-2} \\ M_{m+n-2} & M_{m+n-3} + M_{m+n-4} + M_{m+n-5} & M_{m+n-3} + M_{m+n-4} & M_{m+n-3} \end{pmatrix}.$$

Now, by multiplying the left-side matrices and then by comparing the 2nd rows and 1st columns entries, we get the required identity in (a).

The remaining of identities can be proved by considering again Theorem 3.2 and Theorem 2.1.

The next two theorems provide us the convenience to obtain the powers of Tetranacci and Tetranacci-Lucas matrix sequences.

Theorem 3.3.
For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{M}_n^m = \mathcal{M}_{mn}$,

(b) $\mathcal{M}_{n+1}^m = \mathcal{M}_1^m \mathcal{M}_{mn}$,

(c) $\mathcal{M}_{n-r}\mathcal{M}_{n+r} = \mathcal{M}_n^2 = \mathcal{M}_2^n$.

Proof.

(a) We can write $\mathcal{M}_n^m$ as

$$\mathcal{M}_n^m = \mathcal{M}_n\mathcal{M}_n...\mathcal{M}_n \ (m \text{ times}).$$

Using Theorem 3.2 (a) iteratively, we obtain the required result:

$$\begin{aligned} \mathcal{M}_n^m &= \underbrace{\mathcal{M}_n\mathcal{M}_n...\mathcal{M}_n}_{m \text{ times}} \\ &= \mathcal{M}_{2n}\underbrace{\mathcal{M}_n\mathcal{M}_n...\mathcal{M}_n}_{m-1 \text{ times}} \\ &= \mathcal{M}_{3n}\underbrace{\mathcal{M}_n\mathcal{M}_n...\mathcal{M}_n}_{m-2 \text{ times}} \\ &\ \ \vdots \\ &= \mathcal{M}_{(m-1)n}\mathcal{M}_n \\ &= \mathcal{M}_{mn}. \end{aligned}$$

(b) As a similar approach in (a) we have

$$\mathcal{M}_{n+1}^m = \mathcal{M}_{n+1}.\mathcal{M}_{n+1}...\mathcal{M}_{n+1} = \mathcal{M}_{m(n+1)} = \mathcal{M}_m\mathcal{M}_{mn} = \mathcal{M}_1\mathcal{M}_{m-1}\mathcal{M}_{mn}.$$

Using Theorem 3.2 (a), we can write iteratively $\mathcal{M}_m = \mathcal{M}_1\mathcal{M}_{m-1}$, $\mathcal{M}_{m-1} = \mathcal{M}_1\mathcal{M}_{m-2}$, ..., $\mathcal{M}_2 = \mathcal{M}_1\mathcal{M}_1$. Now it follows that

$$\mathcal{M}_{n+1}^m = \underbrace{\mathcal{M}_1\mathcal{M}_1...\mathcal{M}_1}_{m \text{ times}}\mathcal{M}_{mn} = \mathcal{M}_1^m\mathcal{M}_{mn}.$$

(c) m Theorem 3.2 (a) gives

$$\mathcal{M}_{n-r}\mathcal{M}_{n+r} = \mathcal{M}_{2n} = \mathcal{M}_n\mathcal{M}_n = \mathcal{M}_n^2$$

and also

$$\mathcal{M}_{n-r}\mathcal{M}_{n+r} = \mathcal{M}_{2n} = \underbrace{\mathcal{M}_2\mathcal{M}_2...\mathcal{M}_2}_{n \text{ times}} = \mathcal{M}_2^n.$$

We have analogues results for the matrix sequence $\mathcal{R}_n$.

Theorem 3.4.
For non-negatif integers m, n and r with $n \geq r$, the following identities hold:

(a) $\mathcal{R}_{n-r}\mathcal{R}_{n+r} = \mathcal{R}_n^2$,

(b) $\mathcal{R}_n^m = \mathcal{R}_0^m\mathcal{M}_{mn}$.

Proof.

(a) We use Binet's formula of Tetranacci-Lucas matrix sequence which is given in Theorem 2.2. So

$$\begin{aligned}
& \mathcal{R}_{n-r}\mathcal{R}_{n+r} - \mathcal{R}_n^2 \\
= \; & (A_2\alpha^{n-r} + B_2\beta^{n-r} + C_2\gamma^{n-r} + D_2\delta^{n-r})(A_2\alpha^{n+r} + B_2\beta^{n+r} + C_2\gamma^{n+r} + D_2\delta^{n+r}) \\
& -(A_2\alpha^n + B_2\beta^n + C_2\gamma^n + D_2\delta^n)^2 \\
= \; & D_2A_2(\alpha^{n-r}\delta^{n+r} + \alpha^{n+r}\delta^{n-r} - 2\alpha^n\delta^n) + D_2B_2(\beta^{n-r}\delta^{n+r} + \beta^{n+r}\delta^{n-r} - 2\beta^n\delta^n) \\
& +D_2C_2(\gamma^{n-r}\delta^{n+r} + \gamma^{n+r}\delta^{n-r} - 2\gamma^n\delta^n) + A_2B_2(\alpha^{n-r}\beta^{n+r} + \alpha^{n+r}\beta^{n-r} - 2\alpha^n\beta^n) \\
& +A_2C_2(\alpha^{n-r}\gamma^{n+r} + \alpha^{n+r}\gamma^{n-r} - 2\alpha^n\gamma^n) + B_2C_2(\beta^{n-r}\gamma^{n+r} + \beta^{n+r}\gamma^{n-r} - 2\beta^n\gamma^n) \\
= \; & 0
\end{aligned}$$

is true since

$$D_2A_2 = D_2B_2 = D_2C_2 = A_2B_2 = A_2C_2 = B_2C_2 = (0)$$

by Lemma 3.2). Now we get the result as required.

(b) By Theorem 3.3, we have

$$\mathcal{R}_0^m\mathcal{M}_{mn} = \underbrace{\mathcal{R}_0\mathcal{R}_0...\mathcal{R}_0}_{m \text{ times}}\underbrace{\mathcal{M}_n\mathcal{M}_n...\mathcal{M}_n}_{m \text{ times}}.$$

When we apply Lemma 3.1 (a) iteratively, it follows that

$$\begin{aligned}
\mathcal{R}_0^m\mathcal{M}_{mn} &= (\mathcal{R}_0\mathcal{M}_n)(\mathcal{R}_0\mathcal{M}_n)...(\mathcal{R}_0\mathcal{M}_n) \\
&= \mathcal{R}_n\mathcal{R}_n...\mathcal{R}_n = \mathcal{R}_n^m.
\end{aligned}$$

This completes the proof.

4. Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. Many authors use matrix methods in their work. On the other hand, the matrix sequences have taken so much interest for different type of numbers. See, for example, [14, 15, 19, 20, 22]. In this paper, we defined the matrix sequences of Tetranacci and Tetranacci-Lucas numbers

It is our intention to continue the study and explore some properties of some type of matrix sequences of special numbers, such as matrix sequences of Pentanacci and Pentanacci-Lucas numbers.

We can summarize the sections as follows:

- In section 1, we have presented some background about Tetranacci and Tetranacci-Lucas numbers.
- In section 2, we have defined Tetranacci and Tetranacci-Lucas matrix sequences and then the generating functions, the Binet formulas, and summation formulas over these new matrix sequences have been presented.
- In section 3, we have given some relationship between these matrix sequences.

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