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Generalized Tribonacci Numbers: Summing Formulas

Research Article

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Abstract: In this paper, closed forms of the summation formulas for generalized Tribonacci numbers are presented. As special cases, we give summation formulas of Tribonacci, Tribonacci-Lucas, Padovan, Perrin, Narayana and some other third order recurrance sequences.

MSC: 11B37 • 11B39 • 11B83

Keywords: Tribonacci numbers • Padovan numbers • Perrin numbers • Narayana numbers • Sum formulas

1. Introduction

The generalized Tribonacci sequence $\{W_n(W_0, W_1, W_2; r, s, t)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) (as a third-order generalization of Fibonacci numbers) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}, \quad W_0 = a, W_1 = b, W_2 = c, \ \ n \geq 3 \tag{1}$$

where W_0, W_1, W_2 are arbitrary complex numbers and r, s, t are real numbers.

This sequence has been studied by many authors, see for example [1],[2],[3],[4],[5],[12],[14],[15],[16],[20],[29],[31],[32]. See also [7],[8],[30] for some work on second-order generalization of Fibonacci numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ when $t \neq 0$. Therefore, recurrence (1) holds for all integer n.

If we set $r = s = t = 1$ and $W_0 = 0, W_1 = 1, W_2 = 1$ then $\{W_n\}$ is the well-known Tribonacci sequence and if we set $r = s = t = 1$ and $W_0 = 3, W_1 = 1, W_2 = 3$ then $\{W_n\}$ is the well-known Tribonacci-Lucas sequence.

In fact, the generalized Tribonacci sequence is the generalization of the well-known sequences like Tribonacci, Tribonacci-Lucas, Padovan (Cordonnier), Perrin, Padovan-Perrin, Narayana, third order Jacobsthal and third order Jacobsthal-Lucas. In literature, for example, the following names and notations (see Table 1) are used for the special case of r, s, t and initial values. Note that the sequence $\{C_n\}$ is't in the database of http://oeis.org [[17], yet.

In this work, we investigate summation formulas of generalized Tribonacci numbers. We present some works on summing formulas of the numbers in the Table 2.

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Table 1. A few special case of generalized Tribonacci sequences

Sequences (Numbers)	Notation	OEIS [17]
Tribonacci	$\{T_n\} = \{W_n(0,1,1;1,1,1)\}$	A000073, A057597
Tribonacci-Lucas	$\{K_n\} = \{W_n(3,1,3;1,1,1)\}$	A001644, A073145
third order Pell	$\{P_n^{(3)}\} = \{W_n(0,1,2;2,1,1)\}$	A077939, A077978
third order Pell-Lucas	$\{Q_n^{(3)}\} = \{W_n(3,2,6;2,1,1)\}$	A276225, A276228
third order modified Pell	$\{E_n^{(3)}\} = \{W_n(0,1,1;2,1,1)\}$	A077997, A078049
Padovan (Cordonnier)	$\{P_n\} = \{W_n(1,1,1;0,1,1)\}$	A000931
Perrin (Padovan-Lucas)	$\{E_n\} = \{W_n(3,0,2;0,1,1)\}$	A001608, A078712
Padovan-Perrin	$\{S_n\} = \{W_n(0,0,1;0,1,1)\}$	A000931, A176971
Pell-Padovan	$\{R_n\} = \{W_n(1,1,1;0,2,1)\}$	A066983, A128587
Pell-Perrin	$\{C_n\} = \{W_n(3,0,2;0,2,1)\}$	-
Jacobsthal-Padovan	$\{Q_n\} = \{W_n(1,1,1;0,1,2)\}$	A159284
Jacobsthal-Perrin (-Lucas)	$\{L_n\} = \{W_n(3,0,2;0,1,2)\}$	A072328
Narayana	$\{N_n\} = \{W_n(0,1,1;1,0,1)\}$	A078012
third order Jacobsthal	$\{J_n^{(3)}\} = \{W_n(0,1,1;1,1,2)\}$	A077947
third order Jacobsthal-Lucas	$\{j_n^{(3)}\} = \{W_n(2,1,5;1,1,2)\}$	A226308

Table 2. A few special study of sum formulas

Name of sequence	Papers which deal with summing formulas
Pell and Pell-Lucas	[8],[10],[11]
Generalized Fibonacci	[9],[22],[23],[21]
Generalized Tribonacci	[6],[13],[18],[19]
Generalized Tetranacci	[24],[25],[33]
Generalized Pentanacci	[26],[27]
Generalized Hexanacci	[28]

2. Sum formulas of Generalized Tribonacci Numbers with Positive Subscripts

The following theorem presents some summing formulas of generalized Tribonacci numbers with positive subscripts.

Theorem 2.1.
Let x be a complex number. For $n \geq 0$, we have the following formulas:

(a) *If $tx^3 + sx^2 + rx - 1 \neq 0$, then*

$$\sum_{k=0}^{n} x^k W_k = \frac{\Omega_1}{tx^3 + sx^2 + rx - 1}$$

where

$$\begin{aligned}\Omega_1 = {} & x^{n+3}W_{n+3} - (rx-1)\,x^{n+2}W_{n+2} - (sx^2+rx-1)x^{n+1}W_{n+1} \\ & -x^2W_2 + x\,(rx-1)\,W_1 + (sx^2+rx-1)W_0.\end{aligned}$$

(b) *If $r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{\Omega_2}{r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1}$$

where

$$\begin{aligned}\Omega_2 = {} & -(sx-1)\,x^{n+1}W_{2n+2} + (t+rs)\,x^{n+2}W_{2n+1} + t\,(r+tx)\,x^{n+2}W_{2n} \\ & +x\,(sx-1)\,W_2 - (t+rs)\,x^2W_1 + \left(r^2x - s^2x^2 + 2sx + rtx^2 - 1\right)W_0.\end{aligned}$$

(c) *If $r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\Omega_3}{r^2x - s^2x^2 + t^2x^3 + 2sx + 2rtx^2 - 1}$$

where

$$\begin{aligned}\Omega_3 = {} & (r+tx)\,x^{n+1}W_{2n+2} + \left(s - s^2x + t^2x^2 + rtx\right)x^{n+1}W_{2n+1} \\ & -t\,(sx-1)\,x^{n+1}W_{2n} - x\,(r+tx)\,W_2 + \left(r^2x + sx + rtx^2 - 1\right)W_1 + tx\,(sx-1)\,W_0.\end{aligned}$$

Proof:

(a) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$$

i.e.

$$tW_{n-3} = W_n - rW_{n-1} - sW_{n-2}$$

we obtain

$$\begin{aligned}
tx^0W_0 &= x^0W_3 - rx^0W_2 - sx^0W_1 \\
tx^1W_1 &= x^1W_4 - rx^1W_3 - sx^1W_2 \\
tx^2W_2 &= x^2W_5 - rx^2W_4 - sx^2W_3 \\
tx^3W_3 &= x^3W_6 - rx^3W_5 - sx^3W_4 \\
&\vdots \\
tx^{n-3}W_{n-3} &= x^{n-3}W_n - rx^{n-3}W_{n-1} - sx^{n-3}W_{n-2} \\
tx^{n-2}W_{n-2} &= x^{n-2}W_{n+1} - rx^{n-2}W_n - sx^{n-2}W_{n-1} \\
tx^{n-1}W_{n-1} &= x^{n-1}W_{n+2} - rx^{n-1}W_{n+1} - sx^{n-1}W_n \\
tx^nW_n &= x^nW_{n+3} - rx^nW_{n+2} - sx^nW_{n+1}.
\end{aligned}$$

If we add the equations by side by, we get (a)

(b) and (c) Using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we obtain

$$\begin{aligned}
rx^1W_3 &= x^1W_4 - sx^1W_2 - tx^1W_1 \\
rx^2W_5 &= x^2W_6 - sx^2W_4 - tx^2W_3 \\
rx^3W_7 &= x^3W_8 - sx^3W_6 - tx^3W_5 \\
rx^4W_9 &= x^4W_{10} - sx^4W_8 - tx^4W_7 \\
&\vdots \\
rx^{n-1}W_{2n-1} &= x^{n-1}W_{2n} - sx^{n-1}W_{2n-2} - tx^{n-1}W_{2n-3} \\
rx^nW_{2n+1} &= x^nW_{2n+2} - sx^nW_{2n} - tx^nW_{2n-1}.
\end{aligned}$$

Now, if we add the above equations by side by, we get

$$\begin{aligned}
r(-x^0W_1 + \sum_{k=0}^{n} x^kW_{2k+1}) &= (x^nW_{2n+2} - x^0W_2 - x^{-1}W_0 + \sum_{k=0}^{n} x^{k-1}W_{2k}) \\
&\quad - s(-x^0W_0 + \sum_{k=0}^{n} x^kW_{2k}) - t(-x^{n+1}W_{2n+1} + \sum_{k=0}^{n} x^{k+1}W_{2k+1}).
\end{aligned} \tag{2}$$

Similarly, using the recurrence relation

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3}$$

i.e.

$$rW_{n-1} = W_n - sW_{n-2} - tW_{n-3}$$

we write the following obvious equations;

$$\begin{array}{rcl} rx^1W_2 & = & x^1W_3 - sx^1W_1 - tx^1W_0 \\ rx^2W_4 & = & x^2W_5 - sx^2W_3 - tx^2W_2 \\ rx^3W_6 & = & x^3W_7 - sx^3W_5 - tx^3W_4 \\ rx^4W_8 & = & x^4W_9 - sx^4W_7 - tx^4W_6 \\ rx^5W_{10} & = & x^5W_{11} - sx^5W_9 - tx^5W_8 \\ rx^6W_{12} & = & x^6W_{13} - sx^6W_{11} - tx^6W_{10} \\ & \vdots & \\ rx^{n-1}W_{2n-2} & = & x^{n-1}W_{2n-1} - sx^{n-1}W_{2n-3} - tx^{n-1}W_{2n-4} \\ rx^nW_{2n} & = & x^nW_{2n+1} - sx^nW_{2n-1} - tx^nW_{2n-2} \end{array}$$

Now, if we add the above equations by side by, we obtain

$$\begin{array}{rcl} r(-W_0 + \sum_{k=0}^{n} x^kW_{2k}) & = & (-W_1 + \sum_{k=0}^{n} x^kW_{2k+1}) - s(-x^{n+1}W_{2n+1} + \sum_{k=0}^{n} x^{k+1}W_{2k+1}) \\ & & -t(-x^{n+1}W_{2n} + \sum_{k=0}^{n} x^{k+1}W_{2k}). \end{array} \quad (3)$$

Then, solving the system (2)-(3), the required result of (b) and (c) follow.

