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On Generalized Narayana Numbers

Research Article

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Abstract: In this paper, we introduce and investigate the generalized Narayana sequences and we deal with, in detail, two special cases besides Narayana sequence which we call them Narayana-Lucas and Narayana-Perrin sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

MSC: 11B39 • 11B83

Keywords: Narayana numbers • Narayana-Perrin numbers • Narayana-Lucas numbers

1. Introduction

The Narayana numbers was introduced by the Indian mathematician Narayana in the 14th century, while studying the problem of a herd of cows and calves, see [1],[22] for details. Narayana's cows problem is a problem similar to the Fibonacci's rabbit problem which can be given as follows: A cow produces one calf every year and beginning in its fourth year, each calf produces one calf at the beginning of each year. How many calves are there altogether after 20 years? This problem can be solved in the same way that Fibonacci solved its problem about rabbits (see [16]). If n is the year, then the Narayana problem can be modelled by the recurrence $N_{n+3} = N_{n+2} + N_n$, with $n \geq 0$, $N_0 = 0, N_1 = 1, N_2 = 1$, see [1]. The first few terms are $0, 1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28...$, (the sequence A000930 in [25]). This sequence is called Narayana sequence (also called Fibonacci-Narayana sequence or Narayana's cows sequence).

Recently, there has been considerable interest in the Narayana sequence and its generalizations (for more details, see [1],[2],[3],[7],[10],[13],[21],[34] and the references given therein). For instance, In [3], Bilgici defined a new recurrence which is called generalized order-k Narayana's cows sequence and he gave some identities related to the Narayana's cows numbers by using this generalization and some matrix properties. Didkivska and St'opochkina [7] proved some basic properties of Fibonacci-Narayana numbers (some of these basic properties is given in [10] in English). Flaut and Shpakivskyi [10] studied some properties of generalized and Fibonacci quaternions and Fibonacci-Narayana quaternions. Goy [13] studied some families of Toeplitz-Hessenberg determinants the entries of which are Fibonacci-Narayana (or Narayana's cows) numbers. In particular, he established connection between Fibonacci-Narayana numbers with Fibonacci and Tibonacci numbers. Ramírez and Sirvent [21] defined the k-Narayana sequence of integer numbers as a generalisation of Narayana numbers and studied recurrence relations and some combinatorial properties of these numbers and of the sum of their first n terms using matrix methods.

Note that Narayana sequence (or Fibonacci-Narayana sequence or Narayana's cows sequence) named after a 14th-century Indian mathematician Narayana. In literature, there is a sequence which is also called Narayana sequence

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(named after Canadian mathematician T. V. Narayana (1930–1987)) and is defined by the numbers (the sequence A001263 in [25])

$$N(n,k)=\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$$

where $1\leq k\leq n$. These type of Narayana numbers (in fact, a q-analogue of them) were first studied by MacMahon [17], Article 495 and were later rediscovered by Narayana [18]. It is well known that for any positive integer n,

$$C_n=\sum_{k=1}^{n}N(n,k)$$

where C_n are Catalan numbers and given by $C_n=\frac{1}{n+1}\binom{2n}{n}$ and satisfies the recurrence $C_{n+1}=\frac{4n+2}{n+2}C_n$ where $C_0=1$.

The purpose of this paper is to study a generalisation of Narayana sequence (Narayana's cows sequence). We define and investigate the generalized Narayana sequences and we deal with, in detail, two special cases besides Narayana sequence which we call them Narayana-Lucas and Narayana-Perrin sequences. Before, we recall the generalized Tribonacci sequence and its some properties.

The generalized Tribonacci sequence $\{W_n(W_0,W_1,W_2;r,s,t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) (as a third-order generalization of Fibonacci numbers) is defined as follows:

$$W_n=rW_{n-1}+sW_{n-2}+tW_{n-3},\quad W_0=a,W_1=b,W_2=c,\ \ n\geq 3 \tag{1}$$

where W_0,W_1,W_2 are arbitrary complex (or real) numbers and r,s,t are real numbers.

This sequence has been studied by many authors, see for example [4],[5],[6],[8],[9],[19],[20],[23],[24],[29],[30],[32],[33]. See also [11],[12],[31] for some work on second-order generalization of Fibonacci numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n}=-\frac{s}{t}W_{-(n-1)}-\frac{r}{t}W_{-(n-2)}+\frac{1}{t}W_{-(n-3)}$$

for $n=1,2,3,\ldots$ when $t\neq 0$. Therefore, recurrence (1) holds for all integer n.

