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The Use of Lavrentiev Regularization Method in Fredholm Integral Equations of the First Kind

Research Article

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Abstract: The Fredholm integral equations of the first kind are often considered as ill-posed problems. The conventional way of solving them is to first convert them into the Fredholm integral equations of the second kind by means of a regularization method. This is followed by applying some standard techniques that are available for solving Fredholm integral equations of the second kind. This combination of two methods usually has some significant drawbacks in the sense that it may not produce a solution or produces only one solution after tedious calculations. The aim of this study is to remove these impediments once and for all for separable kernels and provide a closed-form expression for obtaining one or infinitely many solutions using the Lavrentiev regularization method.

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Keywords: Regularization • Fredholm integral equation • ill-posed problems

1. Introduction

Many integral equations originate from partial or ordinary differential equations. Some Fredholm integral equations of the first kind are related to a family of boundary value problems. They also appear in many different branches of science. Therefore, many researchers from different backgrounds have shown interest in them [1–11].
The general form of the linear Fredholm integral equation is as follows:

$$\psi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt,$$

where $f(x), \psi(x)$, and $k(x,t)$ are known, a and b are constants, λ is a parameter, and $u(x)$ is the unknown (desired) function.
If the function $\psi(x) = 1$, then the equation

$$u(x) = f(x) + \lambda \int_a^b k(x,t)u(t)\,dt,$$

is called Fredholm integral equation of the second kind and if the function $\psi(x) = 0$, then the equation

$$f(x) = \lambda \int_a^b k(x,t)u(t)\,dt, \tag{1}$$

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is called Fredholm integral equation of the first kind.
If we replace $u(x)$ appearing under the integral sign by $F(u(x))$, where F is a nonlinear function of u, we get the corresponding nonlinear equations. That is, the equation

$$f(x) = \lambda \int_a^b k(x,t)F(u(t))\,dt.$$

is called nonlinear Fredholm integral equation of the first kind.

Since many physical processes involve the Fredholm integral equations of the first kind, it is important and useful to have a method for finding infinitely many solutions. This is simply because one might be interested in finding a solution satisfying some initial or other physical constraints. Such a case could be handled more efficiently if one has a family of solutions in hand.

In this article, we focus on Fredholm integral equation of the first kind which are often considered to be ill-posed [4, 12]. Hadamard proposed the following properties to classify a problem as an ill-posed or a well-posed problem:

- the problem has a solution,
- the solution is unique, and
- continuous dependence of $u(x)$ and $f(x)$.

If a problem satisfies these properties, it is called well-posed. Otherwise, it is called ill-posed [10, 13].

Wazwaz [14] applied the regularization method to find a solution to the integral equation. Our motivation in this article was to provide an easy algorithm derived from an application of the regularization method. We provide a closed-form expression to obtain solutions without delving into operator approach or some other complicated approaches.

If an integral equation has many solutions, it is desirable to have a systematic procedure to reach them all, or, at least to have a technique to obtain infinitely many of them. Besides many other benefits, having an infinite number of solutions allows one to select a solution satisfying an initial condition or some other physical constraints. To make this point more precise, we provide examples in section 4.

We also want to note that the results obtained for separable kernels have an important place in the theory of integral equations since arbitrary kernels can be approximated by them [15].

2. Basic Assumptions and Preliminaries

The regularization method was introduced by Tikhonov [16, 17] and Philips [18]. It consists of transforming the integral equation of the first kind to the integral equation of the second kind. To achieve this, the Lavrentiev regularization method is employed [19]. To be more precise, the method transforms

$$f(x) = \int_a^b k(x,t)u(t)\,dt, \tag{2}$$

to

$$\alpha u_\alpha(x) = f(x) - \int_a^b k(x,t)u_\alpha(t)\,dt,$$

where α is a positive parameter and x is taken from a compact set of real numbers. Now using one of the existing techniques for second kind Fredholm equations, one could obtain $u_\alpha(x)$.
Under some mild conditions (page 161 in [10]), it was proved in [20, 21] that

$$u(x) = \lim_{\alpha\to 0} u_\alpha(x)$$

As a result of above discussion, once the equation is transformed into the Fredholm equation of the second kind, its solution is obtained by taking the limit of $u_\alpha(x)$ as $\alpha \to 0$.

Throughout the article we assume that the kernel is separable and the equation (2) has a solution. We note that when the kernel is separable, the equation (2) has a solution if and only if f is in the $span\{g_1, g_2, \ldots, g_N\}$.