2.1. The case $x = 1$

The case $x = 1$ of Theorem 2.1 is given in [18], see also [19]. In this subsection, we only consider the case $x = 1, r = 0, s = 2, t = 1$ and we present a theorem which is in different forms than given in [18]).

Observe that setting $x = 1, r = 0, s = 2, t = 1$ (i.e. for the generalized Pell-Padovan case) in Theorem 2.1 (b) and (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas. If $r = 0, s = 2, t = 1$ then we have the following theorem.

Theorem 2.2.
If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k = \frac{1}{2}(W_{n+3} + W_{n+2} - W_{n+1} - W_2 - W_1 + W_0)$.

(b) $\sum_{k=0}^{n} W_{2k} = (n+3)W_{2n+2} - (n+2)W_{2n+1} - (n+3)W_{2n} - 3W_2 + 2W_1 + 4W_0$.

(c) $\sum_{k=0}^{n} W_{2k+1} = -(n+2)W_{2n+2} + (n+3)W_{2n+1} + (n+3)W_{2n} + 2W_2 - 2W_1 - 3W_0$.

Proof:

(a) Taking $r = 0, s = 2, t = 1$ in Theorem 2.1 (a) we obtain (a).

(b) We use Theorem 2.1 (b). If we set $r = 0, s = 2, t = 1$ in Theorem 2.1 (b) then we have
$\sum_{k=0}^{n} x^kW_{2k} = \frac{(2x-1)x^{n+1}W_{2n+2} - x^{n+2}W_{2n+1} - x^{n+3}W_{2n} - x(2x-1)W_2 + x^2W_1 + (4x^2-4x+1)W_0}{-x^3+4x^2-4x+1}$.
For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{array}{rcl} \sum_{k=0}^{n} W_{2k} & = & \left. \frac{\frac{d}{dx}(f_1(x))}{\frac{d}{dx}(-x^3 + 4x^2 - 4x + 1)} \right|_{x=1} \\ & = & (n+3)W_{2n+2} - (n+2)W_{2n+1} - (n+3)W_{2n} - 3W_2 + 2W_1 + 4W_0 \end{array}$$

where

$$f_1(x) = (2x-1)x^{n+1}W_{2n+2} - x^{n+2}W_{2n+1} - x^{n+3}W_{2n} - x(2x-1)W_2 + x^2W_1 + \left(4x^2 - 4x + 1\right)W_0.$$

(c) We use Theorem 2.1 (c). If we set $r = 0, s = 2, t = 1$ in Theorem 2.1 (c) then we have
$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{-x^{n+2}W_{2n+2}-(x^2-4x+2)x^{n+1}W_{2n+1}+(2x-1)x^{n+1}W_{2n}+x^2W_2-(2x-1)W_1-x(2x-1)W_0}{-x^3+4x^2-4x+1}$.
For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}\sum_{k=0}^{n} W_{2k+1} &= \left.\frac{\frac{d}{dx}(f_2(x))}{\frac{d}{dx}(-x^3+4x^2-4x+1)}\right|_{x=1}\\ &= -(n+2)W_{2n+2}+(n+3)W_{2n+1}+(n+3)W_{2n}+2W_2-2W_1-3W_0\end{aligned}$$

where

$$f_2(x) = -x^{n+2}W_{2n+2} - \left(x^2-4x+2\right)x^{n+1}W_{2n+1} + (2x-1)x^{n+1}W_{2n} + x^2W_2 - (2x-1)W_1 - x(2x-1)W_0$$

From the above theorem we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R_1 = 1, R_2 = 1$).

Corollary 2.1.
For $n \geq 0$, Pell-Padovan numbers have the following property:

(a) $\sum_{k=0}^{n} R_k = \frac{1}{2}(R_{n+3} + R_{n+2} - R_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} R_{2k} = (n+3)R_{2n+2} - (n+2)R_{2n+1} - (n+3)R_{2n} + 3$.

(c) $\sum_{k=0}^{n} R_{2k+1} = -(n+2)R_{2n+2} + (n+3)R_{2n+1} + (n+3)R_{2n} - 3$.

Taking $W_n = C_n$ with $C_0 = 3, C_1 = 0, C_2 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 2.2.
For $n \geq 0$, Pell-Perrin numbers have the following property:

(a) $\sum_{k=0}^{n} C_k = \frac{1}{2}(C_{n+3} + C_{n+2} - C_{n+1} + 1)$.

(b) $\sum_{k=0}^{n} C_{2k} = (n+3)C_{2n+2} - (n+2)C_{2n+1} - (n+3)C_{2n} + 6$.

(c) $\sum_{k=0}^{n} C_{2k+1} = -(n+2)C_{2n+2} + (n+3)C_{2n+1} + (n+3)C_{2n} - 5$.

2.2. The case $x = -1$

We now consider the case $x = -1$ in Theorem 2.1.
Taking $x = -1, r = s = t = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.1.
If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{2}((-1)^n(W_{n+3} - 2W_{n+2} + W_{n+1}) + W_2 - 2W_1 + W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{2}((-1)^n W_{2n+2} - (-1)^n W_{2n+1} - W_2 + W_1 + 2W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{2}((-1)^n W_{2n+1} + (-1)^n W_{2n} + W_1 - W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 2.3.
For $n \geq 0$, Tribonacci numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^k T_k = \frac{1}{2}((-1)^n(T_{n+3} - 2T_{n+2} + T_{n+1}) - 1)$.

(b) $\sum_{k=0}^{n}(-1)^k T_{2k} = \frac{1}{2}(-1)^n(T_{2n+2} - T_{2n+1})$.

(c) $\sum_{k=0}^{n}(-1)^k T_{2k+1} = \frac{1}{2}((-1)^n T_{2n+1} + (-1)^n T_{2n} + 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 2.4.
For $n \geq 0$, Tribonacci-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^k K_k = \frac{1}{2}((-1)^n (K_{n+3} - 2K_{n+2} + K_{n+1}) + 4)$.

(b) $\sum_{k=0}^{n}(-1)^k K_{2k} = \frac{1}{2}((-1)^n K_{2n+2} - (-1)^n K_{2n+1} + 4)$.

(c) $\sum_{k=0}^{n}(-1)^k K_{2k+1} = \frac{1}{2}((-1)^n K_{2n+1} + (-1)^n K_{2n} - 2)$.

Taking $x = -1, r = 2, s = 1, t = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.2.
If $r = 2, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{3}((-1)^n (W_{n+3} - 3W_{n+2} + 2W_{n+1}) + W_2 - 3W_1 + 2W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{5}((-1)^n (2W_{2n+2} - 3W_{2n+1} - W_{2n}) - 2W_2 + 3W_1 + 6W_0)$.

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{5}((-1)^n (W_{2n+2} + W_{2n+1} + 2W_{2n}) - W_2 + 4W_1 - 2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1, P_2 = 2$).

Corollary 2.5.
For $n \geq 0$, third-order Pell numbers have the following properties:

(a) $\sum_{k=0}^{n}(-1)^k P_k = \frac{1}{3}((-1)^n (P_{n+3} - 3P_{n+2} + 2P_{n+1}) - 1)$.

(b) $\sum_{k=0}^{n}(-1)^k P_{2k} = \frac{1}{5}((-1)^n (2P_{2n+2} - 3P_{2n+1} - P_{2n}) - 1)$.

(c) $\sum_{k=0}^{n}(-1)^k P_{2k+1} = \frac{1}{5}((-1)^n (P_{2n+2} + P_{2n+1} + 2P_{2n}) + 2)$.

Taking $W_n = Q_n$ with $Q_0 = 3, Q_1 = 2, Q_2 = 6$ in the last proposition, we have the following corollary which presents sum formulas of third-order Pell-Lucas numbers.

Corollary 2.6.
For $n \geq 0$, third-order Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n}(-1)^k Q_k = \frac{1}{3}((-1)^n (Q_{n+3} - 3Q_{n+2} + 2Q_{n+1}) + 6)$.

(b) $\sum_{k=0}^{n}(-1)^k Q_{2k} = \frac{1}{5}((-1)^n (2Q_{2n+2} - 3Q_{2n+1} - Q_{2n}) + 12)$.

(c) $\sum_{k=0}^{n}(-1)^k Q_{2k+1} = \frac{1}{5}((-1)^n (Q_{2n+2} + Q_{2n+1} + 2Q_{2n}) - 4)$.

From the last proposition, we have the following corollary which gives sum formulas of third-order modified Pell numbers (take $W_n = E_n$ with $E_0 = 0, E_1 = 1, E_2 = 1$).