As $\{W_n\}$ is a third order recurrence sequence (difference equation), it's characteristic equation is

$$x^3-rx^2-sx-t=0 \tag{2}$$

whose roots are

$$\begin{aligned}
\alpha &= \alpha(r,s,t)=\frac{r}{3}+A+B\\
\beta &= \beta(r,s,t)=\frac{r}{3}+\omega A+\omega^2 B\\
\gamma &= \gamma(r,s,t)=\frac{r}{3}+\omega^2 A+\omega B
\end{aligned}$$

where

$$\begin{aligned}
A &= \left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}+\sqrt{\Delta}\right)^{1/3},\ B=\left(\frac{r^3}{27}+\frac{rs}{6}+\frac{t}{2}-\sqrt{\Delta}\right)^{1/3}\\
\Delta &= \Delta(r,s,t)=\frac{r^3t}{27}-\frac{r^2s^2}{108}+\frac{rst}{6}-\frac{s^3}{27}+\frac{t^2}{4},\ \ \omega=\frac{-1+i\sqrt{3}}{2}=\exp(2\pi i/3)
\end{aligned}$$

Note that we have the following identities

$$\begin{aligned}
\alpha+\beta+\gamma &= r,\\
\alpha\beta+\alpha\gamma+\beta\gamma &= -s,\\
\alpha\beta\gamma &= t.
\end{aligned}$$

If $\Delta(r,s,t)>0$, then the Equ. (2) has one real (α) and two non-real solutions with the latter being conjugate complex. So, in this case, it is well known that generalized Tribonacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n=\frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)}+\frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)}+\frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{3}$$

where

$$b_1=W_2-(\beta+\gamma)W_1+\beta\gamma W_0,\ b_2=W_2-(\alpha+\gamma)W_1+\alpha\gamma W_0,\ b_3=W_2-(\alpha+\beta)W_1+\alpha\beta W_0.$$

Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n.

In this paper we consider the case $r = 1$, $s = 0, t = 1$ and in this case we write $V_n = W_n$. A generalized Narayana sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$V_n = V_{n-1} + V_{n-3} \tag{4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -V_{-(n-2)} + V_{-(n-3)}$$

for $n = 1,2,3,....$ Therefore, recurrence (4) holds for all integer n.

(3) can be used to obtain Binet formula of generalized Narayana numbers. Binet formula of generalized Narayana numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)}$$

where

$$b_1 = V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0,\ b_2 = V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0,\ b_3 = V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0. \tag{5}$$

Here, α, β and γ are the roots of the cubic equation $x^3 - x^2 - 1 = 0$.

Moreover

$$\begin{aligned}
\alpha &= \frac{1}{3} + \left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3} \\
\beta &= \frac{1}{3} + \omega\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega^2\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3} \\
\gamma &= \frac{1}{3} + \omega^2\left(\frac{29}{54} + \sqrt{\frac{31}{108}}\right)^{1/3} + \omega\left(\frac{29}{54} - \sqrt{\frac{31}{108}}\right)^{1/3}
\end{aligned}$$

where

$$\omega = \frac{-1 + i\sqrt{3}}{2} = \exp(2\pi i/3).$$

Note that

$$\begin{aligned}
\alpha + \beta + \gamma &= 1, \\
\alpha\beta + \alpha\gamma + \beta\gamma &= 0, \\
\alpha\beta\gamma &= 1.
\end{aligned}$$

The first few generalized Narayana numbers with positive subscript and negative subscript are given in the following Table 1

Table 1. A few generalized Narayana numbers

n	V_n	V_{-n}
0	V_0	...
1	V_1	$V_2 - V_1$
2	V_2	$V_1 - V_0$
3	$V_2 + V_0$	$-V_2 + V_1 + V_0$
4	$V_2 + V_1 + V_0$	$V_2 - 2V_1 + V_0$
5	$(2V_2 + V_1 + V_0$	$V_2 - 2V_0$
6	$3V_2 + V_1 + 2V_0$	$-2V_2 + 3V_1$
7	$4V_2 + 2V_1 + 3V_0$	$-2V_1 + 3V_0$
8	$6V_2 + 3V_1 + 4V_0$	$3V_2 - 3V_1 - 2V_0$