3. Main Section

Applying the method of regularization combined with any other standard methods to obtain solutions sometimes can require a tedious work. One of the aims of this article is to obtain a simple formula which provides not only the solution obtained by the regularization method followed by some existing techniques such as direct computation, Adomian decomposition, successive approximations, etc., but also other solutions.

Before considering the general case, we first take the simplest case, namely the case $n = 1$, into consideration and then generalize it.

Case 1: $k(x,t) = g(x)h(t)$.
It is easy to verify that If f is not orthogonal to h, then

$$u(x) = \frac{f(x)}{\int_a^b k(t,t)\,dt}. \tag{3}$$

is a solution for (2).
Applying the regularization method to (2), we get

$$\begin{aligned} u_\alpha(x) &= \frac{f(x)}{\alpha} - \frac{g(x)}{\alpha}\int_a^b h(t)u_\alpha(t)\,dt, \\ &= \frac{1}{\alpha}\Big(f(x) - \lambda g(x)\Big), \end{aligned} \tag{4}$$

where $\lambda = \int_a^b h(t)u_\alpha(t)\,dt$. Substituting (4) for $u_\alpha(t)$ in λ, we obtain

$$\begin{aligned} \lambda &= \int_a^b h(t)\Big[\frac{1}{\alpha}\Big(f(t) - \lambda g(t)\Big)\Big]\,dt, \\ &= \frac{1}{\alpha}\int_a^b f(t)h(t)\,dt - \frac{\lambda}{\alpha}\int_a^b k(t,t)\,dt, \end{aligned} \tag{5}$$

Solving (5) for λ, we have

$$\lambda = \frac{\int_a^b f(t)h(t)\,dt}{\alpha + \int_a^b k(t,t)\,dt}. \tag{6}$$

Plug (6) back into (4), we get

$$\begin{aligned} u_\alpha(x) &= \frac{1}{\alpha}\Big(f(x) - \frac{\int_a^b f(t)h(t)\,dt}{\alpha + \int_a^b k(t,t)\,dt}g(x)\Big), \\ &= \frac{f(x)}{\alpha + \int_a^b k(t,t)\,dt}. \end{aligned}$$

Thus,

$$\begin{aligned} u(x) &= \lim_{\alpha\to 0} u_\alpha(x) = \lim_{\alpha\to 0}\frac{f(x)}{\alpha + \int_a^b k(t,t)\,dt}, \\ &= \frac{f(x)}{\int_a^b k(t,t)\,dt}. \end{aligned}$$

Corollary 3.1.
Adding any function q that is orthogonal to h to (3) produces a solution for (2). That is,

$$u(x) = \frac{f(x)}{\int_a^b k(t,t)\,dt} + q(x),$$

where $q \perp h$.

Notice that when $f \perp h$, the formula (3) cannot be used. To be able to use this formula for the case $f \perp h$, we proceed as follows:
We multiply both sides of

$$f(x) = \int_a^b k(x,t)u(t) = \int_a^b g(x)h(t)u(t)\,dt.$$

by an auxiliary function ϕ_β so that we have

$$f(x)\phi_\beta(x) = \int_a^b k(x,t)\phi_\beta(x)u(t)\,dt.$$

Applying the regularization method, we get

$$u_\alpha(x) = \frac{f(x)\phi_\beta(x)}{\alpha} - \frac{g(x)\phi_\beta(x)}{\alpha}\lambda \tag{7}$$

where $\lambda = \int_a^b h(t)u_\alpha(t)\,dt$. Substituting (7) for $u_\alpha(t)$ in λ and solving the resulting equation for λ, we obtain

$$\lambda = \frac{y_\beta}{\alpha + v_\beta}, \tag{8}$$

where

$$y_\beta = \int_a^b f(t)h(t)\phi_\beta(t)\,dt \quad \text{and} \quad v_\beta = \int_a^b k(t,t)\phi_\beta(t)\,dt.$$

Plug (8) back into (7) and taking the limit as $\alpha \to 0$, we get

$$u(x) = \frac{f(x)\phi_\beta(x)}{v_\beta}. \tag{9}$$

A few important remarks are in order.

Remark 3.1.
The equation (9) produces infinitely many solutions.

Remark 3.2.
ϕ_β is chosen so that $f\phi_\beta$ is not orthogonal to h. With this manipulation the hypothesis in theorem 3.1 is satisfied. Notice also that v_β does not vanish if ϕ_β is defined as it is proposed.