Corollary 2.7.
For $n \geq 0$, third-order modified Pell numbers have the following properties:

(a) $\sum_{k=0}^{n}(-1)^k E_k = \frac{1}{3}((-1)^n (E_{n+3} - 3E_{n+2} + 2E_{n+1}) - 2)$.

(b) $\sum_{k=0}^{n}(-1)^k E_{2k} = \frac{1}{5}((-1)^n (2E_{2n+2} - 3E_{2n+1} - E_{2n}) + 1)$.

(c) $\sum_{k=0}^{n}(-1)^k E_{2k+1} = \frac{1}{5}((-1)^n (E_{2n+2} + E_{2n+1} + 2E_{2n}) + 3)$.

Taking $x = -1, r = 0, s = 1, t = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.3.
If $r = 0, s = 1, t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k = (-1)^n (W_{n+3} - W_{n+2}) + W_2 - W_1.$

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{5}((-1)^n (2W_{2n+2} - W_{2n+1} + W_{2n}) - 2W_2 + W_1 + 4W_0).$

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{5}((-1)^n (-W_{2n+2} + 3W_{2n+1} + 2W_{2n}) + W_2 + 2W_1 - 2W_0).$

From the last proposition, we have the following corollary which gives sum formulas of Padovan numbers (take $W_n = P_n$ with $P_0 = 1, P = 1, P_2 = 1$).

Corollary 2.8.
For $n \geq 0$, Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^k P_k = (-1)^n (P_{n+3} - P_{n+2}).$

(b) $\sum_{k=0}^{n}(-1)^k P_{2k} = \frac{1}{5}((-1)^n (2P_{2n+2} - P_{2n+1} + P_{2n}) + 3).$

(c) $\sum_{k=0}^{n}(-1)^k P_{2k+1} = \frac{1}{5}((-1)^n (-P_{2n+2} + 3P_{2n+1} + 2P_{2n}) + 1).$

Taking $W_n = E_n$ with $E_0 = 3, E_2 = 0, E_2 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Perrin numbers.

Corollary 2.9.
For $n \geq 0$, Perrin numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^k E_k = (-1)^n (E_{n+3} - E_{n+2}) + 2.$

(b) $\sum_{k=0}^{n}(-1)^k E_{2k} = \frac{1}{5}((-1)^n (2E_{2n+2} - E_{2n+1} + E_{2n}) + 8).$

(c) $\sum_{k=0}^{n}(-1)^k E_{2k+1} = \frac{1}{5}((-1)^n (-E_{2n+2} + 3E_{2n+1} + 2E_{2n}) - 4).$

Taking $W_n = S_n$ with $S_0 = 0, S_2 = 0, S_2 = 1$ in the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers.

Corollary 2.10.
For $n \geq 0$, Padovan-Perrin numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^k S_k = (-1)^n (S_{n+3} - S_{n+2}) + 1.$

(b) $\sum_{k=0}^{n}(-1)^k S_{2k} = \frac{1}{5}((-1)^n (2S_{2n+2} - S_{2n+1} + S_{2n}) - 2).$

(c) $\sum_{k=0}^{n}(-1)^k S_{2k+1} = \frac{1}{5}((-1)^n (-S_{2n+2} + 3S_{2n+1} + 2S_{2n}) + 1).$

If $x = -1, r = 0, s = 2, t = 1$ then $tx^3 + sx^2 + rx - 1 = 0$ so we can't use Theorem 2.1 (a) directly. Observe that setting $x = -1, r = 0, s = 2, t = 1$ (i.e. for the generalized Pell-Padovan case) in Theorem 2.1 (a) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule provides the evaluation of the sum formulas. If $r = 0, s = 2, t = 1$ then we have the following theorem.

Theorem 2.3.
If $r = 0, s = 2, t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k = (-1)^n (-(n+3) W_{n+3} + (n+2) W_{n+2} + (n+5) W_{n+1}) - 2W_2 + W_1 + 4W_0.$

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{10}(3(-1)^n W_{2n+2} - (-1)^n W_{2n+1} + (-1)^n W_{2n} - 3W_2 + W_1 + 9W_0).$

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{10}((-1)^n (-W_{2n+2} + 7W_{2n+1} + 3W_{2n}) + 3W_1 - 3W_0 + W_2).$

Proof:

(a) We use Theorem 2.1 (a). If we set $r=0, s=2, t=1$ in Theorem 2.1 (a) then we have

$$\sum_{k=0}^{n} x^k W_k = \frac{x^{n+3}W_{n+3}+x^{n+2}W_{n+2}-x^{n+1}\left(2x^2-1\right)W_{n+1}-x^2W_2-xW_1+\left(2x^2-1\right)W_0}{x^3+2x^2-1}.$$

For $x=-1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k W_k &= \left.\frac{\frac{d}{dx}(x^{n+3}W_{n+3}+x^{n+2}W_{n+2}-x^{n+1}\left(2x^2-1\right)W_{n+1}-x^2W_2-xW_1+\left(2x^2-1\right)W_0)}{\frac{d}{dx}(x^3+2x^2-1)}\right|_{x=-1} \\ &= (-1)^n\left(-(n+3)W_{n+3}+(n+2)W_{n+2}+(n+5)W_{n+1}\right)-2W_2+W_1+4W_0.\end{aligned}$$

(b) Taking $r=0, s=2, t=1$ in Theorem 2.1 (b) we obtain (b).

(c) Taking $r=0, s=2, t=1$ in Theorem 2.1 (c) we obtain (c).

From the above theorem we have the following corollary which gives sum formulas of Pell-Padovan numbers (take $W_n=R_n$ with $R_0=1, R_1=1, R_2=1$).

Corollary 2.11.
For $n\geq 0$, Pell-Padovan numbers have the following property:

(a) $\sum_{k=0}^{n}(-1)^k R_k = (-1)^n\left(-(n+3)R_{n+3}+(n+2)R_{n+2}+(n+5)R_{n+1}\right)+3.$

(b) $\sum_{k=0}^{n}(-1)^k R_{2k} = \frac{1}{10}(3(-1)^n R_{2n+2}-(-1)^n R_{2n+1}+(-1)^n R_{2n}+7).$

(c) $\sum_{k=0}^{n}(-1)^k R_{2k+1} = \frac{1}{10}((-1)^n(-R_{2n+2}+7R_{2n+1}+3R_{2n})+1).$

Taking $W_n=C_n$ with $C_0=3, C_1=0, C_2=2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Perrin numbers.

Corollary 2.12.
For $n\geq 0$, Pell-Perrin numbers have the following property:

(a) $\sum_{k=0}^{n}(-1)^k C_k = (-1)^n\left(-(n+3)C_{n+3}+(n+2)C_{n+2}+(n+5)C_{n+1}\right)+8.$

(b) $\sum_{k=0}^{n}(-1)^k C_{2k} = \frac{1}{10}(3(-1)^n C_{2n+2}-(-1)^n C_{2n+1}+(-1)^n C_{2n}+21).$

(c) $\sum_{k=0}^{n}(-1)^k C_{2k+1} = \frac{1}{10}((-1)^n(-C_{2n+2}+7C_{2n+1}+3C_{2n})-7).$

Taking $x=-1, r=0, s=1, t=2$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.4.
If $r=0, s=1, t=2$ then for $n\geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k = \frac{1}{2}((-1)^n W_{n+3}-(-1)^n W_{n+2}+W_2-W_1).$

(b) $\sum_{k=0}^{n}(-1)^k W_{2k} = \frac{1}{4}((-1)^n(W_{2n+2}-W_{2n+1}+2W_{2n})-W_2+W_1+2W_0).$

(c) $\sum_{k=0}^{n}(-1)^k W_{2k+1} = \frac{1}{4}((-1)^n(-W_{2n+2}+3W_{2n+1}+2W_{2n})+W_2+W_1-2W_0)$

Taking $W_n=Q_n$ with $Q_0=1, Q_1=1, Q_2=1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Padovan numbers.

Corollary 2.13.
For $n\geq 0$, Jacobsthal-Padovan numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^k Q_k = \frac{1}{2}(-1)^n(Q_{n+3}-Q_{n+2}).$

(b) $\sum_{k=0}^{n}(-1)^{k}Q_{2k}=\frac{1}{4}((-1)^{n}(Q_{2n+2}-Q_{2n+1}+2Q_{2n})+2)$.

(c) $\sum_{k=0}^{n}(-1)^{k}Q_{2k+1}=\frac{1}{4}(-1)^{n}(-Q_{2n+2}+3Q_{2n+1}+2Q_{2n})$.

From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Perrin numbers (take $W_n=L_n$ with $L_0=3,L_1=0,L_2=2$).

Corollary 2.14.
For $n\geq 0$, Jacobsthal-Perrin numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^{k}L_{k}=\frac{1}{2}((-1)^{n}L_{n+3}-(-1)^{n}L_{n+2}+2)$.

(b) $\sum_{k=0}^{n}(-1)^{k}L_{2k}=\frac{1}{4}((-1)^{n}(L_{2n+2}-L_{2n+1}+2L_{2n})+4)$.

(c) $\sum_{k=0}^{n}(-1)^{k}L_{2k+1}=\frac{1}{4}((-1)^{n}(-L_{2n+2}+3L_{2n+1}+2L_{2n})-4)$.

Taking $x=-1,r=1,s=0,t=1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.5.
If $r=1,s=0,t=1$ then for $n\geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^{k}W_{k}=\frac{1}{3}((-1)^{n}(W_{n+3}-2W_{n+2}+2W_{n+1})+W_2-2W_1+2W_0)$.