Now we define three special case of the sequence $\{V_n\}$. Narayana sequence $\{N_n\}_{n\geq 0}$, Narayana-Lucas sequence $\{U_n\}_{n\geq 0}$ and Narayana-Perrin sequence $\{H_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$N_{n+3} = N_{n+2} + N_n, \quad N_0 = 0, N_1 = 1, N_2 = 1, \tag{6}$$

$$U_{n+3}=U_{n+2}+U_n, \quad U_0=3, U_1=1, U_2=1, \tag{7}$$

and

$$H_{n+3}=H_{n+2}+H_n, \quad H_0=3, H_1=0, H_2=2, \tag{8}$$

The sequences $\{N_n\}_{n\geq 0}, \{U_n\}_{n\geq 0}$ and $\{H_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$N_{-n}=-N_{-(n-2)}+N_{-(n-3)} \tag{9}$$

and

$$U_{-n}=-U_{-(n-2)}+U_{-(n-3)} \tag{10}$$

and

$$H_{-n}=-H_{-(n-2)}+H_{-(n-3)} \tag{11}$$

for $n=1,2,3,\ldots$ respectively. Therefore, recurrences (9), (10) and (11) hold for all integer n.

Note that N_n is the sequence A000930 in [25] associated with the Narayana's cows sequence and the sequence A078012 in [25] associated with the expansion of $(1-x)/(1-x-x^3)$ and U_n is the sequence A001609 in [25].

Next, we present the first few values of the Narayana, Narayana-Lucas and Narayana-Perrin numbers with positive and negative subscripts:

Table 2. The first few values of the special third-order numbers with positive and negative subscripts

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
N_n	0	1	1	1	2	3	4	6	9	13	19	28	41	60
N_{-n}		0	1	0	−1	1	1	−2	0	3	−2	−3	5	1
U_n	3	1	1	4	5	6	10	15	21	31	46	67	98	144
U_{-n}		0	−2	3	2	−5	1	7	−6	−6	13	0	−19	13
H_n	3	0	2	5	5	7	12	17	24	36	53	77	113	166
H_{-n}		2	−3	1	5	−4	−4	9	0	−13	9	13	−22	−4

For all integers n, Narayana, Narayana-Lucas and Narayana-Perrin numbers (using initial conditions in (5)) can be expressed using Binet's formulas as

$$N_n=\frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$U_n=\alpha^n+\beta^n+\gamma^n,$$

and

$$H_n=\frac{(3+2\alpha)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)}+\frac{(3+2\beta)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)}+\frac{(3+2\gamma)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)},$$

respectively. Note that

$$H_n=\frac{3+2\alpha}{3+\alpha^2}\alpha^n+\frac{3+2\beta}{3+\beta^2}\beta^n+\frac{3+2\gamma}{3+\gamma^2}\gamma^n.$$

2. Generating Functions

Next, we give the ordinary generating function $\sum_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1.
Suppose that $f_{V_n}(x)=\sum_{n=0}^{\infty} V_n x^n$ is the ordinary generating function of the generalized Narayana sequence $\{V_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} V_n x^n$ is given by

$$\sum_{n=0}^{\infty} V_n x^n=\frac{V_0+V_1x+(V_2-V_0)x^2}{1-x-x^3}. \tag{12}$$

Proof: Using the definition of generalized Narayana numbers, and substracting $x\sum_{n=0}^{\infty} V_n x^n$ and $x^3\sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
(1-x-x^3)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - x\sum_{n=0}^{\infty} V_n x^n - x^3\sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=0}^{\infty} V_n x^{n+1} - \sum_{n=0}^{\infty} V_n x^{n+3} \\
&= \sum_{n=0}^{\infty} V_n x^n - \sum_{n=1}^{\infty} V_{n-1} x^{n} - \sum_{n=3}^{\infty} V_{n-3} x^{n} \\
&= (V_0 + V_1 x + V_2 x^2) - (V_0 x + V_1 x^2) + \sum_{n=3}^{\infty}(V_n - V_{n-1} - V_{n-3})x^n \\
&= V_0 + V_1 x + V_2 x^2 - V_0 x - V_1 x^2 \\
&= V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2}{1-x-x^3}.$$

The previous lemma gives the following results as particular examples.