Case 2: $k(x,t) = \sum_{k=1}^{n} g_k(x)h_k(t),\ n \geq 2.$
We now consider

$$f(x) = \int_a^b \left(\sum_{k=1}^{n} g_k(x)h_k(t)\right)u(t)\,dt. \tag{10}$$

We apply the regularization method and obtain

$$\begin{aligned} u_\alpha(x) &= \frac{f(x)}{\alpha} - \frac{1}{\alpha}\sum_{k=1}^{n} g_k(x)\int_a^b h_k(t)u_\alpha(t)\,dt \\ &= \frac{1}{\alpha}\Big(f(x) - \sum_{k=1}^{n}\beta_k g_k(x)\Big), \end{aligned} \tag{11}$$

where $\beta_k = \int_a^b h_k(t)u_\alpha(t)\,dt$ for $k = 1,2,\dots,n$.
Substituting (11) for $u_\alpha(t)$ in β_k, we get

$$\begin{aligned} \beta_k &= \frac{1}{\alpha}\int_a^b h_k(t)\Big(f(t) - \sum_{j=1}^{n}\beta_j g_j(t)\Big)\,dt \\ &= \frac{1}{\alpha}\int_a^b f(t)h_k(t)\,dt - \frac{1}{\alpha}\sum_{j=1}^{n}\beta_j\int_a^b g_j(t)h_k(t)\,dt \end{aligned} \tag{12}$$

For $i, j, k = 1, 2, \ldots, n$, we let $F_{n\times 1} = [f_k]$ and $A_{n\times n} = [a_{ij}]$, where

$$f_k = \int_a^b f(t) h_k(t)\, dt$$

and

$$a_{ij} = \int_a^b h_i(t) g_j(t)\, dt$$

so that (12) can be written as

$$(A + \alpha I)_{n\times n} \beta_{n\times 1} = F_{n\times 1},$$

where $\beta^T = (\beta_1, \beta_2, \ldots \beta_n)$. We assume that A invertible. This assumption covers most cases of practical interest. Thus, we have

$$\beta = ((A + \alpha I))^{-1} F. \tag{13}$$

Substituting (13) into (11), we obtain

$$u_\alpha(x) = \frac{f(x) - \left((A + \alpha I)^{-1} F\right) \cdot G(x)}{\alpha} \tag{14}$$

where $G^T(x) = (g_1(x), g_2(x), \ldots, g_n(x))$ and $\cdot$ is used as standard dot product of two vectors. Observing the last equation, we claim that
if $f(x) = (A^{-1}F) \cdot G(x)$, then $u(x) = ((A^{-1})^2 F) \cdot G(x)$ is a solution of (10). We begin by assuming that $f(x) = (A^{-1}F) \cdot G(x)$. Then (14) becomes

$$u_\alpha(x) = \left(\frac{A^{-1} - (A + \alpha I)^{-1}}{\alpha}\right) F \cdot G(x)$$

We now need to compute the following limit:

$$\lim_{\alpha\to 0} u_\alpha(x) = \lim_{\alpha\to 0} \left(\frac{A^{-1} - (A + \alpha I)^{-1}}{\alpha}\right) F \cdot G(x) \tag{15}$$

If we let the Jordan decomposition of the matrix A be PJP^{-1}, then (15) becomes

$$\lim_{\alpha\to 0} u_\alpha(x) = P \lim_{\alpha\to 0} \left(\frac{J^{-1} - (J + \alpha I)^{-1}}{\alpha}\right) P^{-1} F \cdot G(x) \tag{16}$$

We take this limit by considering each of the blocks in Jordan decomposition form of A seperately. Thus, we consider the Jordan block

$$J_m(\lambda) = \begin{pmatrix} \lambda & 1 & 0 & \ldots & 0 \\ 0 & \lambda & 1 & \ldots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & \lambda & 1 \\ 0 & \ldots & 0 & 0 & \lambda \end{pmatrix}_{m\times m} \tag{17}$$

and its inverse

$$J_m^{-1}(\lambda) = \begin{pmatrix} 1/\lambda & -1/\lambda^2 & 1/\lambda^3 & \ldots & (-1)^{1-m}/\lambda^m \\ 0 & 1/\lambda & -1/\lambda^2 & \ldots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 1/\lambda^3 \\ 0 & \ldots & 0 & 1/\lambda & -1/\lambda^2 \\ 0 & \ldots & 0 & 0 & 1/\lambda \end{pmatrix}_{m\times m} \tag{18}$$

