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On extending $S_{\aleph}$-fibrations to $C_{\aleph}$-fibrations in bitopological semigroups

Research Article

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Abstract: In this paper, we start by giving the concepts of bitopological semigroups and study some their properties. Then we extend the concepts of $S_{\aleph}$-fibrations in the homotopy theory for topological semigroups to bitopological semigroups by giving the concept of $C_{\aleph}$-fibrations. Furthermore, we study some properties on $C_{\aleph}$-fibrations such as a restriction property, composition property, a product property, and a covering homotopy theorem.

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1. Introduction

Throughout this paper, by all X_τ we mean all topological spaces (X,τ) which will be assumed Hausdorff spaces. In 1955, W. Hurewicz [6] introduced the concepts of fibrations in homotopy theory for topological spaces (i.e., spaces) which have played very important roles for investigating the mutual relations of among the objects. In 1963, J.C. Kelly [7] introduced the notion of bitopological spaces. Such spaces equipped with its two (arbitrary) topologies. The reader is suggested to refer [7] for the detail definitions and notations. Furthermore, Kelly was extended some of the standard results of separation axioms in a topological space to a bitopological space. By all $X_{\tau_{12}}$ we mean all bitopological spaces (X,τ_1,τ_2). There are many ways in which a space can be regarded as a bitopological space. In our work, we use a space O_ρ as a bitopological space $O_{\rho\rho}$.

The concept of homotopy theory for topological semigroups have been introduced by Zvonko in 2002, [9]. In this theory, he was extended some of concepts with their standard results in a topological space to a topological semigroup. For example, the retract, the contractible spaces, the fibrations, and the cofibrations.

This paper is organized as follows. It consists of five sections. After this Introduction, Section 2 is devoted to some preliminaries. In Section 3 we give the concepts of bitopological semigroups, S_1-maps, S_2-maps, Sp-maps, c-bitopological semigroups, and c-maps. Some their properties are proved. In Section 4 we define an $C_{\aleph}$-fibration and study some its basic properties. In Section 5 we introduce the covering homotopy theorem for S-maps into $C_{\aleph}$-fibrations.

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2. Preliminaries

For two bitopological spaces $X_{\tau_{12}}$ and $Y_{\rho_{12}}$, a *pairwise continuous map* (or simply *p-map*), $h : X_{\tau_{12}} \to Y_{\rho_{12}}$ is a function from X into Y that is continuous function (i.e., a map) from a space X_{τ_1} into a space Y_{ρ_1} and from X_{τ_2} into Y_{ρ_2}, [7]. As a result of these definitions, many topological ideas can be extended to bitopological spaces by [7]. For example, given a bitopological space $X_{\tau_{12}}$ and $A \subset X$, the *bitopological subspace* is found by individually restricting the two topologies: $(A, \tau_1|_A, \tau_2|_A)$; the usual product of two bitopological spaces $X_{\tau_{12}}$ and $Y_{\rho_{12}}$, is the usual product set with the usual product topologies, $(X \times Y, \tau_1 \times \rho_1, \tau_2 \times \rho_2)$.

The most of the following definitions and results have been worked out previously by Zvonko, [9]. A *topological semigroup* or an *S-space* is a pair $(X_\tau, *)$ consisting a topological space X_τ and a map $* : X_\tau \times X_\tau \to X_\tau$ from the product space $X_\tau \times X_\tau$ into X_τ such that $*(x, *(y, z)) = *(*(x, y), z)$ for all $x, y, z \in X$. That is, an S-space is a topological space with a continuous associative multiplication. We denote the class of all S-spaces by $\aleph$.

An S-space $(A, *')$ is called an *S-subspace* of $(X_\tau, *)$ if A is a subspace of X_τ, the map $*$ takes the product $A \times A$ into A, and $*'(x, y) = *(x, y)$ for all $x, y \in A$. It is natural to denote the multiplication of an S-subspace with the same symbol used for the multiplication on the S-space under consideration.

For every space X_τ, the *natural S-space* is an S-space (X_τ, π_i), where π_i is a continuous associative multiplication on X_τ given by $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$ for all $x, y \in X$. We denote the class of all natural S-spaces (X_τ, π) by $\mathcal{N}_\pi$, where $\pi = \pi_1, \pi_2$.

Let $(X_\tau, *)$ and $(O_\rho, \circ)$ be two S-spaces. The function $f : (X_\tau, *) \to (O_\rho, \circ)$ is called a *homomorphism* or *an S-map* if f is a map of a space X_τ into O_ρ and

$$f(*(x, y)) = \circ(f(x), f(y)) \quad \text{for all } x, y \in X.$$

Recall [9] that the usual composition and the usual product of two S-maps are S-maps and that the function $f : X_\tau \to O_\rho$ of a natural S-space (X_τ, π) into (O_ρ, π) is an S-map if and only if it is continuous.

For every a space X_τ, by $P(X_\tau)$ we mean the space of all paths from the unit closed interval $I = [0, 1]$ into X_τ with the compact-open topology. Recall [9] that for every an S-space $(X_\tau, *)$, $(P(X_\tau), p(*))$ is an S-space where $p(*) : P(X_\tau) \times P(X_\tau) \to P(X_\tau)$ is a map defined by

$$p(*)(\alpha, \beta)(t) = *(\alpha(t), \beta(t)) \quad \text{for all } \alpha, \beta \in P(X_\tau),\ t \in I.$$

The shorter notion for this S-space will be $P(X_\tau, *)$.