Corollary 2.1.
Generated functions of Narayana, Narayana-Lucas and Narayana-Perrin numbers are

$$\sum_{n=0}^{\infty} N_n x^n = \frac{x}{1-x-x^3}$$

and

$$\sum_{n=0}^{\infty} U_n x^n = \frac{3-2x}{1-x-x^3}$$

and

$$\sum_{n=0}^{\infty} H_n x^n = \frac{3-3x+2x^2}{1-x-x^3}$$

respectively.

3. Obtaining Binet Formula From Generating Function

We next find Binet formula of generalized Narayana numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3.1.
(Binet formula of generalized Narayana numbers)

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)} \tag{13}$$

where

$$\begin{aligned}
d_1 &= V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1), \\
d_2 &= V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1), \\
d_3 &= V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1).
\end{aligned}$$

Proof: Let

$$h(x) = 1 - x - x^3.$$

Then for some α, β and γ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)$$

i.e.,

$$1 - x - x^3 = (1-\alpha x)(1-\beta x)(1-\gamma x) \tag{14}$$

Hence $\frac{1}{\alpha}, \frac{1}{\beta}$, ve $\frac{1}{\gamma}$ are the roots of $h(x)$. This gives α, β, and γ as the roots of

$$h(\frac{1}{x}) = 1 - \frac{1}{x} - \frac{1}{x^3} = 0.$$

This implies $x^3 - x^2 - 1 = 0$. Now, by (12) and (14), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)}.$$

Then we write

$$\frac{V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2}{(1-\alpha x)(1-\beta x)(1-\gamma x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)}. \tag{15}$$

So

$$V_0 + (V_1 - V_0)x + (V_2 - V_1)x^2 = A_1(1-\beta x)(1-\gamma x) + A_2(1-\alpha x)(1-\gamma x) + A_3(1-\alpha x)(1-\beta x).$$

If we consider $x = \frac{1}{\alpha}$, we get $V_0 + (V_1 - V_0)\frac{1}{\alpha} + (V_2 - V_1)\frac{1}{\alpha^2} = A_1(1 - \frac{\beta}{\alpha})(1 - \frac{\gamma}{\alpha})$. This gives

$$A_1 = \frac{\alpha^2(V_0 + (V_1 - V_0)\frac{1}{\alpha} + (V_2 - V_1)\frac{1}{\alpha^2})}{(\alpha-\beta)(\alpha-\gamma)} = \frac{V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1)}{(\alpha-\beta)(\alpha-\gamma)}.$$

Similarly, we obtain

$$A_2 = \frac{V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1)}{(\beta-\alpha)(\beta-\gamma)}, A_3 = \frac{V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1)}{(\gamma-\alpha)(\gamma-\beta)}.$$

Thus (15) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n$$

where

$$\begin{aligned}
A_1 &= \frac{V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1)}{(\alpha-\beta)(\alpha-\gamma)}, \\
A_2 &= \frac{V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1)}{(\beta-\alpha)(\beta-\gamma)}, \\
A_3 &= \frac{V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1)}{(\gamma-\alpha)(\gamma-\beta)}.
\end{aligned}$$

and then we get (13).

Note that from (5) and (13) we have

$$\begin{aligned}
V_2 - (\beta+\gamma)V_1 + \beta\gamma V_0 &= V_0\alpha^2 + (V_1 - V_0)\alpha + (V_2 - V_1), \\
V_2 - (\alpha+\gamma)V_1 + \alpha\gamma V_0 &= V_0\beta^2 + (V_1 - V_0)\beta + (V_2 - V_1), \\
V_2 - (\alpha+\beta)V_1 + \alpha\beta V_0 &= V_0\gamma^2 + (V_1 - V_0)\gamma + (V_2 - V_1).
\end{aligned}$$

Next, using Theorem 3.1, we present the Binet formulas of Narayana, Narayana-Lucas and Narayana-Perrin sequences.