(b) $\sum_{k=0}^{n}(-1)^{k}W_{2k}=(-1)^{n}(W_{2n+2}-W_{2n+1})-W_2+W_1+W_0$.

(c) $\sum_{k=0}^{n}(-1)^{k}W_{2k+1}=(-1)^{n}W_{2n}+W_1-W_0$.

From the last proposition, we have the following corollary which presents sum formulas of Narayana numbers (take $W_n=N_n$ with $N_0=0,N_1=1,N_2=1$).

Corollary 2.15.
For $n\geq 0$, Narayana numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^{k}N_{k}=\frac{1}{3}((-1)^{n}(N_{n+3}-2N_{n+2}+2N_{n+1})-1)$.

(b) $\sum_{k=0}^{n}(-1)^{k}N_{2k}=(-1)^{n}(N_{2n+2}-N_{2n+1})$.

(c) $\sum_{k=0}^{n}(-1)^{k}N_{2k+1}=(-1)^{n}N_{2n}+1$.

Taking $x=-1,r=1,s=1,t=2$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.6.
If $r=1,s=1,t=2$ then for $n\geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^{k}W_{k}=\frac{1}{3}((-1)^{n}(W_{n+3}-2W_{n+2}+W_{n+1})+W_2-2W_1+W_0)$.

(b) $\sum_{k=0}^{n}(-1)^{k}W_{2k}=\frac{1}{5}((-1)^{n}(2W_{2n+2}-3W_{2n+1}+2W_{2n})-2W_2+3W_1+3W_0)$.

(c) $\sum_{k=0}^{n}(-1)^{k}W_{2k+1}=\frac{1}{5}((-1)^{n}(-W_{2n+2}+4W_{2n+1}+4W_{2n})+W_2+W_1-4W_0)$.

Taking $W_n=J_n$ with $J_0=0,J_1=1,J_2=1$ in the last proposition, we have the following corollary which presents sum formulas of third order Jacobsthal numbers.

Corollary 2.16.
For $n\geq 0$, third order Jacobsthal numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^{k}J_{k}=\frac{1}{3}((-1)^{n}(J_{n+3}-2J_{n+2}+J_{n+1})-1)$.

(b) $\sum_{k=0}^{n}(-1)^{k}J_{2k}=\frac{1}{5}((-1)^{n}(2J_{2n+2}-3J_{2n+1}+2J_{2n})+1)$.

(c) $\sum_{k=0}^{n}(-1)^{k}J_{2k+1}=\frac{1}{5}((-1)^{n}(-J_{2n+2}+4J_{2n+1}+4J_{2n})+2)$.

From the last proposition, we have the following corollary which gives sum formulas of third order Jacobsthal-Lucas numbers (take $W_n = j_n$ with $j_0 = 2, j_1 = 1, j_2 = 5$).

Corollary 2.17.
For $n \geq 0$, third order Jacobsthal-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n}(-1)^k j_k = \frac{1}{3}((-1)^n (j_{n+3} - 2j_{n+2} + j_{n+1}) + 5)$.

(b) $\sum_{k=0}^{n}(-1)^k j_{2k} = \frac{1}{5}((-1)^n (2j_{2n+2} - 3j_{2n+1} + 2j_{2n}) - 1)$.

(c) $\sum_{k=0}^{n}(-1)^k j_{2k+1} = \frac{1}{5}((-1)^n (-j_{2n+2} + 4j_{2n+1} + 4j_{2n}) - 2)$.

2.3. The Case $x = 1 + i$

We now consider the complex case $x = 1 + i$ in Theorem 2.1.
Taking $x = 1 + i, r = s = t = 1$ in Theorem 2.1 (a), (b) and (c), we obtain the following proposition.

Proposition 2.7.
If $r = s = t = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(1+i)^k W_k = \frac{1}{-2+5i}((1+i)^n (3(1-i) W_{n+1} + 2W_{n+2} + 2(-1+i) W_{n+3}) - 2iW_2 - (1-i) W_1 + 3iW_0)$.

(b) $\sum_{k=0}^{n}(1+i)^k W_{2k} = \frac{1}{7i}((1+i)^n ((1-i)W_{2n+2} + 4iW_{2n+1} + 2(-1+2i)W_{2n}) - (1-i) W_2 - 4iW_1 + (2+3i) W_0)$

(c) $\sum_{k=0}^{n}(1+i)^k W_{2k+1} = \frac{1}{7i}((1+i)^n ((1+3i)W_{2n+2} + (-1+3i)W_{2n+1} + (1-i)W_{2n}) - (1+3i) W_2 + (1+4i) W_1 - (1-i) W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 2.18.
For $n \geq 0$, Tribonacci numbers have the following properties.

(a) $\sum_{k=0}^{n}(1+i)^k T_k = \frac{1}{-2+5i}((1+i)^n (3(1-i) T_{n+1} + 2T_{n+2} + 2(-1+i) T_{n+3}) - 1 - i)$.

(b) $\sum_{k=0}^{n}(1+i)^k T_{2k} = \frac{1}{7i}((1+i)^n ((1-i)T_{2n+2} + 4iT_{2n+1} + 2(-1+2i)T_{2n}) - 1 - 3i)$.

(c) $\sum_{k=0}^{n}(1+i)^k T_{2k+1} = \frac{1}{7i}((1+i)^n ((1+3i)T_{2n+2} + (-1+3i)T_{2n+1} + (1-i)T_{2n}) + i)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which presents sum formulas of Tribonacci-Lucas numbers.

Corollary 2.19.
For $n \geq 0$, Tribonacci-Lucas numbers have the following properties.

(a) $\sum_{k=0}^{n}(1+i)^k K_k = \frac{1}{-2+5i}((1+i)^n (3(1-i) K_{n+1} + 2K_{n+2} + 2(-1+i) K_{n+3}) - 1 + 4i)$.

(b) $\sum_{k=0}^{n}(1+i)^k K_{2k} = \frac{1}{7i}((1+i)^n ((1-i)K_{2n+2} + 4iK_{2n+1} + 2(-1+2i)K_{2n}) + 3 + 8i)$.

(c) $\sum_{k=0}^{n}(1+i)^k K_{2k+1} = \frac{1}{7i}((1+i)^n ((1+3i)K_{2n+2} + (-1+3i)K_{2n+1} + (1-i)K_{2n}) - 5 - 2i)$.

Corresponding sums of the other third order linear sequences can be calculated similarly when $x = 1 + i$.

3. Sum formulas of Generalized Tribonacci Numbers with Negative Subscripts

The following theorem presents some summing formulas (identities) of generalized Tribonacci numbers with negative subscripts.

Theorem 3.1.
Let x be a complex number. For $n \geq 1$, we have the following formulas:

(a) *If $t+rx^2+sx-x^3 \neq 0$, then*

$$\sum_{k=1}^{n} x^k W_{-k} = \frac{\Omega_4}{t+rx^2+sx-x^3}$$

where

$$\begin{aligned}\Omega_4 &= -\left(t+rx^2+sx\right)x^{n+1}W_{-n-1}-(t+sx)x^{n+2}W_{-n-2}-tx^{n+3}W_{-n-3} \\ &\quad +xW_2-x(r-x)W_1+x\left(-s-rx+x^2\right)W_0.\end{aligned}$$

(b) *If $2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k} = \frac{\Omega_5}{2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx}$$

where

$$\begin{aligned}\Omega_5 &= -(t+rx)x^{n+1}W_{-2n+1}+\left(r^2x+rt+sx-x^2\right)x^{n+1}W_{-2n}+t(s-x)x^{n+1}W_{-2n-1} \\ &\quad -x(s-x)W_2+x(t+rs)W_1+x\left(-r^2x-rt-2sx+s^2+x^2\right)W_0.\end{aligned}$$

(c) *If $2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx \neq 0$ then*

$$\sum_{k=1}^{n} x^k W_{-2k+1} = \frac{\Omega_6}{2sx^2-s^2x+r^2x^2+t^2-x^3+2rtx}$$

where

$$\begin{aligned}\Omega_6 &= (s-x)x^{n+2}W_{-2n+1}-(t+rs)x^{n+2}W_{-2n}-t(t+rx)x^{n+1}W_{-2n-1} \\ &\quad +x(t+rx)W_2+x\left(-r^2x-rt-sx+x^2\right)W_1-tx(s-x)W_0.\end{aligned}$$

Proof:

(a) Using the recurrence relation

$$W_{-n+3} = r \times W_{-n+2} + s \times W_{-n+1} + t \times W_{-n} \Rightarrow W_{-n} = -\frac{s}{t}W_{-(n-1)} - \frac{r}{t}W_{-(n-2)} + \frac{1}{t}W_{-(n-3)}$$

i.e.

$$tW_{-n} = W_{-n+3} - rW_{-n+2} - sW_{-n+1}$$

or

$$W_{-n} = \frac{1}{t}W_{-n+3} - \frac{r}{t}W_{-n+2} - \frac{s}{t}W_{-n+1}$$

we obtain

$$\begin{aligned} tx^nW_{-n} &= x^nW_{-n+3}-rx^nW_{-n+2}-sx^nW_{-n+1} \\ tx^{n-1}W_{-n+1} &= x^{n-1}W_{-n+4}-rx^{n-1}W_{-n+3}-sx^{n-1}W_{-n+2} \\ tx^{n-2}W_{-n+2} &= x^{n-2}W_{-n+5}-rx^{n-2}W_{-n+4}-sx^{n-2}W_{-n+3} \\ &\vdots \\ tx^3W_{-3} &= x^3W_0-r\times x^3W_{-1}-s\times x^3W_{-2} \\ tx^2W_{-2} &= x^2W_1-r\times x^2W_0-s\times x^2W_{-1} \\ tx^1W_{-1} &= x^1W_2-r\times x^1W_1-s\times x^1W_0. \end{aligned}$$

If we add the equations by side by, we get (a).