Notice that since A is inverible, $0 \notin \sigma\{A\}$. Using (17) and (18), we have

$$\frac{J_m^{-1}(\lambda)-(J_m(\lambda)+\alpha I)^{-1}}{\alpha}=\frac{1}{\alpha}\begin{pmatrix}1/\lambda & -1/\lambda^2 & 1/\lambda^3 & \dots & (-1)^{1-m}/\lambda^m\\ 0 & 1/\lambda & -1/\lambda^2 & \dots & \vdots\\ \vdots & \vdots & \ddots & \ddots & 1/\lambda^3\\ 0 & \dots & 0 & 1/\lambda & -1/\lambda^2\\ 0 & \dots & 0 & 0 & 1/\lambda\end{pmatrix}$$
$$-\frac{1}{\alpha}\begin{pmatrix}1/(\lambda+\alpha) & -1/(\lambda+\alpha)^2 & 1/(\lambda+\alpha)^3 & \dots & (-1)^{1-m}/(\lambda+\alpha)^m\\ 0 & 1/(\lambda+\alpha) & -1/(\lambda+\alpha)^2 & & \vdots\\ 0 & 0 & \ddots & \ddots & 1/(\lambda+\alpha)^3\\ \vdots & \vdots & 0 & 1/(\lambda+\alpha) & -1/(\lambda+\alpha)^2\\ 0 & \dots & 0 & 0 & 1/(\lambda+\alpha)\end{pmatrix}$$

We take a sample entry from kth diagonal of above difference and evaluate its limit

$$\begin{aligned}\lim_{\alpha\to 0}\frac{1}{\alpha}\left(\frac{(-1)^{1-k}}{\lambda^k}-\frac{(-1)^{1-k}}{(\lambda+\alpha)^k}\right)&=\lim_{\alpha\to 0}\frac{(-1)^{1-k}(\lambda+\alpha)^k-(-1)^{1-k}\lambda^k}{\alpha\lambda^k(\lambda+\alpha)^k}\\ &=\lim_{\alpha\to 0}\frac{(-1)^{1-k}\sum_{i=0}^{k}\binom{k}{i}\lambda^i\alpha^{k-i}-(-1)^{1-k}\lambda^k}{\alpha\lambda^k(\lambda+\alpha)^k}\\ &=\lim_{\alpha\to 0}\frac{-(-1)^{1-k}\sum_{i=0}^{k-1}\binom{k}{i}\lambda^i(-\alpha)^{k-i}}{\alpha\lambda^k(\lambda-\alpha)^k}\\ &=\frac{(-1)^{1-k}\binom{k}{k-1}\lambda^{k-1}}{\lambda^{2k}}=\frac{(-1)^{1-k}k}{\lambda^{k+1}}\end{aligned}$$

Thus, we have

$$\begin{aligned}\lim_{\alpha\to 0}\frac{J_m^{-1}(\lambda)-(J_m(\lambda)+\alpha I)^{-1}}{\alpha}&=\begin{pmatrix}1/\lambda^2 & -2/\lambda^3 & 3/\lambda^4 & \dots & (-1)^{1-m}m/\lambda^{m+1}\\ 0 & 1/\lambda^2 & -2/\lambda^3 & \dots & \vdots\\ 0 & 0 & \ddots & \ddots & 3/\lambda^4\\ \vdots & \vdots & 0 & 1/\lambda^2 & -2/\lambda^3\\ 0 & \dots & 0 & 0 & 1/\lambda^2\end{pmatrix}\\ &=\left(J_m^{-1}(\lambda)\right)^2\end{aligned}$$

This fact holds for each Jordan block $J_m(\lambda)$ in the Jordan canonical form J of A. From the properties of inversion of block matrices, we deduce that

$$\lim_{\alpha\to 0}\left(\frac{J^{-1}-(J+\alpha I)^{-1}}{\alpha}\right)=\left(J^{-1}\right)^2$$

Since

$$P\left(J^{-1}\right)^2P^{-1}=\left(A^{-1}\right)^2$$

(16) gives the desired solution.
Alternatively, we can use the fact in matrix theory that

$$(I-B)^{-1}=\sum_{k=0}^{\infty}B^k \tag{19}$$

for $||B||<1$, where $||\cdot||$ is a matrix norm.
It is celar that

$$(A+\alpha I)^{-1}=(I+\alpha A^{-1})^{-1}A^{-1}$$

Now, let $B=-\alpha A^{-1}$, use the fact (19), and substitute these into (15) to get

$$\lim_{\alpha\to 0}\left(\frac{A^{-1}-(A+\alpha I)^{-1}}{\alpha}\right)=(A^{-1})^2$$

We conclude this in the following theorem.