Corollary 3.1.
Binet formulas of Narayana, Narayana-Lucas and Narayana-Perrin sequences are

$$N_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)},$$

and

$$U_n = \alpha^n + \beta^n + \gamma^n,$$

and

$$H_n = \frac{(3+2\alpha)\alpha^{n-1}}{(\alpha-\beta)(\alpha-\gamma)} + \frac{(3+2\beta)\beta^{n-1}}{(\beta-\alpha)(\beta-\gamma)} + \frac{(3+2\gamma)\gamma^{n-1}}{(\gamma-\alpha)(\gamma-\beta)}$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [15]. Take $k = i = 3$ in Corollary 3.1 in [15]. Let

$$\begin{aligned}
\Lambda &= \begin{pmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{pmatrix}, \\
\Lambda_2 &= \begin{pmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{pmatrix}, \Lambda_3 = \begin{pmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{pmatrix}.
\end{aligned}$$

Then the Binet formula for Narayana numbers is

$$\begin{aligned}
N_n &= \frac{1}{\det(\Lambda)} \sum_{j=1}^{3} N_{4-j} \det(\Lambda_j) = \frac{1}{\det(\Lambda)} (N_3 \det(\Lambda_1) + N_2 \det(\Lambda_2) + N_1 \det(\Lambda_3)) \\
&= \frac{1}{\det(\Lambda)} (\det(\Lambda_1) + \det(\Lambda_2) + \det(\Lambda_3)) \\
&= \left(\begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}
\end{aligned}$$

Similarly, we obtain the Binet formula for Narayana-Lucas and Narayana-Perrin numbers as

$$\begin{aligned}
Q_n &= \frac{1}{\Lambda} (Q_3 \det(\Lambda_1) + Q_2 \det(\Lambda_2) + Q_1 \det(\Lambda_3)) \\
&= \left(4 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} + \begin{vmatrix} \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^2 & \beta & \beta^{n-1} \\ \gamma^2 & \gamma & \gamma^{n-1} \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}
\end{aligned}$$

and

$$\begin{aligned}
E_n &= \frac{1}{\Lambda} (E_1 \det(\Lambda_1) + E_2 \det(\Lambda_2) + E_1 \det(\Lambda_3)) \\
&= \left(5 \begin{vmatrix} \alpha^{n-1} & \alpha & 1 \\ \beta^{n-1} & \beta & 1 \\ \gamma^{n-1} & \gamma & 1 \end{vmatrix} + 2 \begin{vmatrix} \alpha^2 & \alpha^{n-1} & 1 \\ \beta^2 & \beta^{n-1} & 1 \\ \gamma^2 & \gamma^{n-1} & 1 \end{vmatrix} \right) / \begin{vmatrix} \alpha^2 & \alpha & 1 \\ \beta^2 & \beta & 1 \\ \gamma^2 & \gamma & 1 \end{vmatrix}
\end{aligned}$$

respectively.

4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following Theorem gives generalization of this result to the generalized Narayana sequence $\{V_n\}_{n\geq 0}$.

Theorem 4.1 (Simson Formula of Generalized Narayana Numbers).
For all integers n, we have

$$\begin{vmatrix} V_{n+2} & V_{n+1} & V_n \\ V_{n+1} & V_n & V_{n-1} \\ V_n & V_{n-1} & V_{n-2} \end{vmatrix} = \begin{vmatrix} V_2 & V_1 & V_0 \\ V_1 & V_0 & V_{-1} \\ V_0 & V_{-1} & V_{-2} \end{vmatrix}. \tag{16}$$

Proof: (16) is given in Soykan [28].

The previous theorem gives the following results as particular examples.

Corollary 4.1.
For all integers n, Simson formula of Narayana, Narayana-Lucas and Narayana-Perrin numbers are given as

$$\begin{vmatrix} N_{n+2} & N_{n+1} & N_n \\ N_{n+1} & N_n & N_{n-1} \\ N_n & N_{n-1} & N_{n-2} \end{vmatrix} = -1,$$

and

$$\begin{vmatrix} U_{n+2} & U_{n+1} & U_n \\ U_{n+1} & U_n & U_{n-1} \\ U_n & U_{n-1} & U_{n-2} \end{vmatrix} = -31,$$

and

$$\begin{vmatrix} H_{n+2} & H_{n+1} & H_n \\ H_{n+1} & H_n & H_{n-1} \\ H_n & H_{n-1} & H_{n-2} \end{vmatrix} = -53,$$

respectively.

5. Some Identities

In this section, we obtain some identities of Narayana, Narayana-Lucas and Narayana-Perrin numbers. First, we can give a few basic relations between $\{U_n\}$ and $\{N_n\}$.