Definition 2.1.
[9] The S-maps $f, g : (X_\tau, *) \to (O_\rho, \circ)$ are called S-homotopic *and write $f \simeq_s g$ provided there is an S-map $H : (X_\tau, *) \to P(O_\rho, \circ)$ called an* S-homotopy *such that $H(x)(0) = f(x)$ and $H(x)(1) = g(x)$ for all $x \in X$.*

Throughout this paper, for every an S-homotopy $H : (X_\tau, *) \to P(O_\rho, \circ)$ and for every $t \in I$, by H_t (or $[H]_t$) we mean the S-map, [9], $H_t : (X_\tau, *) \to (O_\rho, \circ)$ which given by $H_t(x) = H(x)(t)$ for all $x \in X$. Also for every an S-homotopy $H : (X_\tau, *) \to P[P(O)_{\rho^c}, p(\circ)]$ and for every $r, t \in I$, by H_{rt} (or $[H]_{rt}$) we mean the S-map $H_{rt} : (X_\tau, *) \to (O_\rho, \circ)$ which given by $H_{rt}(x) = [H(x)(r)](t)$ for all $x \in X$, where ρ^c is the compact-open topology on $P(O)$ which is induced by ρ.

Theorem 2.1.
The relation of S-homotopy $\simeq_s$ is an equivalence relation on the set of all S-maps of $(X_\tau, *)$ into $(O_\rho, \circ)$.

Theorem 2.2.
If the S-maps $f, g : (X_\tau, *) \to (O_\rho, \circ)$ are S-homotopic then the relations $f \circ h \simeq_s g \circ h$ and $k \circ f \simeq_s k \circ g$ hold for all S-maps h into $(X_\tau, *)$ and k from $(O_\rho, \circ)$.

If the S-maps $f, g : (X_\tau, *) \to (O_\rho, \circ)$ are S-homotopic then the maps $f, g : X_\tau \to O_\rho$ are homotopic. The S-maps $f, g : (X_\tau, \pi) \to (O_\rho, \pi)$ are S-homotopic if and only if the maps $f, g : X_\tau \to O_\rho$ are homotopic.

Definition 2.2.
*[9] An S-map $f : (X_\tau, *) \to (O_\rho, \circ)$ is called an $S_{\aleph}$-*fibration *if for every an-space $(Y_\omega, \star) \in \aleph$ and all S-maps $g : (Y_\omega, \star) \to (X_\tau, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ with $G_0 = f \circ g$, there is an S-homotopy $H : (Y_\omega, \star) \to P(X_\tau, *)$ such that $H_0 = g$ and $f \circ H_t = G_t$ for all $t \in I$.*

Theorem 2.3.
The map $f : X_\tau \to O_\rho$ is a Hurewicz fibration if and only if the S-map $f : (X_\tau, \pi) \to (O_\rho, \pi)$ is an $S_{\mathcal{N}_\pi}$−fibration.

3. Bitopological semigroups

By a *bitopological semigroups* we mean a pair $(X_{\tau_{12}},*)$ consisting of a bitopological space $X_{\tau_{12}}$ and the associative multiplication $*$ on X such that $*$ is an p-map from the product bitopological space $(X\times X,\tau_1\times\tau_1,\tau_2\times\tau_2)$ into $X_{\tau_{12}}$.

Example 3.1.
Let $\mathbb{R}$ be the set of real numbers with the usual topology $\tau_1:=\tau_u$ and the discrete topology $\tau_2:=\tau_D$. Define the associative multiplication $*:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $*(x,y)=x+y+xy$ for all $x,y\in\mathbb{R}$. It's clear that $*$ is a continuous function from a space $(\mathbb{R}\times\mathbb{R},\tau_1\times\tau_1)$ into a space $(\mathbb{R},\tau_1)$ and is a continuous function from a space $(\mathbb{R}\times\mathbb{R},\tau_2\times\tau_2)$ into a space $(\mathbb{R},\tau_2)$. That is, $*$ is an p-map and a pair $(\mathbb{R}_{\tau_{12}},*)$ is a bitopological semigroup.

Let $(X_{\tau_{12}},*)$ be a bitopological semigroup and $B\subseteq X$. By $B|_{\tau_{12}}$ we mean the bitopological subspace $(B,\tau_1|_B,\tau_2|_B)$ of $X_{\tau_{12}}$ where $\tau_1|_B$ and $\tau_2|_B$ are the relative topologies of τ_1 and τ_2 on B, respectively. If the p-map $*$ takes the product $B\times B$ into B then the pair $(B|_{\tau_{12}},*)$ will be a bitopological semigroup and called an *b-subspace* of $(X_{\tau_{12}},*)$.

Let $(X_{\tau_{12}},*)$ and $(Y_{\rho_{12}},\circ)$ be two bitopological semigroups. We shall write $h:(X_{\tau_{12}},*)\to(Y_{\rho_{12}},\circ)$ and say that h is an S_i*-map* from $(X_{\tau_{12}},*)$ into $(Y_{\rho_{12}},\circ)$ provided h is an S-map from as a function an S-space $(X_{\tau_i},*)$ into an S-space $(Y_{\rho_i},\circ)$, where $i=1,2$. We say that h is an *Sp-map* if it is an S_1-map and S_2-map.

It is easy to check that the composition of Sp- S-maps (resp., S_i-map) is an Sp-map (resp., S_i-maps) and the usual product of Sp-maps (resp., S_i-maps) is an Sp-map (resp., S_i-map). The identity p-map id on any bitopological semigroup $(X_{\tau_{12}},*)$ is an Sp-map from $(X_{\tau_{12}},*)$ into itself.

There are many ways in which S-spaces and bitopological spaces can be regarded as bitopological semigroups. In our work, for any S-space $(X_\tau,*)$ can be regarded as a bitopological semigroup $(X_{\tau\tau},*)$ and for any bitopological space $X_{\tau_{12}}$ can be regarded as bitopological semigroup $(X_{\tau_{12}},\pi)$.