Theorem 3.1.
If A is invertible and $f(x) = (A^{-1}F)\cdot G(x)$, *then*

$$u(x) = \left((A^{-1})^2 F\right)\cdot G(x) \tag{20}$$

is a solution of (10).

Remark 3.3.
By introducing an auxiliary function ϕ_β as discussed in the first case, one can derive other solutions from (20).

For nonlinear equations, we follow a similar approach as in the linear case. By assuming that $v(x) = F(u(x))$ is invertible, we'll obtain $u(x)$ by setting that $u(x) = F^{-1}(v(x))$.

4. Illustrative Examples

In the following, we provide some examples to illustrate how the proposed formula can be used to obtain solutions of Fredholm integral equations with various types of separable kernels.

Example 1:
Consider the following Fredholm integral equation of the first kind:

$$\frac{\pi}{2}\cos(x) = \int_0^\pi \cos(x)\sin(t)u(t)dt. \tag{21}$$

In this case,

$$f(x) = \tfrac{\pi}{2}\cos(x),\ g(x) = \cos(x), \text{ and } h(t) = \sin(t).$$

This case f is orthogonal to h. Thus we cannot directly use (3). However, if we follow the foregoing remark and let that $\phi_\beta = \cos(\beta x)$,
(9) gives

$$u(x) = \frac{\pi(\beta^2-4)\cos(x)\cos(\beta x)}{2\left(\cos(\beta\pi)-1\right)},\ \beta \neq 2k,\ k\in\mathbb{Z}.$$

To illustrate some solutions, we plug some values for β. That is, If $\beta = 1/2$, then

$$u(x) = \frac{15\pi}{8}\left[\cos(x)\cos(x/2)\right],$$

which is a solution.
If $\beta = 1$, then

$$u(x) = \frac{3\pi}{4}\cos^2(x),$$

which is another solution.
We want to point out that as long as ϕ_β satisfies the assumptions provided in the previous section, (9) will provide solutions for the integral equation.
For instance, if we let $\phi_\beta = e^{\beta x}$, then it is easy to show that

$$u(x) = \frac{\pi(\beta^2+4)\cos(x)e^{\beta x}}{2(1-e^{\beta\pi})}.$$

If $\beta = 1$, then

$$u(x) = \frac{5\pi}{2(1-e^{\pi})}e^{x}\cos(x),$$

which is a solution of (21).
If $\beta = 2$, then

$$u(x) = \frac{4\pi}{1-e^{2\pi}}e^{2x}\cos(x),$$

which is another solution of (21).

Example 2:
Consider the following nonlinear Fredholm integral equation of the first kind [14]:

$$e^x = \int_0^1 e^{x-2t} u^2(t) dt. \tag{22}$$

Set $y(x) = u^2(x)$ so that $u(x) = \pm\sqrt{y(x)}$ and consider

$$e^x = \int_0^1 e^{x-2t} y(t) dt.$$

$f(x) = e^x$ $g(x) = e^x$, and $h(t) = e^{-2t}$. Since f is not orthogonal to h,

$$y(x) = \frac{e^{x+1}}{e-1}$$

(see theorem 3.1). Let $\phi_\beta(x) = e^{\beta x}$. Then

$$y(x) = \begin{cases} e^{2x}, & \beta = 1, \\ \frac{(\beta-1)e^{(\beta+1)x}}{e^{\beta-1}-1}, & \beta \neq 1. \end{cases}$$

It is easy to verify that

$$u(x) = \pm\sqrt{\frac{e^{x+1}}{e-1}},$$

is a pair of solution of (22).
There are also other pair of solutions such as

$$u(x) = \pm\sqrt{e^{2x}}, \pm\sqrt{\frac{e^{3x}}{e-1}}, \pm\sqrt{\frac{2e^{4x}}{e^2-1}}, \pm\sqrt{\frac{3e^{5x}}{e^3-1}}, \ldots$$