(b) and (c) Using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n}$$

we obtain

$$\begin{aligned}
sx^{n}W_{-2n+1} &= x^{n}W_{-2n+3}-rx^{n}W_{-2n+2}-tx^{n}W_{-2n}\\
sx^{n-1}W_{-2n+3} &= x^{n-1}W_{-2n+5}-rx^{n-1}W_{-2n+4}-tx^{n-1}W_{-2n+2}\\
sx^{n-2}W_{-2n+5} &= x^{n-2}W_{-2n+7}-rx^{n-2}W_{-2n+6}-tx^{n-2}W_{-2n+4}\\
sx^{n-3}W_{-2n+7} &= x^{n-3}W_{-2n+9}-rx^{n-3}W_{-2n+8}-tx^{n-3}W_{-2n+6}\\
&\vdots\\
sx^{3}W_{-5} &= x^{3}W_{-3}-rx^{3}W_{-4}-tx^{3}W_{-6}\\
sx^{2}W_{-3} &= x^{2}W_{-1}-rx^{2}W_{-2}-tx^{2}W_{-4}\\
sx^{1}W_{-1} &= x^{1}W_{1}-rx^{1}W_{0}-tx^{1}W_{-2}
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
s\sum_{k=1}^{n}x^{k}W_{-2k+1} &= (-x^{n+1}W_{-2n+1}+x^{1}W_{1}+\sum_{k=1}^{n}x^{k+1}W_{-2k+1})\\
&\quad -r(-x^{n+1}W_{-2n}+x^{1}W_{0}+\sum_{k=1}^{n}x^{k+1}W_{-2k})-t(\sum_{k=1}^{n}x^{k}W_{-2k})
\end{aligned} \tag{4}$$

and so

$$\begin{aligned}
s\sum_{k=1}^{n}x^{k}W_{-2k+1} &= (-x^{n+1}W_{-2n+1}+x^{1}W_{1}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k+1})\\
&\quad -r(-x^{n+1}W_{-2n}+x^{1}W_{0}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k})-t(\sum_{k=1}^{n}x^{k}W_{-2k}).
\end{aligned}$$

Similarly, using the recurrence relation

$$W_{-n+3}=rW_{-n+2}+sW_{-n+1}+tW_{-n}$$

i.e.

$$sW_{-n+1}=W_{-n+3}-rW_{-n+2}-tW_{-n}$$

we obtain

$$\begin{aligned}
sx^{n}W_{-2n} &= x^{n}W_{-2n+2}-rx^{n}W_{-2n+1}-tx^{n}W_{-2n-1}\\
sx^{n-1}W_{-2n+2} &= x^{n-1}W_{-2n+4}-rx^{n-1}W_{-2n+3}-tx^{n-1}W_{-2n+1}\\
sx^{n-2}W_{-2n+4} &= x^{n-2}W_{-2n+6}-rx^{n-2}W_{-2n+5}-tx^{n-2}W_{-2n+3}\\
sx^{n-3}W_{-2n+6} &= x^{n-3}W_{-2n+8}-rx^{n-3}W_{-2n+7}-tx^{n-3}W_{-2n+5}\\
&\vdots\\
sx^{4}W_{-8} &= x^{4}W_{-6}-rx^{4}W_{-7}-tx^{4}W_{-9}\\
sx^{3}W_{-6} &= x^{3}W_{-4}-rx^{3}W_{-5}-tx^{3}W_{-7}\\
sx^{2}W_{-4} &= x^{2}W_{-2}-rx^{2}W_{-3}-tx^{2}W_{-5}\\
sx^{1}W_{-2} &= x^{1}W_{0}-rx^{1}W_{-1}-tx^{1}W_{-3}.
\end{aligned}$$

If we add the equations by side by, we get

$$\begin{aligned}
s\sum_{k=1}^{n}x^{k}W_{-2k} &= (-x^{n+1}W_{-2n}+x^{1}W_{0}+\sum_{k=1}^{n}x^{k+1}W_{-2k})-r(\sum_{k=1}^{n}x^{k}W_{-2k+1})\\
&\quad -t(x^{n}W_{-2n-1}-x^{0}W_{-1}+\sum_{k=1}^{n}x^{k-1}W_{-2k+1})
\end{aligned}$$

and so

$$s\sum_{k=1}^{n}x^{k}W_{-2k} = (-x^{n+1}W_{-2n}+x^{1}W_{0}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k})-r(\sum_{k=1}^{n}x^{k}W_{-2k+1}) -t(x^{n}W_{-2n-1}-x^{0}W_{-1}+x^{-1}\sum_{k=1}^{n}x^{k}W_{-2k+1})$$

Since

$$W_{-1}=(-\frac{s}{t}W_{0}-\frac{r}{t}W_{1}+\frac{1}{t}W_{2})$$

it follows that

$$s\sum_{k=1}^{n}x^{k}W_{-2k} = (-x^{n+1}W_{-2n}+x^{1}W_{0}+x^{1}\sum_{k=1}^{n}x^{k}W_{-2k})-r(\sum_{k=1}^{n}x^{k}W_{-2k+1}) -t(x^{n}W_{-2n-1}-x^{0}(-\frac{s}{t}W_{0}-\frac{r}{t}W_{1}+\frac{1}{t}W_{2})+x^{-1}\sum_{k=1}^{n}x^{k}W_{-2k+1}) \quad (5)$$

Then, solving system (4)-(5) the required result of (b) and (c) follow.

3.1. The case $x = 1$

The case $x = 1$ of Theorem 3.1 is given in [18], see also [19]. In this subsection, we only consider the case $x = 1, r = 0, s = 2, t = 1$ and we present a theorem which is in different forms than given in [18]).

Observe that setting $x = 1, r = 0, s = 2, t = 1$ (i.e. for the generalized Pell-Padovan case) in Theorem 3.1 (b) and (c) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule however provides the evaluation of the sum formulas. If $r = 0, s = 2, t = 1$ then we have the following theorem.

Theorem 3.2.
If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} W_{-k} = \frac{1}{2}(-3W_{-n-1}-3W_{-n-2}-W_{-n-3}+W_{2}+W_{1}-W_{0})$.

(b) $\sum_{k=1}^{n} W_{-2k} = -(n+1)W_{-2n+1}+(n+1)W_{-2n}+nW_{-2n-1}+W_{1}-W_{0}$.

(c) $\sum_{k=1}^{n} W_{-2k+1} = (n+1)W_{-2n+1}-(n+2)W_{-2n}-(n+1)W_{-2n-1}+W_{2}-W_{1}$.

Proof:

(a) Taking $r = 0, s = 2, t = 1$ in Theorem 3.1 (a) we obtain (a).

(b) We use Theorem 3.1 (b). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (b) then we have
$\sum_{k=0}^{n} x^{k}W_{-2k} = \frac{-x^{n+1}W_{-2n+1}+(2x-x^{2})x^{n+1}W_{-2n}-(x-2)x^{n+1}W_{-2n-1}+x(x-2)W_{2}+xW_{1}+x(x^{2}-4x+4)W_{0}}{-x^{3}+4x^{2}-4x+1}$.
For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\sum_{k=0}^{n}W_{-2k} = \left.\frac{\frac{d}{dx}(g_{1}(x))}{\frac{d}{dx}(-x^{3}+4x^{2}-4x+1)}\right|_{x=1} = -(n+1)W_{-2n+1}+(n+1)W_{-2n}+nW_{-2n-1}+W_{1}-W_{0}.$$

where

$$g_{1}(x) = -x^{n+1}W_{-2n+1}+\left(2x-x^{2}\right)x^{n+1}W_{-2n}-(x-2)x^{n+1}W_{-2n-1}+x(x-2)W_{2}+xW_{1}+x\left(x^{2}-4x+4\right)W_{0}$$

(c) We use Theorem 3.1 (c). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (c) then we have
$\sum_{k=0}^{n} x^{k}W_{-2k+1} = \frac{-(x-2)x^{n+2}W_{-2n+1}-x^{n+2}W_{-2n}-x^{n+1}W_{-2n-1}+xW_{2}-x(2x-x^{2})W_{1}+x(x-2)W_{0}}{-x^{3}+4x^{2}-4x+1}$.
For $x = 1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we obtain

$$\sum_{k=0}^{n}W_{-2k+1} = \left.\frac{\frac{d}{dx}(g_{2}(x))}{\frac{d}{dx}(-x^{3}+4x^{2}-4x+1)}\right|_{x=1} = (n+1)W_{-2n+1}-(n+2)W_{-2n}-(n+1)W_{-2n-1}+W_{2}-W_{1}.$$

where

$$g_{2}(x) = -(x-2)x^{n+2}W_{-2n+1}-x^{n+2}W_{-2n}-x^{n+1}W_{-2n-1}+xW_{2}-x\left(2x-x^{2}\right)W_{1}+x(x-2)W_{0}.$$

From the last theorem, we have the following corollary which gives sum formula of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R = 1, R_2 = 1$).

Corollary 3.1.
For $n \geq 1$, Pell-Padovan numbers have the following property:

(a) $\sum_{k=1}^{n} R_{-k} = \frac{1}{2}(-3R_{-n-1} - 3R_{-n-2} - R_{-n-3} + 1)$.

(b) $\sum_{k=1}^{n} R_{-2k} = -(n+1)R_{-2n+1} + (n+1)R_{-2n} + nR_{-2n-1}$.