Lemma 5.1.
The following equalities are true:

$$\begin{aligned} U_n &= 3N_{n+4} - 5N_{n+3} + 2N_{n+2}, \\ U_n &= -2N_{n+3} + 2N_{n+2} + 3N_{n+1}, \\ U_n &= 3N_{n+1} - 2N_n, \\ U_n &= N_n + 3N_{n-2}, \end{aligned} \tag{17}$$

and

$$\begin{aligned} 31N_n &= -3U_{n+4} + U_{n+3} + 11U_{n+2}, \\ 31N_n &= -2U_{n+3} + 11U_{n+2} - 3U_{n+1}, \\ 31N_n &= 9U_{n+2} - 3U_{n+1} - 2U_n, \\ 31N_n &= 6U_{n+1} - 2U_n + 9U_{n-1}, \\ 31N_n &= 4U_n + 9U_{n-1} + 6U_{n-2}. \end{aligned}$$

Proof: Note that all the identities hold for all integers n. We prove (17). To show (17), writing

$$U_n = a \times N_{n+4} + b \times N_{n+3} + c \times N_{n+2}$$

and solving the system of equations

$$\begin{aligned} U_0 &= a \times N_4 + b \times N_3 + c \times N_2 \\ U_1 &= a \times N_5 + b \times N_4 + c \times N_3 \\ U_2 &= a \times N_6 + b \times N_5 + c \times N_4 \end{aligned}$$

we find that $a = 3, b = -5, c = 2$. The other equalities can be proved similarly.

Note that all the identities in the above lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{H_n\}$ and $\{N_n\}$.

Lemma 5.2.
The following equalities are true:

$$\begin{aligned} H_n &= N_{n+4} - 4N_{n+3} + 5N_{n+2}, \\ H_n &= -3N_{n+3} + 5N_{n+2} + N_{n+1}, \\ H_n &= 2N_{n+2} + N_{n+1} - 3N_n, \\ H_n &= 3N_{n+1} - 3N_n + 2N_{n-1}, \\ H_n &= 2N_{n-1} + 3N_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 53N_n &= 4H_{n+4} - 10H_{n+3} + 15H_{n+2}, \\ 53N_n &= -6H_{n+3} + 15H_{n+2} + 4H_{n+1}, \\ 53N_n &= 9H_{n+2} + 4H_{n+1} - 6H_n, \\ 53N_n &= 13H_{n+1} - 6H_n + 9H_{n-1}, \\ 53N_n &= 7H_n + 9H_{n-1} + 13H_{n-2}. \end{aligned}$$

Thirdly, we give a few basic relations between $\{U_n\}$ and $\{H_n\}$.

Lemma 5.3.
The following equalities are true:

$$\begin{aligned} 53U_n &= -26H_{n+4} + 65H_{n+3} - 18H_{n+2}, \\ 53U_n &= 39H_{n+3} - 18H_{n+2} - 26H_{n+1}, \\ 53U_n &= 21H_{n+2} - 26H_{n+1} + 39H_n, \\ 53U_n &= -5H_{n+1} + 39H_n + 21H_{n-1}, \\ 53U_n &= 34H_n + 21H_{n-1} - 5H_{n-2}, \end{aligned}$$

and

$$\begin{aligned} 31H_n &= 25U_{n+4} + 2U_{n+3} - 40U_{n+2}, \\ 31H_n &= 27U_{n+3} - 40U_{n+2} + 25U_{n+1}, \\ 31H_n &= -13 \times U_{n+2} + 25U_{n+1} + 27U_n, \\ 31H_n &= 12U_{n+1} + 27U_n - 13U_{n-1}, \\ 31H_n &= 39U_n - 13U_{n-1} + 12U_{n-2}. \end{aligned}$$

We now present a few special identities for the Narayana-Lucas sequence $\{U_n\}$.

Theorem 5.1.
(Catalan's identity) For all integers n and m, the following identity holds

$$U_{n+m}U_{n-m} - U_n^2 = (N_{n+m} + 3N_{n+m-2})(N_{n-m} + 3N_{n-m-2}) - (N_n + 3N_{n-2})^2.$$

Proof: We use the identity

$$U_n = N_n + 3N_{n-2}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the Narayana-Lucas sequnce.