Example 3:
Consider the following Fredholm integral equation of the first kind

$$x + 6x^2 = 5\int_0^1 (xt + x^2t^2)u(t)\, dt \tag{23}$$

Following the notations introduced in **Case 2**, we have

$$A = \begin{pmatrix} 5/3 & 5/4 \\ 5/4 & 1 \end{pmatrix}, F = \begin{pmatrix} 11/6 \\ 29/20 \end{pmatrix}, \text{ and } G(x) = \begin{pmatrix} 5x \\ 5x^2 \end{pmatrix}.$$

Then,

$$A^{-1} = \begin{pmatrix} 48/5 & -12 \\ -12 & 16 \end{pmatrix} \text{ and } (A^{-1}F)\cdot G(x) = x + 6x^2.$$

Since $f(x) = (A^{-1}F)\cdot G(x) = x + 6x^2$, all hypotheses of the theorem 3.2 are satisfied. Thus,

$$\begin{aligned} u(x) &= ((A^{-1})^2 F)\cdot G(x) \\ &= -\frac{312}{5}x + 84x^2 \end{aligned}$$

It is clear to see that this is a solution to (23). Like in Case 1, other solutions can be obtained by introducing a suitable ϕ_β. For instance, letting that $\phi_\beta = x^\beta$, we obtain infinitely many solutions for different values of β. That is,

$$u(x) = 156x^4 - 124x^3, \frac{1302x^6}{5} - 216x^5, \frac{2016x^8}{5} - \frac{1722x^7}{5}, \ldots$$

is a set of solutions for (23).

Example 4
Consider the following Fredholm integral equation of the first kind

$$3\cos(x) = \int_0^{\pi/2} \sin(x-t)u(t)\, dt \tag{24}$$

In this case,

$$A = \begin{pmatrix} \frac{1}{2} & -\frac{\pi}{4} \\ \frac{\pi}{4} & -\frac{1}{2} \end{pmatrix}, F = \begin{pmatrix} 3\pi/4 \\ 3/2 \end{pmatrix}, \text{ and } G(x) = \begin{pmatrix} \sin(x) \\ -\cos(x) \end{pmatrix}.$$

Then, we have

$$A^{-1} = \frac{4}{\pi^2 - 4} \begin{pmatrix} -2 & \pi \\ -\pi & 2 \end{pmatrix} \text{ and } (A^{-1}F) \cdot G(x) = 3\cos(x).$$

Thus, we get

$$\begin{aligned} u(x) &= ((A^{-1})^2 F) \cdot G(x) \\ &= -\frac{24}{4-\pi^2} \big(\frac{\pi}{2} \sin(x) - \cos(x) \big) \end{aligned}$$

It is easy to verify that this is a solution to (24). If we let that $\phi_\beta = \cos(\beta x)$, we get other solutions. These, for instance, are

$$u(x) = 9(\cos^2(x) - \sin(2x)), 18\cos(4x)\cos(x), 90\cos(8x)\cos(x), \ldots$$

Example 5

Consider the following Fredholm integral equation of the first kind [10]:

$$e^{x+1} = \int_0^1 \big(4te^x + 3\big)\, u(t)\, dt \tag{25}$$

We now have

$$A = \begin{pmatrix} 4 & \frac{3}{2} \\ 4(e-1) & 3 \end{pmatrix}, F = \begin{pmatrix} e \\ e(e-1) \end{pmatrix}, \text{ and } G(x) = \begin{pmatrix} 4e^x \\ 3 \end{pmatrix}.$$

Then,

$$A^{-1} = \frac{1}{18-6e} \begin{pmatrix} 3 & -\frac{3}{2} \\ 4(1-e) & 4 \end{pmatrix} \text{ and } A^{-1}F \cdot G(x) = e^{x+1}.$$

Thus, we get

$$\begin{aligned} u(x) &= ((A^{-1})^2 F) \cdot G(x) \\ &= \frac{1}{2(e-3)} \big(e^2 - e - e^{x+1} \big) \end{aligned}$$

It is easy to verify that this is a solution to (25). There are also other solutions.

5. Conclusion

Fredholm integral equations of the first kind are usually considered to be ill-posed problems. That means solution may not exist, or if it exists, it may not be unique. That also means that a small change on the function $f(x)$ may result in a big change in the solution $u(x)$. For practical purpose, it is therefore important to have a family of solutions in hand if we are certain that there is at least one. In this work, we investigate the regularization method further and provide an alternative way for obtaining solutions for some classes of Fredholm integeral equations with separable kernels. We also want to note that this idea can easily be generalized and used for solving higher dimensional Fredholm integral equations of the first kind.

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