(c) $\sum_{k=1}^{n} R_{-2k+1} = (n+1)R_{-2n+1} - (n+2)R_{-2n} - (n+1)R_{-2n-1}$.

Taking $W_n = C_n$ with $C_0 = 3, C = 0, C_2 = 2$ in the last theorem, we have the following corollary which gives sum formulas of Pell-Perrin numbers.

Corollary 3.2.
For $n \geq 1$, Pell-Perrin numbers have the following property:

(a) $\sum_{k=1}^{n} C_{-k} = \frac{1}{2}(-3C_{-n-1} - 3C_{-n-2} - C_{-n-3} - 1)$.

(b) $\sum_{k=1}^{n} C_{-2k} = -(n+1)C_{-2n+1} + (n+1)C_{-2n} + nC_{-2n-1} - 3$.

(c) $\sum_{k=1}^{n} C_{-2k+1} = (n+1)C_{-2n+1} - (n+2)C_{-2n} - (n+1)C_{-2n-1} + 2$.

3.2. The case $x = -1$

We now consider the case $x = -1$ in Theorem 3.1.
Taking $x = -1, r = s = t = 1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.1.
If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} (-1)^k W_{-k} = \frac{1}{2}((-1)^n W_{-n-1} + (-1)^n W_{-n-3} - W_2 + 2W_1 - W_0)$.

(b) $\sum_{k=1}^{n} (-1)^k W_{-2k} = \frac{1}{2}((-1)^n W_{-2n} - (-1)^n W_{-2n-1} + W_2 - W_1 - 2W_0)$.

(c) $\sum_{k=1}^{n} (-1)^k W_{-2k+1} = \frac{1}{2}((-1)^n W_{-2n+1} - (-1)^n W_{-2n} - W_1 + W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 3.3.
For $n \geq 1$, Tribonacci numbers have the following properties.

(a) $\sum_{k=1}^{n} (-1)^k T_{-k} = \frac{1}{2}((-1)^n T_{-n-1} + (-1)^n T_{-n-3} + 1)$.

(b) $\sum_{k=1}^{n} (-1)^k T_{-2k} = \frac{1}{2}(-1)^n (T_{-2n} - T_{-2n-1})$.

(c) $\sum_{k=1}^{n} (-1)^k T_{-2k+1} = \frac{1}{2}((-1)^n T_{-2n+1} - (-1)^n T_{-2n} - 1)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the above proposition, we have the following corollary which gives sum formulas of Tribonacci-Lucas numbers.

Corollary 3.4.
For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:

(a) $\sum_{k=1}^{n} (-1)^k K_{-k} = \frac{1}{2}((-1)^n K_{-n-1} + (-1)^n K_{-n-3} - 4)$.

(b) $\sum_{k=1}^{n} (-1)^k K_{-2k} = \frac{1}{2}((-1)^n K_{-2n} - (-1)^n K_{-2n-1} - 4)$.

(c) $\sum_{k=1}^{n} (-1)^k K_{-2k+1} = \frac{1}{2}((-1)^n K_{-2n+1} - (-1)^n K_{-2n} + 2)$.

Taking $x=-1, r=2, s=1, t=1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.2.
If $r=2, s=1, t=1$ then for $n\geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^{k}W_{-k}=\frac{1}{3}(2(-1)^{n}W_{-n-1}+(-1)^{n}W_{-n-3}-W_{2}+3W_{1}-2W_{0})$.

(b) $\sum_{k=1}^{n}(-1)^{k}W_{-2k}=\frac{1}{5}((-1)^{n}(-W_{-2n+1}+4W_{-2n}-2W_{-2n-1})+2W_{2}-3W_{1}-6W_{0})$.

(c) $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1}=\frac{1}{5}((-1)^{n}(2W_{-2n+1}-3W_{-2n}-W_{-2n-1})+W_{2}-4W_{1}+2W_{0})$.

From the last proposition, we have the following corollary which gives sum formulas of third-order Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1, P_2=2$).

Corollary 3.5.
For $n\geq 1$, third-order Pell numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^{k}P_{-k}=\frac{1}{3}(2(-1)^{n}P_{-n-1}+(-1)^{n}P_{-n-3}+1)$.

(b) $\sum_{k=1}^{n}(-1)^{k}P_{-2k}=\frac{1}{5}((-1)^{n}(-P_{-2n+1}+4P_{-2n}-2P_{-2n-1})+1)$.

(c) $\sum_{k=1}^{n}(-1)^{k}P_{-2k+1}=\frac{1}{5}((-1)^{n}(2P_{-2n+1}-3P_{-2n}-P_{-2n-1})-2)$.

Taking $W_n=Q_n$ with $Q_0=3, Q_1=2, Q_2=6$ in the last proposition, we have the following corollary which gives sum formulas of third-order Pell-Lucas numbers.

Corollary 3.6.
For $n\geq 1$, third-order Pell-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^{k}Q_{-k}=\frac{1}{3}(2(-1)^{n}Q_{-n-1}+(-1)^{n}Q_{-n-3}-6)$.

(b) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k}=\frac{1}{5}((-1)^{n}(-Q_{-2n+1}+4Q_{-2n}-2Q_{-2n-1})-12)$.

(c) $\sum_{k=1}^{n}(-1)^{k}Q_{-2k+1}=\frac{1}{5}((-1)^{n}(2Q_{-2n+1}-3Q_{-2n}-Q_{-2n-1})+4)$.

From the last proposition, we have the following corollary which presents sum formulas of third-order modified Pell numbers (take $W_n=E_n$ with $E_0=0, E_1=1, E_2=1$).

Corollary 3.7.
For $n\geq 1$, third-order modified Pell numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^{k}E_{-k}=\frac{1}{3}(2(-1)^{n}E_{-n-1}+(-1)^{n}E_{-n-3}+2)$.

(b) $\sum_{k=1}^{n}(-1)^{k}E_{-2k}=\frac{1}{5}((-1)^{n}(-E_{-2n+1}+4E_{-2n}-2E_{-2n-1})-1)$.

(c) $\sum_{k=1}^{n}(-1)^{k}E_{-2k+1}=\frac{1}{5}((-1)^{n}(2E_{-2n+1}-3E_{-2n}-E_{-2n-1})-3)$.

Taking $x=-1, r=0, s=1, t=1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.3.
If $r=0, s=1, t=1$ then for $n\geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^{k}W_{-k}=(-1)^{n}W_{-n-3}-W_{2}+W_{1}$.

(b) $\sum_{k=1}^{n}(-1)^{k}W_{-2k}=\frac{1}{5}((-1)^{n}(W_{-2n+1}+2W_{-2n}-2W_{-2n-1})+2W_{2}-W_{1}-4W_{0})$.

(c) $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1}=\frac{1}{5}((-1)^{n}(2W_{-2n+1}-W_{-2n}+W_{-2n-1})-W_{2}-2W_{1}+2W_{0})$.

Taking $W_n=P_n$ with $P_0=1, P_1=1, P_2=1$ in the last proposition, we have the following corollary which gives sum formulas of Padovan numbers.

Corollary 3.8.
For $n \geq 1$, Padovan numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^k P_{-k} = (-1)^n P_{-n-3}$.

(b) $\sum_{k=1}^{n}(-1)^k P_{-2k} = \frac{1}{5}((-1)^n (P_{-2n+1} + 2P_{-2n} - 2P_{-2n-1}) - 3)$.

(c) $\sum_{k=1}^{n}(-1)^k P_{-2k+1} = \frac{1}{5}((-1)^n (2P_{-2n+1} - P_{-2n} + P_{-2n-1}) - 1)$.

From the last proposition, we have the following corollary which presents sum formulas of Perrin numbers (take $W_n = E_n$ with $E_0 = 3, E = 0, E_2 = 2$).

Corollary 3.9.
For $n \geq 1$, Perrin numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^k E_{-k} = (-1)^n E_{-n-3} - 2$.

(b) $\sum_{k=1}^{n}(-1)^k E_{-2k} = \frac{1}{5}((-1)^n (E_{-2n+1} + 2E_{-2n} - 2E_{-2n-1}) - 8)$.

(c) $\sum_{k=1}^{n}(-1)^k E_{-2k+1} = \frac{1}{5}((-1)^n (2E_{-2n+1} - E_{-2n} + E_{-2n-1}) + 4)$.

Taking $W_n = S_n$ with $S_0 = 0, S_1 = 0, S_2 = 1$ in the last proposition, we have the following corollary which gives sum formulas of Padovan-Perrin numbers.

Corollary 3.10.
For $n \geq 1$, Padovan-Perrin numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^k S_{-k} = (-1)^n S_{-n-3} - 1$.

(b) $\sum_{k=1}^{n}(-1)^k S_{-2k} = \frac{1}{5}((-1)^n (S_{-2n+1} + 2S_{-2n} - 2S_{-2n-1}) + 2)$.

(c) $\sum_{k=1}^{n}(-1)^k S_{-2k+1} = \frac{1}{5}((-1)^n (2S_{-2n+1} - S_{-2n} + S_{-2n-1}) - 1)$.

If $x = -1, r = 0, s = 2, t = 1$ then $t + rx^2 + sx - x^3 = 0$ so we can't use Theorem 3.1 (a) directly. Observe that setting $x = -1, r = 0, s = 2, t = 1$ (i.e. for the generalized Pell-Padovan case) in Theorem 3.1 (a) makes the right hand side of the sum formulas to be an indeterminate form. Application of L'Hospital rule provides the evaluation of the sum formulas. If $r = 0, s = 2, t = 1$ then we have the following theorem.