By an *c-bitopological semigroups* we mean a triple $(X_{\tau_{12}},*,\mathscr{X})$ consisting of bitopological semigroups $(X_{\tau_{12}},*)$ and an S-map $\mathscr{X}:(X_{\tau_2},*)\to(X_{\tau_1},*)$ from an S-space $(X_{\tau_2},*)$ into an S-space $(X_{\tau_1},*)$.

Let $(X_{\tau_{12}},*,\mathscr{X})$ and $(Y_{\rho_{12}},\circ,\mathscr{Y})$ be two c-bitopological semigroups. It is easy to check that the usual product

$$(X_{\tau_{12}},*,\mathscr{X})\times(Y_{\rho_{12}},\circ,\mathscr{Y})=((X\times O,\tau_1\times\rho_1,\tau_2\times\rho_2),*\times\circ,\mathscr{X}\times\mathscr{Y})$$

of $(X_{\tau_{12}},*,\mathscr{X})$ and $(Y_{\rho_{12}},\circ,\mathscr{Y})$ is an c-bitopological semigroups.

In our work, for any S-space $(O_\rho,\circ)$ can be regarded as an c-bitopological semigroup $(O_{\rho\rho},\circ,id)$ where id is the identity S-map on $(O_\rho,\circ)$. That is, $(O_\rho,\circ):=(O_{\rho\rho},\circ,id)$.

Definition 3.1.
Let $(X_{\tau_{12}},*,\mathscr{X})$ be an c-bitopological semigroup and $(O_\rho,\circ)$ be an S-space. An c-map *from $(X_{\tau_{12}},*,\mathscr{X})$ into $(O_\rho,\circ)$ is a pair*

$$f_{12}=(f_1,f_2):(X_{\tau_{12}},*,\mathscr{X})\to(O_\rho,\circ)$$

of an S_1-map $f_1:(X_{\tau_1},*)\to(O_\rho,\circ)$ and S_2-map $f_2:(X_{\tau_2},*)\to(O_\rho,\circ)$ such that $f_1\circ\mathscr{X}=f_2$.

There are many ways in which S-maps $f:(X_\tau,*)\to(O_\rho,\circ)$ can be regarded as c-maps $f_{12}=(f,f)$ from an c-bitopological semigroup $(X_{\tau\tau},*,id)$ into an S-space $(O_\rho,\circ)$.

Example 3.2.
Let $(X_{\tau_{12}},*,\mathscr{X})$ be an c-bitopological semigroup. For every S-space $(O_\rho,\circ)$, the Sp-map

$$f_{12}=(f,f):((X\times O)_{\tau_{12}\times\rho},*\times\circ,\mathscr{X}\times id)\to(O_\rho,\circ)$$

is an c-map, where $f(x,y)=y$ for all $x\in X$, $y\in O$. We observe that

$$[f\circ(\mathscr{X}\times id)](x,y)=f(\mathscr{X}(x),y)=y=f(x,y)$$

for all $x\in X$, $y\in O$.

Theorem 3.1.
The usual product

$$f_{12}\times f'_{12}=(f_1\times f'_1,f_2\times f'_2):(X_{\tau_{12}},*,\mathscr{X})\times(X'_{\tau'_{12}},*',\mathscr{X}')\to(O_\rho,\circ)\times(O'_{\rho'},\circ')$$

of the two c-maps $f_{12}:(X_{\tau_{12}},*,\mathscr{X})\to(O_\rho,\circ)$ and $f'_{12}:(X'_{\tau'_{12}},*',\mathscr{X}')\to(O'_{\rho'},\circ')$ is an c-map.

Proof. It's clear that $f_1 \times f_1'$ is an S_1-map and $f_2 \times f_2'$ is an S_2-map. Since f_{12} and f_{12}' are c-maps, then $f_1 \circ \mathscr{X} = f_2$ and $f_1' \circ \mathscr{X}' = f_2'$. Hence

$$(f_1 \times f_1') \circ (\mathscr{X} \times \mathscr{X}') = (f_1 \circ \mathscr{X}) \times (f_1' \circ \mathscr{X}') = (f_2 \times f_2').$$

That is, $f_{12} \times f_{12}'$ is an c-map. □

Theorem 3.2.
The composition

$$f' \circ f_{12} = (f' \circ f_1, f' \circ f_2) : (X_{\tau_{12}}, *, \mathscr{X}) \to (O'_{\rho'}, \circ')$$

of an c-map $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ and an S-map $f' : (O_\rho, \circ) \to (O'_{\rho'}, \circ')$ is an c-map.

Proof. We observe that $f' \circ f_1$ and $f' \circ f_2$ are S-maps. Since f_{12} is an c-map, then $f_1 \circ \mathscr{X} = f_2$. Hence

$$(f' \circ f_1) \circ \mathscr{X} = f \circ (f_1 \circ \mathscr{X}) = f' \circ f_2.$$

That is, $f' \circ f_{12}$ is an c-map. □

Recall [9] that for an S-map $f : (X_\tau, *) \to (O_\rho, \circ)$, the function: $\alpha \to f \circ \alpha$ for all $\alpha \in P(X_\tau)$ from $P(X_\tau, *)$ into $P(O_\rho, \circ)$ is an S-map, denoted by $\widehat{f}$. Then for every c-bitopological semigroup $(X_{\tau_{12}}, *, \mathscr{X})$, the triple $(P(X)_{\tau_{12}^c}, p(*), \widehat{\mathscr{X}})$ is an c-bitopological semigroup where τ_1^c and τ_2^c are compact-open topologies on $P(X)$ which induced by τ_1 and τ_2, respectively. The shorter notion for this c-bitopological semigroup will be $P(X_{\tau_{12}}, *, \mathscr{X})$.