Corollary 5.1.
(Cassini's identity) For all integers numbers n and m, the following identity holds

$$U_{n+1}U_{n-1} - U_n^2 = (N_{n+1} + 3N_{n-1})(N_{n-1} + 3N_{n-3}) - (N_n + 3N_{n-2})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $U_n = N_n + 3N_{n-2}$. The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham's identities of Narayana-Lucas sequence $\{U_n\}$.

Theorem 5.2.
Let n and m be any integers. Then the following identities are true:

(a) *(d'Ocagne's identity)*

$$U_{m+1}U_n - U_mU_{n+1} = (N_{m+1}+3N_{m-1})(N_n+3N_{n-2}) - (N_m+3N_{m-2})(N_{n+1}+3N_{n-1}).$$

(b) *(Gelin-Cesàro's identity)*

$$U_{n+2}U_{n+1}U_{n-1}U_{n-2} - U_n^4 = (N_{n+2}+3N_n)(N_{n+1}+3N_{n-1})(N_{n-1}+3N_{n-3})(N_{n-2}+3N_{n-4}) - (N_n+3N_{n-2})^4.$$

(c) *(Melham's identity)*

$$U_{n+1}U_{n+2}U_{n+6} - U_{n+3}^3 = (N_{n+1}+3N_{n-1})(N_{n+2}+3N_n)(N_{n+6}+3N_{n+4}) - (N_{n+3}+3N_{n+1})^3.$$

Proof: Use the identity $U_n = N_n + 3N_{n-2}$.

6. Linear Sums

The following proposition presents some formulas of generalized Narayana numbers with positive subscripts.

Proposition 6.1.
If $r=1, s=0, t=1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} V_k = V_{n+3} - V_2$.

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{3}(V_{2n+2} + V_{2n+1} + 2V_{2n} - V_2 - V_1 + V_0)$.

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{3}(2V_{2n+2} + 2V_{2n+1} + V_{2n} - 2V_2 + V_1 - V_0)$.

Proof: It is given in [26], see also [27].

From the last proposition, we have the following corollary which presents sum formulas of Narayana numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 6.1.
For $n \geq 0$, Narayana numbers have the following properties.

(a) $\sum_{k=0}^{n} N_k = N_{n+3} - 1$.

(b) $\sum_{k=0}^{n} N_{2k} = \frac{1}{3}(N_{2n+2} + N_{2n+1} + 2N_{2n} - 2)$.

(c) $\sum_{k=0}^{n} N_{2k+1} = \frac{1}{3}(2N_{2n+2} + 2N_{2n+1} + N_{2n} - 1)$.

Taking $V_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 6.2.
For $n \geq 0$, Narayana-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n} U_k = U_{n+3} - 1$.

(b) $\sum_{k=0}^{n} U_{2k} = \frac{1}{3}(U_{2n+2} + U_{2n+1} + 2U_{2n} + 1)$.

(c) $\sum_{k=0}^{n} U_{2k+1} = \frac{1}{3}(2U_{2n+2} + 2U_{2n+1} + U_{2n} - 4)$.

From the last proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $V_n = H_n$ with $H_0 = 3, H_1 = 0, H_2 = 2$).

Corollary 6.3.
For $n \geq 0$, Narayana-Perrin numbers have the following properties.

(a) $\sum_{k=0}^{n} H_k = H_{n+3} - 1.$

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{3}(H_{2n+2} + H_{2n+1} + 2H_{2n} - 2).$

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{3}(2H_{2n+2} + 2H_{2n+1} + H_{2n} - 1).$

The following proposition presents some formulas of generalized Narayana numbers with negative subscripts.

Proposition 6.2.
If $r = 1, s = 0, t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} V_{-k} = -2V_{-n-1} - V_{-n-2} - V_{-n-3} + V_2.$

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{3}(-2V_{-2n+1} + V_{-2n} - V_{-2n-1} + V_2 + V_1 - V_0).$

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{3}(-V_{-2n+1} - V_{-2n} - 2V_{-2n-1} + 2V_2 - V_1 + V_0).$

Proof: It is given in [26], see also [27].

From the above proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $V_n = N_n$ with $N_0 = 0, N_1 = 1, N_2 = 1$).

Corollary 6.4.
For $n \geq 1$, Narayana numbers have the following properties.