Theorem 3.3.
If $r = 0, s = 2, t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^k W_{-k} = (-1)^n (-(n+3) W_{-n-1} + (n+4) W_{-n-2} + (n+3) W_{-n-3}) - W_2 + 2W_1 - W_0$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{10}((-1)^n (W_{-2n+1} + 3W_{-2n} - 3W_{-2n-1}) + 3W_2 - W_1 - 9W_0)$.

(c) $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{10}((-1)^n (3W_{-2n+1} - W_{-2n} + W_{-2n-1}) - W_2 - 3W_1 + 3W_0)$.

Proof:

(a) We use Theorem 3.1 (a). If we set $r = 0, s = 2, t = 1$ in Theorem 3.1 (a) then we have
$\sum_{k=1}^{n} x^k W_{-k} = \frac{(2x+1)x^{n+1}W_{-n-1}+(2x+1)x^{n+2}W_{-n-2}+x^{n+3}W_{-n-3}-xW_2-x^2W_1-x(x^2-2)W_0}{x^3-2x-1}$.
For $x = -1$, the right hand side of the above sum formulas is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}\sum_{k=1}^{n}(-1)^k W_{-k} &= \left.\frac{\frac{d}{dx}(h(x))}{\frac{d}{dx}(x^3-2x-1)}\right|_{x=-1} \\ &= (-1)^n (-(n+3) W_{-n-1} + (n+4) W_{-n-2} + (n+3) W_{-n-3}) - W_2 + 2W_1 - W_0.\end{aligned}$$

where

$$h(x) = (2x+1) x^{n+1} W_{-n-1} + (2x+1) x^{n+2} W_{-n-2} + x^{n+3} W_{-n-3} - xW_2 - x^2 W_1 - x\left(x^2-2\right) W_0.$$

(b) Taking $r = 0, s = 2, t = 1$ in Theorem 3.1 (b) we obtain (b).

(c) Taking $r = 0, s = 2, t = 1$ in Theorem 3.1 (c) we obtain (c).

From the last theorem, we have the following corollary which gives sum formula of Pell-Padovan numbers (take $W_n = R_n$ with $R_0 = 1, R = 1, R_2 = 1$).

Corollary 3.11.
For $n \geq 1$, Pell-Padovan numbers have the following property:

(a) $\sum_{k=1}^{n}(-1)^k R_{-k} = (-1)^n\,(-(n+3)\,R_{-n-1} + (n+4)\,R_{-n-2} + (n+3)\,R_{-n-3})$.

(b) $\sum_{k=1}^{n}(-1)^k R_{-2k} = \frac{1}{10}((-1)^n\,(R_{-2n+1} + 3R_{-2n} - 3R_{-2n-1}) - 7)$.

(c) $\sum_{k=1}^{n}(-1)^k R_{-2k+1} = \frac{1}{10}((-1)^n\,(3R_{-2n+1} - R_{-2n} + R_{-2n-1}) - 1)$.

Taking $W_n = C_n$ with $C_0 = 3, C = 0, C_2 = 2$ in the last theorem, we have the following corollary which gives sum formulas of Pell-Perrin numbers.

Corollary 3.12.
For $n \geq 1$, Pell-Perrin numbers have the following property:

(a) $\sum_{k=1}^{n}(-1)^k C_{-k} = (-1)^n\,(-(n+3)\,C_{-n-1} + (n+4)\,C_{-n-2} + (n+3)\,C_{-n-3}) - 5$.

(b) $\sum_{k=1}^{n}(-1)^k C_{-2k} = \frac{1}{10}((-1)^n\,(C_{-2n+1} + 3C_{-2n} - 3C_{-2n-1}) - 21)$.

(c) $\sum_{k=1}^{n}(-1)^k C_{-2k+1} = \frac{1}{10}((-1)^n\,(3C_{-2n+1} - C_{-2n} + C_{-2n-1}) + 7)$.

Taking $x = -1, r = 0, s = 1, t = 2$ in Theorem 3.1 we obtain the following proposition.

Proposition 3.4.
If $r = 0, s = 1, t = 2$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^k W_{-k} = \frac{1}{2}((-1)^n\,(W_{-n-1} - W_{-n-2} + 2W_{-n-3}) - W_2 + W_1)$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-2k} = \frac{1}{4}((-1)^n\,(W_{-2n+1} + W_{-2n} - 2W_{-2n-1}) + W_2 - W_1 - 2W_0)$.

(c) $\sum_{k=1}^{n}(-1)^k W_{-2k+1} = \frac{1}{4}((-1)^n\,(W_{-2n+1} - W_{-2n} + 2W_{-2n-1}) - W_2 - W_1 + 2W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Padovan numbers (take $W_n = Q_n$ with $Q_0 = 1, Q_1 = 1, Q_2 = 1$).

Corollary 3.13.
For $n \geq 1$, Jacobsthal-Padovan numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^k Q_{-k} = \frac{1}{2}(-1)^n\,(Q_{-n-1} - Q_{-n-2} + 2Q_{-n-3})$.

(b) $\sum_{k=1}^{n}(-1)^k Q_{-2k} = \frac{1}{4}((-1)^n\,(Q_{-2n+1} + Q_{-2n} - 2Q_{-2n-1}) - 2)$.

(c) $\sum_{k=1}^{n}(-1)^k Q_{-2k+1} = \frac{1}{4}(-1)^n\,(Q_{-2n+1} - Q_{-2n} + 2Q_{-2n-1})$.

Taking $W_n = L_n$ with $L_0 = 3, L_1 = 0, L_2 = 2$ in the last proposition, we have the following corollary which gives sum formulas of Jacobsthal-Perrin numbers.

Corollary 3.14.
For $n \geq 1$, Jacobsthal-Perrin numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^k L_{-k} = \frac{1}{2}((-1)^n\,(L_{-n-1} - L_{-n-2} + 2L_{-n-3}) - 2)$.

(b) $\sum_{k=1}^{n}(-1)^k L_{-2k} = \frac{1}{4}((-1)^n\,(L_{-2n+1} + L_{-2n} - 2L_{-2n-1}) - 4)$.

(c) $\sum_{k=1}^{n}(-1)^k L_{-2k+1} = \frac{1}{4}((-1)^n\,(L_{-2n+1} - L_{-2n} + 2L_{-2n-1}) + 4)$.

Taking $x=-1, r=1, s=0, t=1$ in Theorem 3.1, we obtain the following proposition.

Proposition 3.5.
If $r=1, s=0, t=1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^{k}W_{-k}=\frac{1}{3}((-1)^{n}(2W_{-n-1}-W_{-n-2}+W_{-n-3})-W_{2}+2W_{1}-2W_{0})$.

(b) $\sum_{k=1}^{n}(-1)^{k}W_{-2k}=(-1)^{n}(W_{-2n}-W_{-2n-1})+W_{2}-W_{1}-W_{0}$.

(c) $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1}=(-1)^{n}(W_{-2n+1}-W_{-2n})-W_{1}+W_{0}$.

From the above proposition, we have the following corollary which gives sum formulas of Narayana numbers (take $W_n=N_n$ with $N_0=0, N_1=1, N_2=1$).

Corollary 3.15.
For $n \geq 1$, Narayana numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^{k}N_{-k}=\frac{1}{3}((-1)^{n}(2N_{-n-1}-N_{-n-2}+N_{-n-3})+1)$.

(b) $\sum_{k=1}^{n}(-1)^{k}N_{-2k}=(-1)^{n}(N_{-2n}-N_{-2n-1})$.

(c) $\sum_{k=1}^{n}(-1)^{k}N_{-2k+1}=(-1)^{n}(N_{-2n+1}-N_{-2n})-1$.

Taking $x=-1, r=1, s=1, t=2$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.6.
If $r=1, s=1, t=2$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^{k}W_{-k}=\frac{1}{3}((-1)^{n}(2W_{-n-1}-W_{-n-2}+2W_{-n-3})-W_{2}+2W_{1}-W_{0})$.

(b) $\sum_{k=1}^{n}(-1)^{k}W_{-2k}=\frac{1}{5}((-1)^{n}(W_{-2n+1}+W_{-2n}-4W_{-2n-1})+2W_{2}-3W_{1}-3W_{0})$.

(c) $\sum_{k=1}^{n}(-1)^{k}W_{-2k+1}=\frac{1}{5}((-1)^{n}(2W_{-2n+1}-3W_{-2n}+2W_{-2n-1})-W_{2}-W_{1}+4W_{0})$.

Taking $W_n=J_n$ with $J_0=0, J_1=1, J_2=1$ in the last proposition, we have the following corollary which gives sum formulas of third order Jacobsthal numbers.

Corollary 3.16.
For $n \geq 1$, third order Jacobsthal numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^{k}J_{-k}=\frac{1}{3}((-1)^{n}(2J_{-n-1}-J_{-n-2}+2J_{-n-3})+1)$.

(b) $\sum_{k=1}^{n}(-1)^{k}J_{-2k}=\frac{1}{5}((-1)^{n}(J_{-2n+1}+J_{-2n}-4J_{-2n-1})-1)$.

(c) $\sum_{k=1}^{n}(-1)^{k}J_{-2k+1}=\frac{1}{5}((-1)^{n}(2J_{-2n+1}-3J_{-2n}+2J_{-2n-1})-2)$.

From the last proposition, we have the following corollary which gives sum formulas of third order Jacobsthal-Lucas numbers (take $W_n=j_n$ with $j_0=2, j_1=1, j_2=5$).

Corollary 3.17.
For $n \geq 1$, third order Jacobsthal-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^{k}j_{-k}=\frac{1}{3}((-1)^{n}(2j_{-n-1}-j_{-n-2}+2j_{-n-3})-5)$.