Theorem 3.3.
The pair $\widehat{f}_{12} = (\widehat{f}_1, \widehat{f}_2) : P(X_{\tau_{12}}, *, \mathscr{X}) \to P(O_\rho, \circ)$ is an c-map for every c-map $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$.

Proof. We already noticed above that the functions $\widehat{f}_1$ and $\widehat{f}_2$ are S-maps, since f_1 and f_2 are S-maps. Since f_{12} is an c-map, then $f_1 \circ \mathscr{X} = f_2$. Hence

$$(\widehat{f}_1 \circ \widehat{\mathscr{X}})(\alpha)(t) = [(f_1 \circ \mathscr{X}) \circ \alpha](t) = (f_2 \circ \alpha)(t) = \widehat{f}_2(\alpha)(t)$$

for all $\alpha \in P(X_{\tau_2})$, $t \in I$. That is, $\widehat{f}_1 \circ \widehat{\mathscr{X}} = \widehat{f}_2$. Hence $\widehat{f}_{12}$ is an c-map. □

Theorem 3.4.
Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to P(O_\rho, \circ)$ be an c-map. Then for every $t \in I$, the pair $[f_{12}]_t = ([f_1]_t, [f_2]_t) : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ is an c-map.

Proof. Consider an S-map $f_1 : P(X_{\tau_1}, *) \to P(O_\rho, \circ)$. Recall [9] that for every $t \in I$, there is a natural evaluation S-map $\mathscr{E}_t : P(O_\rho, \circ) \to (O_\rho, \circ)$ given by $\mathscr{E}_t(\alpha) = \alpha(t)$ for all $\alpha \in P(O_\rho)$. Then for every $t \in I$, the composition $\mathscr{E}_t \circ f_1$ is an S-map; thus

$$[f_1]_t(x) = f_1(x)(t) = (\mathscr{E}_t \circ f_1)(x)$$

for every $x \in X$. Then for every $t \in I$, $[f_1]_t$ is an S-map. Similarly, for every $t \in I$, $[f_2]_t$ is an S-map.

Now since $f_1 \circ \mathscr{X} = f_2$, then for every $t \in I$,

$$[f_1]_t \circ \mathscr{X} = (\mathscr{E}_t \circ f_1) \circ \mathscr{X} = \mathscr{E}_t \circ (f_1 \circ \mathscr{X}) = \mathscr{E}_t \circ f_2 = [f_2]_t.$$

That is, for every $t \in I$, $[f_{12}]_t$ is an c-map. □

Let $(X_{\tau_{12}}, *, \mathscr{B})$ be an c-bitopological semigroup and $(B|_{\tau_{12}}, *)$ be an b-subspace of $(X_{\tau_{12}}, *)$. The c-bitopological semigroup $(B|_{\tau_{12}}, *, \mathscr{B})$ is called an *c-subspace* of $(X_{\tau_{12}}, *, \mathscr{X})$ provided $\mathscr{B}(b) = \mathscr{X}(b)$ for all $b \in B$.

In the following theorem, we study the restriction property of c-maps on c-subspaces.

Theorem 3.5.
Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an c-map and $(B|_\rho, \circ)$ be an S-subspace of $(O_\rho, \circ)$ such that $f_1^{-1}(B) = f_2^{-1}(B)$. Then the triple $(B^-|_{\tau_{12}}, *, \mathscr{X}|_{B^-})$ is an c-subspace of $(X_{\tau_{12}}, *, \mathscr{X})$ and a pair $f_{12}|_{B^-} = (f_1|_{B^-}, f_2|_{B^-})$ is an c-map from an c-bitopological semigroup $(B^-|_{\tau_{12}}, *, \mathscr{X}|_{B^-})$ into $(B|_\rho, \circ)$, where $B^- = f_1^{-1}(B)$.

Proof. Its clear that $B^-|_{\tau_{12}}$ is a bitopological subspace of $X_{\tau_{12}}$. Since f_{12} is an c-map then for $x \in B^-$,

$$f_1[\mathscr{X}|_{B^-}(x)] = f_1[\mathscr{X}(x)] = f_2(x) \in f_2(B^-) \subseteq B.$$

That is, $\mathscr{X}|_{B^-}(x) \in f_1^{-1}(B) = B^-$. Then $\mathscr{X}|_{B^-}$ is a well-defined S-map taking B^- into B^-. Then $(B^-|_{\tau_{12}}, *, \mathscr{X}|_{B^-})$ is an c-subspace of $(X_{\tau_{12}}, *, \mathscr{X})$.

Since f_1 is an S_1-map and f_2 is an S_2-map, then $f_1|_{B^-}$ is an S_1-map and $f_2|_{B^-}$ is an S_2-map, respectively. Since $f_1 \circ \mathscr{X} = f_2$, then

$$(f_1|_{B^-} \circ \mathscr{X}|_{B^-})(x) = f_1|_{B^-}[\mathscr{X}(x)] = (f_1 \circ \mathscr{X})(x) = f_2(x) = f_2|_{B^-}(x)$$

for all $x \in B^-$. That is, $f_1|_{B^-} \circ \mathscr{X}|_{B^-} = f_2|_{B^-}$. Hence $f_{12}|_{B^-}$ is an c-map. □

4. $C_\aleph$-fibrations

In this section, we introduce the concept of $C_\aleph$-fibration and study some its basic properties such as a restriction property, a composition property, and a product property,. Furthermore, we show that the $C_\aleph$-fibrations are generalized the notions for $S_\aleph$-fibrations.

We first give the concept of the $C_\aleph$-fibration property by an S-maps in the homotopy theory for topological spaces.