(a) $\sum_{k=1}^{n} N_{-k} = -2N_{-n-1} - N_{-n-2} - N_{-n-3} + 1.$

(b) $\sum_{k=1}^{n} N_{-2k} = \frac{1}{3}(-2N_{-2n+1} + N_{-2n} - N_{-2n-1} + 2).$

(c) $\sum_{k=1}^{n} N_{-2k+1} = \frac{1}{3}(-N_{-2n+1} - N_{-2n} - 2N_{-2n-1} + 1).$

Taking $V_n = U_n$ with $U_0 = 3, U_1 = 1, U_2 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Narayana-Lucas numbers.

Corollary 6.5.
For $n \geq 1$, Narayana-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} U_{-k} = -2U_{-n-1} - U_{-n-2} - U_{-n-3} + 1$

(b) $\sum_{k=1}^{n} U_{-2k} = \frac{1}{3}(-2U_{-2n+1} + U_{-2n} - U_{-2n-1} - 1)$

(c) $\sum_{k=1}^{n} U_{-2k+1} = \frac{1}{3}(-U_{-2n+1} - U_{-2n} - 2U_{-2n-1} + 4)$

From the above proposition, we have the following corollary which gives sum formulas of Narayana-Perrin numbers (take $V_n = H_n$ with $H_0 = 2, H_1 = 0, H_2 = 2$).

Corollary 6.6.
For $n \geq 1$, Narayana-Perrin numbers have the following properties.

(a) $\sum_{k=1}^{n} H_{-k} = -2H_{-n-1} - H_{-n-2} - H_{-n-3} + 2$

(b) $\sum_{k=1}^{n} H_{-2k} = \frac{1}{3}(-2H_{-2n+1} + H_{-2n} - H_{-2n-1} - 1)$

(c) $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{3}(-H_{-2n+1} - H_{-2n} - 2H_{-2n-1} + 7)$

7. Matrices related with Generalized Narayana numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{18}$$

For matrix formulation (18), see [14]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ r & s & t \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \end{pmatrix}.$$

We define the square matrix A of order 3 as:

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. From (4) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix} \tag{19}$$

and from (18) (or using (19) and induction) we have

$$\begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V_n = N_n$ in (19) we have

$$\begin{pmatrix} N_{n+2} \\ N_{n+1} \\ N_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} N_{n+1} \\ N_n \\ N_{n-1} \end{pmatrix}. \tag{20}$$

We also define

$$B_n = \begin{pmatrix} N_{n+1} & N_{n-1} & N_n \\ N_n & N_{n-2} & N_{n-1} \\ N_{n-1} & N_{n-3} & N_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix}.$$

Theorem 7.1.
For all integer $m, n \geq 0$, we have

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof:

(a) By expanding the vectors on the both sides of (20) to 3-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \\ V_{n-2} & V_{n-4} & V_{n-3} \end{pmatrix} \\ &= \begin{pmatrix} V_{n+1} & V_{n-1} & V_n \\ V_n & V_{n-2} & V_{n-1} \\ V_{n-1} & V_{n-3} & V_{n-2} \end{pmatrix} = C_n. \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^mC_1 = A^{n-1}C_1A^m = C_nB_m$$

and similarly

$$C_{n+m} = B_mC_n.$$

Some properties of matrix A^n can be given as

$$A^n = A^{n-1} + A^{n-3}$$

and

$$A^{n+m} = A^nA^m = A^mA^n$$

and

$$\det(A^n) = 1$$

for all integer m and n.

Theorem 7.2.
For $m, n \geq 0$ we have

$$V_{n+m} = V_nN_{m+1} + V_{n-1}N_{m-1} + V_{n-2}N_m \tag{21}$$

Proof: From the equation $C_{n+m} = C_nB_m = B_mC_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and C_nB_m. This completes the proof.

Remark 7.1.
By induction, it can be proved that for all integers $m, n \leq 0$, (21) holds. So for all integers m, n, (21) is true.

Corollary 7.1.
For all integers m, n, we have

$$\begin{aligned} N_{n+m} &= N_nN_{m+1} + N_{n-1}N_{m-1} + N_{n-2}N_m, && (22) \\ U_{n+m} &= U_nN_{m+1} + U_{n-1}N_{m-1} + U_{n-2}N_m, && (23) \\ H_{n+m} &= H_nN_{m+1} + H_{n-1}N_{m-1} + H_{n-2}N_m. && (24) \end{aligned}$$

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