(b) $\sum_{k=1}^{n}(-1)^{k}j_{-2k}=\frac{1}{5}((-1)^{n}(j_{-2n+1}+j_{-2n}-4j_{-2n-1})+1)$.

(c) $\sum_{k=1}^{n}(-1)^{k}j_{-2k+1}=\frac{1}{5}((-1)^{n}(2j_{-2n+1}-3j_{-2n}+2j_{-2n-1})+2)$.

3.3. The Case $x = 1 + i$

We now consider the complex case $x = 1 + i$ in Theorem 3.1. The following theorem presents some summing formulas of generalized Tribonacci numbers with negative subscripts.

Taking $x = 1 + i, r = s = t = 1$ in Theorem 3.1 (a), (b) and (c), we obtain the following proposition.

Proposition 3.7.
If $r = s = t = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} (1+i)^k W_{-k} = \frac{1}{4+i}((1+i)^n((1-5i)W_{-n-1} + 2(1-2i)W_{-n-2} + 2(1-i)W_{-n-3}) + (1+i)W_2 - (1-i)W_1 - (3+i)W_0)$.

(b) $\sum_{k=1}^{n} (1+i)^k W_{-2k} = \frac{1}{4+5i}((1+i)^n(-(1+3i)W_{-2n+1} + 3(1+i)W_{-2n} + (1-i)W_{-2n-1}) - (1-i)W_2 + (2+2i)W_1 - (2+4i)W_0)$.

(c) $\sum_{k=1}^{n} (1+i)^k W_{-2k+1} = \frac{1}{4+5i}((1+i)^n(2W_{-2n+1} - 4iW_{-2n} - (1+3i)W_{-2n-1}) + (1+3i)W_2 - (3+3i)W_1 - (1-i)W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Tribonacci numbers (take $W_n = T_n$ with $T_0 = 0, T_1 = 1, T_2 = 1$).

Corollary 3.18.
For $n \geq 1$, Tribonacci numbers have the following properties.

(a) $\sum_{k=1}^{n} (1+i)^k T_{-k} = \frac{1}{4+i}((1+i)^n((1-5i)T_{-n-1} + 2(1-2i)T_{-n-2} + 2(1-i)T_{-n-3}) + 2i)$.

(b) $\sum_{k=1}^{n} (1+i)^k T_{-2k} = \frac{1}{4+5i}((1+i)^n(-(1+3i)T_{-2n+1} + 3(1+i)T_{-2n} + (1-i)T_{-2n-1}) + 1 + 3i)$.

(c) $\sum_{k=1}^{n} (1+i)^k T_{-2k+1} = \frac{1}{4+5i}((1+i)^n(2T_{-2n+1} - 4iT_{-2n} - (1+3i)T_{-2n-1}) - 2)$.

Taking $W_n = K_n$ with $K_0 = 3, K_1 = 1, K_2 = 3$ in the last proposition, we have the following corollary which gives sum formulas of Tribonacci-Lucas numbers.

Corollary 3.19.
For $n \geq 1$, Tribonacci-Lucas numbers have the following properties:

(a) $\sum_{k=1}^{n} (1+i)^k K_{-k} = \frac{1}{4+i}((1+i)^n((1-5i)K_{-n-1} + 2(1-2i)K_{-n-2} + 2(1-i)K_{-n-3}) - 7 + i)$.

(b) $\sum_{k=1}^{n} (1+i)^k K_{-2k} = \frac{1}{4+5i}((1+i)^n(-(1+3i)K_{-2n+1} + 3(1+i)K_{-2n} + (1-i)K_{-2n-1}) - 7 - 7i)$.

(c) $\sum_{k=1}^{n} (1+i)^k K_{-2k+1} = \frac{1}{4+5i}((1+i)^n(2K_{-2n+1} - 4iK_{-2n} - (1+3i)K_{-2n-1}) - 3 + 9i)$.

Corresponding sums of the other third order linear sequences can be calculated similarly when $x = 1 + i$.

References

[1] I. Bruce, A modified Tribonacci sequence, Fibonacci Quarterly, 22(3) (1984), 244–246.
[2] M. Catalani, Identities for Tribonacci-related sequences, arXiv:math/0209179, 2012.
[3] E. Choi, Modular Tribonacci Numbers by Matrix Method, Journal of the Korean Society of Mathematical Education Series B: Pure and Applied. Mathematics. 20(3) (2013), 207–221.
[4] M. Elia, Derived Sequences, The Tribonacci Recurrence and Cubic Forms, Fibonacci Quarterly, 39(2) (2001), 107-115.
[5] M.C. Er, Sums of Fibonacci Numbers by Matrix Methods, Fibonacci Quarterly, 22(3) (1984), 204-207.
[6] R. Frontczak, Sums of Tribonacci and Tribonacci-Lucas Numbers, International Journal of Mathematical Analysis, 12(1) (2018), 19-24.
[7] A. D. Godase, M. B. Dhakne, On the properties of k-Fibonacci and k-Lucas numbers, Int. J. Adv. Appl. Math. and Mech. 2(1) (2014) 100-106.
[8] H. Gökbaş, H. Köse, Some sum formulas for products of Pell and Pell-Lucas numbers, Int. J. Adv. Appl. Math. and Mech. 4(4) (2017) 1–4.

[9] R.T. Hansen., General Identities for Linear Fibonacci and Lucas Summations, Fibonacci Quarterly, 16(2) (1978), 121-28.
[10] T. Koshy, Fibonacci and Lucas Numbers with Applications, A Wiley-Interscience Publication, New York, 2001.
[11] T. Koshy, Pell and Pell-Lucas Numbers with Applications, Springer, New York, 2014.
[12] P.Y. Lin, De Moivre-Type Identities For The Tribonacci Numbers, Fibonacci Quarterly, 26 (1988), 131-134.
[13] T. Parpar, k'ncı Mertebeden Rekürans Bağıntısının Özellikleri ve Bazı Uygulamaları, Selçuk Üniversitesi, Fen Bilimleri Enstitüsü, Yüksek Lisans Tezi, 2011.
[14] S. Pethe, Some Identities for Tribonacci sequences, Fibonacci Quarterly, 26(2), (1988), 144–151.
[15] A. Scott, T. Delaney, V. Hoggatt Jr, The Tribonacci sequence, Fibonacci Quarterly, 15(3) (1977), 193–200.
[16] A. Shannon, Tribonacci numbers and Pascal's pyramid, Fibonacci Quarterly, 15(3) (1977), pp. 268 and 275.
[17] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/
[18] Y. Soykan, Summing Formulas For Generalized Tribonacci Numbers, arXiv:1910.03490v1 [math.GM], 2019.
[19] Y. Soykan, Summing Formulas For Generalized Tribonacci Numbers, Universal Journal of Mathematics and Applications, accepted.
[20] Y. Soykan, Tribonacci and Tribonacci-Lucas Sedenions. Mathematics 7(1), 74, 2019.
[21] Y. Soykan, On Summing Formulas for Horadam Numbers, Asian Journal of Advanced Research and Reports 8(1) (2020), 45-61, , DOI: 10.9734/AJARR/2020/v8i130192.
[22] Y. Soykan, On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, Advances in Research, 20(2) (2019), 1-15.
[23] Y. Soykan, Corrigendum: On Summing Formulas For Generalized Fibonacci and Gaussian Generalized Fibonacci Numbers, 2020.
https://www.researchgate.net/publication/337063487_Corrigendum_On_Summing_Formulas_For_Generalized_Fibonacci_and_Gaussian_Generalized_Fibonacci_Numbers.
[24] Y. Soykan, Matrix Sequences of Tribonacci and Tribonacci-Lucas Numbers, arXiv:1809.07809v1 [math.NT] 20 Sep 2018.
[25] Y. Soykan, Summation Formulas For Generalized Tetranacci Numbers, Asian Journal of Advanced Research and Reports, 7(2) (2019), 1-12. doi.org/10.9734/ajarr/2019/v7i230170.
[26] Y. Soykan, Linear Summing Formulas of Generalized Pentanacci and Gaussian Generalized Pentanacci Numbers, Journal of Advanced in Mathematics and Computer Science, 33(3) (20199, 1-14.
[27] Y. Soykan, Sum Formulas For Generalized Fifth-Order Linear Recurrence Sequences, Journal of Advances in Mathematics and Computer Science, 34(5), 1-14, 2019, Article no.JAMCS.53303, ISSN: 2456-9968, DOI: 10.9734/JAMCS/2019/v34i530224.
[28] Y. Soykan, On Summing Formulas of Generalized Hexanacci and Gaussian Generalized Hexanacci Numbers, Asian Research Journal of Mathematics, 14(4) (2019), 1-14, Article no.ARJOM.50727.
[29] W. Spickerman, Binet's formula for the Tribonacci sequence, Fibonacci Quarterly, 20 (1982), 118–120.
[30] S. Uygun, The binomial transforms of the generalized (s,t)-Jacobsthal matrix sequence, Int. J. Adv. Appl. Math. and Mech. 6(3) (2019) 14–20.
[31] Yalavigi, C. C., Properties of Tribonacci numbers, Fibonacci Quarterly, 10(3) (1972), 231–246.
[32] Yilmaz, N., Taskara, N., Tribonacci and Tribonacci-Lucas Numbers via the Determinants of Special Matrices, Applied Mathematical Sciences, 8(39) (2014), 1947-1955.
[33] Waddill, M. E., The Tetranacci Sequence and Generalizations, Fibonacci Quarterly, (1992), 9-20.