Definition 4.1.
Let $f : (X_\tau, *) \to (O_\rho, \circ)$ and $h : (X'_{\tau'}, *') \to (X_\tau, *)$ be two S-maps. An S-map f is said to be have the $C_\aleph$-fibration property by an S-map h *provided for every $(Y_\omega, \star) \in \aleph$ and given two S-maps $g : (Y_\omega, \star) \to (X'_{\tau'}, *')$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ with $G_0 = f \circ (h \circ g)$, there exists an S-homotopy $H : (Y_\omega, \star) \to P(X_\tau, *)$ such that $H_0 = h \circ g$ and $f \circ H_t = G_t$ for all $t \in I$.*

Definition 4.2.
An c-map $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ is called an $C_\aleph$-fibration *if an S_1-map $f_1 : (X_{\tau_1}, *) \to (O_\rho, \circ)$ has the $C_\aleph$-fibration property by an S-map $\mathscr{X} : (X_{\tau_2}, *) \to (X_{\tau_1}, *)$.*

Theorem 4.1.
An c-map $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ is called an $C_\aleph$-fibration *if for every $(Y_\omega, \star) \in \aleph$ and given two S-maps $g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ with $G_0 = f_2 \circ g$, there exists an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $H_0 = \mathscr{X} \circ g$ and $f_1 \circ H_t = G_t$ for all $t \in I$.*

Proof. It is obvious by Definitions (4.1), (4.2) and the definition of an c-map. □

By $\mathscr{J}_1$ and $\mathscr{J}_2$ we mean the usual first and the second projection S-maps (or maps), respectively.

Example 4.1.
In Example (3.1), the c-map f_{12} is an $C_\aleph$-fibration. For every $(Y_\omega, \star) \in \aleph$ and for given two S-maps $g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ with $G_0 = f_2 \circ g$, define an S-homotopy $H : (Y, \omega) \to P(X \times O, \tau_1 \times \rho)$ by

$$H(y)(t) = [\mathscr{X}[\mathscr{J}_1(g(y))], G(y)(t)]$$

for all $y \in Y$, $t \in I$. We observe that

$$\begin{aligned} H_0(y) &= [\mathscr{X}[\mathscr{J}_1(g(y))], G(y)(0)] = [\mathscr{X}[\mathscr{J}_1(g(y))], (f_2 \circ g)(y)] \\ &= [\mathscr{X}[\mathscr{J}_1(g(y))], \mathscr{J}_2(g(y))] = (\mathscr{X} \times id)[\mathscr{J}_1(g(y)), \mathscr{J}_2(g(y))] \\ &= (\mathscr{X} \times id)[g(y)] = [(\mathscr{X} \times id) \circ g](y) \end{aligned}$$

for all $y \in Y$ and $f_1 \circ H_t = G_t$ for all $t \in I$.

In the following theorem and its corollary, we study the $C_\aleph$-fibration property and the $C_\aleph$-fibration property for the restriction S-maps of $C_\aleph$-fibrations $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ on $f_1^{-1}(B)$ and $f_2^{-1}(B)$ for every S-subspace $(B|_\rho, \circ)$ of $(O_\rho, \circ)$.

Theorem 4.2.
Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an $C_\aleph$-fibration and $(B|_\rho, \circ)$ be a subspace of $(O_\rho, \circ)$. Then the restriction map $f_1|_{B_1} : (B_1, \tau_1|_{B_1}) \to (B|_\rho, \circ)$ has the $C_\aleph$-fibration property by an S-map $\mathscr{X}|_{B_2} : (B_2|_{\tau_2}, *) \to (B_1|_{\tau_1}, *)$, where $B_i = f_i^{-1}(B)$, $(i = 1, 2)$.

Proof. Since f_{12} is an c-map then for $x \in B_2$,

$$f_1[\mathscr{X}|_{B_2}(x)] = f_1[\mathscr{X}(x)] = f_2(x) \in f_2(B_2) \subseteq B.$$

That is, $\mathscr{X}|_{B_2}(x) \in f_1^{-1}(B) = B_1$. Then $\mathscr{X}|_{B_2}$ is a well-defined S-map taking B_2 into B_1.

Now let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to (B_2|_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(B|_\rho, \circ)$ be two S-maps with $G_0 = f_1|_{B_1} \circ (\mathscr{X}|_{B_2} \circ g)$. Let $j_2 : (B_2|_{\tau_2}, *) \to (X_{\tau_2}, *)$ and $j : (B|_\rho, \circ) \to (O_\rho, \circ)$ be inclusion S-maps. Then

$$[j \circ G]_0 = (f_1 \circ \mathscr{X}) \circ (j_2 \circ g) = f_2 \circ (j_2 \circ g).$$

Since f_{12} is an $C_\aleph$-fibration, then there exists an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $H_0 = \mathscr{X} \circ (j_2 \circ g) = \mathscr{X}|_{B_2} \circ g$ and

$$(f_1 \circ H)(y)(t) = (j \circ G)(y)(t) = G(y)(t)$$

for all $y \in Y$, $t \in I$. By the last part, note that $H(y)(t) \in B_1$ for all $y \in Y$, $t \in I$. That is, we can consider H as homotopy :$(Y_\omega, \star) \to P(B_1|_{\tau_1}, *)$. Hence $f_1|_{B_1}$ has the $C_\aleph$-fibration property by an S-map $\mathscr{X}|_{B_2}$. □

The proof of the following corollary is obvious by the last theorem and Theorems (4.1) and (3.5).

Corollary 4.1.
Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an $C_\aleph$-fibration and $(B|_\rho, \circ)$ be an S-subspace of $(O_\rho, \circ)$ such that $f_1^{-1}(B) = f_2^{-1}(B)$. Then the restriction c-map

$$f_{12}|_{B^-} : (B^-|_{\tau_{12}}, *, \mathscr{X}|_{B^-}) \to (B|_\rho, \circ)$$

is an $C_\aleph$-fibration, where $B^- = f_1^{-1}(B)$.

The following result shows that the the composition of an $C_\aleph$-fibration and any $S_\aleph$-fibration will be an $C_\aleph$-fibration.

Theorem 4.3.
The composition c-map $f' \circ f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O'_{\rho'}, \circ')$ of an $C_\aleph$-fibration $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ and an $S_\aleph$-fibration $f' : (O_\rho, \circ) \to (O'_{\rho'}, \circ')$ is an $C_\aleph$-fibration.

Proof. Let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(O'_{\rho'}, \circ')$ be two S-maps with $G_0 = (f' \circ f_2) \circ g = f' \circ (f_2 \circ g)$. Since f' is an $S_\aleph$-fibration, then there exists an S-homotopy $F : (Y_\omega, \star) \to P(O_\rho, \circ)$ such that $F_0 = f_2 \circ g$ and $f' \circ F_t = G_t$ for all $t \in I$. By the first part and f_{12} is an $C_\aleph$-fibration, there exists an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $H_0 = \mathscr{X} \circ g$ and $f_1 \circ H_t = F_t$ for all $t \in I$. Then $(f' \circ f_1) \circ H_t = f' \circ (f_1 \circ H_t) = f' \circ F_t = G_t$ for all $t \in I$. Hence $f' \circ f_{12}$ is an $C_\aleph$-fibration. □

Theorem 4.4.
The usual product c-map

$$f_{12} \times f'_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \times (X'_{\tau'_{12}}, *', \mathscr{X}') \to (O_\rho, \circ) \times (O'_{\rho'}, \circ')$$

of two $C_\aleph$-fibrations $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ and $f'_{12} : (X', \tau'_1, \tau'_2)_{\mathscr{X}'} \to (O'_{\rho'}, \circ')$ is an $C_\aleph$-fibration.

Proof. Let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to [(X \times X')_{\tau_2 \times \tau'_2}, * \times *']$ and

$$G : (Y_\omega, \star) \to P[(O \times O')_{\rho \times \rho'}, \circ \times \circ']$$

be two S-maps with $G_0 = (f_2 \times f'_2) \circ g$. For an $C_\aleph$-fibration f_{12}, consider the two S-maps $\mathscr{J}_1 \circ g$ and $\mathscr{J}_1 \circ G$. Since

$$[\mathscr{J}_1 \circ G]_0 = \mathscr{J}_1 \circ G_0 = \mathscr{J}_1 \circ [(f_2 \times f'_2) \circ g] = f_2 \circ (\mathscr{J}_1 \circ g),$$

then there exists an S-homotopy $F : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $F_0 = \mathscr{X} \circ (\mathscr{J}_1 \circ g)$ and $f_1 \circ F_t = \mathscr{J}_1 \circ G_t$ for all $t \in I$. For an $C_\aleph$-fibration f'_{12}, similarly, then there exists an S-homotopy $F' : (Y_\omega, \star) \to P(X', \tau'_1)$ such that $F'_0 = \mathscr{X}' \circ (\mathscr{J}_2 \circ g)$ and $f'_1 \circ F'_t = \mathscr{J}_2 \circ G_t$ for all $t \in I$.

Define an S-homotopy $H : (Y_\omega, \star) \to P[(X \times X')_{\tau_1 \times \tau'_1}, * \times *']$ by $H_t = F_t \times F'_t$ for all $t \in I$. We observe that

$$H_0 = [\mathscr{X} \circ (\mathscr{J}_1 \circ g)] \times [\mathscr{X}' \circ (\mathscr{J}_2 \circ g)] = (\mathscr{X} \times \mathscr{X}') \circ [(\mathscr{J}_1 \circ g) \times (\mathscr{J}_2 \circ g)] = (\mathscr{X} \times \mathscr{X}') \circ g$$

and

$$(f_1 \times f'_1) \circ H_t = (f_1 \circ F_t) \times (f'_1 \circ F'_t) = (\mathscr{J}_1 \circ G_t) \times (\mathscr{J}_2 \circ G_t) = G_t$$

for all $t \in I$. Hence $f_{12} \times f'_{12}$ is an $C_\aleph$-fibration.

□

Theorem 4.5.
Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an c-map. If f_1 or f_2 is $S_\aleph$-fibration then f_{12} is an $C_\aleph$-fibration.

Proof. Firstly, let f_1 be an $S_\aleph$-fibration. Let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ be two S-maps with $G_0 = f_2 \circ g = f_1 \circ (\mathscr{X} \circ g)$. Since $\mathscr{X} \circ g$ is an S-map from $(Y_\omega, \star)$ into $(X_{\tau_1}, *)$ and f_1 is an $S_\aleph$-fibration, then there exists an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $H_0 = \mathscr{X} \circ g$ and $f_1 \circ H_t = G_t$ for all $t \in I$. That is f_{12} is an $C_\aleph$-fibration.

The other case, let $f_2 : (X_{\tau_1}, *) \to (O_\rho, \circ)$ be an $S_\aleph$-fibration. Let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ be two S-maps with $G_0 = f_2 \circ g$. Then there exists an S-homotopy $F : (Y_\omega, \star) \to P(X_{\tau_2}, *)$ such that $F_0 = g$ and $f_2 \circ F_t = G_t$ for all $t \in I$. Define an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ by $H_t = \mathscr{X} \circ F_t$ for all $t \in I$. Then $H_0 = \mathscr{X} \circ F_0 = \mathscr{X} \circ g$ and

$$f_1 \circ H_t = f_1 \circ (\mathscr{X} \circ F_t) = f_2 \circ F_t = G_t$$

for all $t \in I$. That is f_{12} is an $C_\aleph$-fibration. □

Theorem 4.6.
Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an $C_\aleph$-fibration.If there exists an S-map $\mathscr{X}' : (X_{\tau_1}, *) \to (X_{\tau_2}, *)$ such that $\mathscr{X} \circ \mathscr{X}' = id$ then f_1 is an $S_\aleph$-fibration as an S-map: $(X_{\tau_1}, *) \to (O_\rho, \circ)$.

Proof. Let $(Y_\omega, \star) \in \aleph$ and let $g : (Y_\omega, \star) \to (X_{\tau_1}, *)$ and $G : (Y_\omega, \star) \to P(O_\rho, \circ)$ be two S-maps with $G_0 = f_1 \circ g = f_2 \circ (\mathscr{X}' \circ g)$. Since $\mathscr{X}' \circ g$ is an S-map from $(Y_\omega, \star)$ into $(X_{\tau_2}, *)$ and f_{12} is an $C_\aleph$-fibration, then there is an S-homotopy $H : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $H_0 = \mathscr{X} \circ (\mathscr{X}' \circ g) = g$ and $f_1 \circ H_t = G_t$ for all $t \in I$. That is f_1 is an $S_\aleph$-fibration. □

5. Covering homotopy property

In this section, we introduce a covering homotopy theorem for S-maps into $C_\aleph$-fibrations.

Theorem 5.1.
Let $f_{12} : (X_{\tau_{12}}, *, \mathscr{X}) \to (O_\rho, \circ)$ be an $C_\aleph$-fibration and let $M, M' : (Y_\omega, \star) \to P(X_{\tau_2}, *)$ be two S-maps. Let $M_0 \simeq_s M'_0$ and $\widehat{f_2} \circ M \simeq_s \widehat{f_2} \circ M'$ by two S-homotopies $R : (Y_\omega, \star) \to P(X_{\tau_2}, *)$ and $G : (Y_\omega, \star) \to P[P(O)_{\rho^c}, p(\circ)]$, respectively. If $G_{0t} = f_2 \circ R_t$ for all $t \in I$, then there exists an S-homotopy $F : (Y_\omega, \star) \to P[P(X)_{\tau_1^c}, p(*)]$ between $\widehat{\mathscr{X}} \circ M$ and $\widehat{\mathscr{X}} \circ M'$ such that $F_{0t} = \mathscr{X} \circ R_t$ and $f_1 \circ F_{rt} = G_{rt}$ for all $r, t \in I$.

Proof. Let $A = (I \times \{0\}) \cup (\{0\} \times I) \cup (I \times \{1\}) \subset I \times I$. By hypothesis, we can define an S-map $L_{(s,t)} : (Y_\omega, \star) \to (X_{\tau_2}, *)$ by

$$L_{(s,t)}(y) = \begin{cases} M(y)(s), & t = 0; \\ R(y)(t), & s = 0; \\ M'(y)(s), & t = 1 \end{cases}$$

for all $y \in Y$, $(s, t) \in A$. Recall ([3], P. 100) that there is a homeomorphism $\mathscr{H} : I \times I \to I \times I$ taking A onto $I \times \{0\}$. By hypothesis, note that

$$f_2[L_{(s,t)}(y)] = [G(y)(s)](t)$$

for all $y \in Y$, $(s, t) \in A$.

For every $s \in I$, define an S-map $g^s : (Y_\omega, \star) \to (X_{\tau_2}, *)$ and an S-homotopy $G^s : (Y_\omega, \star) \to P(O_\rho, \circ)$ by $g^s(y) = L_{\mathscr{H}^{-1}(s,0)}(y)$ and

$$G^s(y)(t) = [G(y)((\mathscr{J}_1 \circ \mathscr{H}^{-1})(s, t))]((\mathscr{J}_2 \circ \mathscr{H}^{-1})(s, t))$$

for all $y \in Y$, $t \in I$. We observe that for every $s \in I$,

$$\begin{aligned} G^s(y)(0) &= [G(y)((\mathscr{J}_1 \circ \mathscr{H}^{-1})(s, 0))]((\mathscr{J}_2 \circ \mathscr{H}^{-1})(s, 0)) \\ &= f_2[L_{((\mathscr{J}_1 \circ \mathscr{H}^{-1})(s,0), (\mathscr{J}_2 \circ \mathscr{H}^{-1})(s,0))}(y)] = f_2[L_{\mathscr{H}^{-1}(s,0)}(y)] \\ &= (f_2 \circ g^s)(y) \end{aligned}$$

for all $y \in Y$. Then for every $s \in I$, since f_{12} is an $C_\aleph$-fibration, then there exists an S-homotopy $F^s : (Y_\omega, \star) \to P(X_{\tau_1}, *)$ such that $F^s_0 = \mathscr{X} \circ g^s$ and $f_1 \circ F^s_t = G^s_t$ for all $t \in I$. Define an S-homotopy $F : (Y_\omega, \star) \to P[P(X)_{\tau_1^c}, p(*)]$ by

$$F_{st}(y) = [F^{(\mathscr{J}_1 \circ \mathscr{H})(s,t)}(y)]((\mathscr{J}_2 \circ \mathscr{H})(s, t))$$

for all $y \in Y, s, t \in I$. Note that

$$\begin{aligned} F_{s0}(y) &= [F^{(\mathscr{J}_1\circ\mathscr{H})(s,0)}(y)]((\mathscr{J}_2\circ\mathscr{H})(s,0)) = [F^{(\mathscr{J}_1\circ\mathscr{H})(s,0)}(y)](0) \\ &= \mathscr{X}[g^{(\mathscr{J}_1\circ\mathscr{H})(s,0)}(y)] = \mathscr{X}\{L_{\mathscr{H}^{-1}((\mathscr{J}_1\circ\mathscr{H})(s,0),0)}(y)\} \\ &= \mathscr{X}\{L_{\mathscr{H}^{-1}((\mathscr{J}_1\circ\mathscr{H})(s,0),(\mathscr{J}_2\circ\mathscr{H})(s,0))}(y)\} \\ &= \mathscr{X}\{L_{\mathscr{H}^{-1}(\mathscr{H}(s,0))}(y)\} = \mathscr{X}\{L_{(s,0)}(y)\} = (\mathscr{X}\circ M)(y)(s) \end{aligned}$$

and

$$\begin{aligned} F_{s1}(y) &= [F^{(\mathscr{J}_1\circ\mathscr{H})(s,0)}(y)]((\mathscr{J}_2\circ\mathscr{H})(s,1)) = [F^{(\mathscr{J}_1\circ\mathscr{H})(s,1)}(y)](0) \\ &= \mathscr{X}[g^{(\mathscr{J}_1\circ\mathscr{H})(s,1)}(y)] = \mathscr{X}\{L_{\mathscr{H}^{-1}((\mathscr{J}_1\circ\mathscr{H})(s,1),0)}(y)\} \\ &= \mathscr{X}\{L_{\mathscr{H}^{-1}((\mathscr{J}_1\circ\mathscr{H})(s,1),(\mathscr{J}_2\circ\mathscr{H})(s,1))}(y)\} \\ &= \mathscr{X}\{L_{\mathscr{H}^{-1}(\mathscr{H}(s,1))}(y)\} = \mathscr{X}\{L_{(s,1)}(y)\} = (\mathscr{X}\circ M')(y)(s) \end{aligned}$$

for all $s \in I, y \in Y$. That is, F is an S-homotopy between $\mathscr{X}\circ M$ and $\mathscr{X}\circ M'$. Also note that

$$\begin{aligned} F_{0t}(y) &= [F^{(\mathscr{J}_1\circ\mathscr{H})(0,t)}(y)]((\mathscr{J}_2\circ\mathscr{H})(0,t)) = [F^{(\mathscr{J}_1\circ\mathscr{H})(t,0)}(y)](0) \\ &= \mathscr{X}[g^{(\mathscr{J}_1\circ\mathscr{H})(t,0)}(y)] = \mathscr{X}\{L_{\mathscr{H}^{-1}((\mathscr{J}_1\circ\mathscr{H})(t,0),0)}(y)\} \\ &= \mathscr{X}\{L_{\mathscr{H}^{-1}((\mathscr{J}_1\circ\mathscr{H})(t,0),(\mathscr{J}_2\circ\mathscr{H})(t,0))}(y)\} \\ &= \mathscr{X}\{L_{\mathscr{H}^{-1}(\mathscr{H}(t,0))}(y)\} = \mathscr{X}\{L_{(t,0)}(y)\} = (\mathscr{X}\circ R_t)(y) \end{aligned}$$

and

$$\begin{aligned} (f_1\circ F_{st})(y) &= f_1\{[F^{(\mathscr{J}_1\circ\mathscr{H})(s,t)}(y)][(\mathscr{J}_2\circ\mathscr{H})(s,t)]\} = G^{(\mathscr{J}_1\circ\mathscr{H})(s,t)}(y)((\mathscr{J}_2\circ\mathscr{H})(s,t)) \\ &= [G(y)((\mathscr{J}_1\circ\mathscr{H}^{-1})((\mathscr{J}_1\circ\mathscr{H})(s,t),(\mathscr{J}_2\circ\mathscr{H})(s,t)))] \\ &\quad \{(\mathscr{J}_2\circ\mathscr{H}^{-1})((\mathscr{J}_1\circ\mathscr{H})(s,t),(\mathscr{J}_2\circ\mathscr{H})(s,t))\} \\ &= [G(y)((\mathscr{J}_1\circ\mathscr{H}^{-1})(\mathscr{H}(s,t)))]\{(\mathscr{J}_2\circ\mathscr{H}^{-1})(\mathscr{H}(s,t))\} \\ &= [G(y)(\mathscr{J}_1(s,t))]\{\mathscr{J}_2(s,t)\} \\ &= [G(y)(s)](t) = G_{st}(y) \end{aligned}$$

for all $s, t \in I$, $y \in Y$. That is, $F_{0t} = \mathscr{X}\circ R_t$ and $f_1\circ F_{rt} = G_{rt}$ for all $r, t \in I$. □

Corollary 5.1.
Let $f_{12}:(X_{\tau_{12}},*,\mathscr{X})\to(O_\rho,\circ)$ be an $C_{\aleph}$-fibration. Let $M, M':(Y_\omega,\star)\to P(X_{\tau_2},*)$ be two S-maps such that $M_0 = M'_0$ and $\widehat{f}_2\circ M = \widehat{f}_2\circ M'$. Then there exists an S-homotopy $F:(Y_\omega,\star)\to P[P(X)_{\tau_1^c}, p(*)]$ between $\widehat{\mathscr{X}}\circ M$ and $\widehat{\mathscr{X}}\circ M'$ such that $F_{0t} = \mathscr{X}\circ M_0 = \mathscr{X}\circ M'_0$ for all $t\in I$ and $f_1\circ F_{rt} = f_2\circ M_r$ for all $r,t\in I$.

Proof. Define an S-homotopy $R:(Y_\omega,\star)\to P(X_{\tau_2},*)$ by $R(y)(t) = M_0(y)$ and an S-homotopy $G:(Y_\omega,\star)\to P[P(O)_{\rho^c}, p(\circ)]$ by $[G(y)(s)](t) = (f_2\circ M_s)(y)$ for all $s,t\in I$, $y\in Y$. Then by using the above theorem, one can get the desired S-homotopy. □

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