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The solution of simultaneous time and space fractional order boundary value problems

Research Article

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Abstract: In this talk we present an analytic method for the solution of fractional order boundary value problems. We see how to deal with a double fractional derivative operator and solve fractional Heat, Wave and Laplace equations. The proposed method is the composition of the classical method of separation of variables technique and Udita N. Katugampola's fractional derivative.

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1. Introduction

When the concept of fractional derivative took place, Many mathematicians have taken keen interest in it as it is the generalized calculus. Now the scenario has changed, every property or application of calculus has the corresponding effect of fractional derivative. For the brief history of fractional calculus, one may see [1], [2]. In starting there were a few definitions for fractional derivative. Later many modified forms of the definitions have been made. R. Khalil [4] and colleagues have given a new definition of fractional derivative known as Conformable derivative in a classical sense using a limit approach. It simply converts a fractional derivative to an ordinary derivative with the product of the independent variable. It satisfies most of the properties of a derivative. Many researchers [5],[6],[7], [9] and [11] have presented many applications and many properties based on this definition. Udita. N. Katugampola [8] has presented the general form of a Conformable fractional derivative using exponential function. Now researchers are curious to study a non integer ordered derivative in the various field of science and engineering.
Fractional boundary value problems[3] are now generalising the space of boundary value problems. Though the theory is yet not complete for the justification, still it has become one of the most interesting branch of fractional calculus to solve numerous problems in physics and engineering like fractional Numen and Dirichlet problems, fractional heat and wave equation, fractional Laplace equation. Heat conduction in the underground water flow, thermo-elasticity, plasma physics, etc.. While modeling physical phenomena, these problems can be solved after converting into a non local problem with integral boundary conditions.

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Likewise partial differential equations, fractional order partial differential equations also has series solution or numerical solution. The series solution of well-known Heat, Wave[10], Cahn–Allen and Cahn–Hilliard equations, Conformable space time-fractional Kawahara equation has already obtained (in time fractional domain only) with help of conformable fractional derivative.

The main interest of the present paper is to provide an analytical solution of fractional order boundary value problems including Heat and Wave equation involving fractional derivatives in both time and space variables simultaneously.

A function $f(x)$ with it's α derivative is said to be solution of simple fractional boundary value problem if it satisfy $D^{\alpha}f(x) = f(x, g(x)); x \in I$ with the conditions $f(0) = f_0, f'(0) = f^*$ on I.

If $f : [0,\infty) \to \mathbb{R}$ then, α order Katugampola fractional derivative of f is defined and denoted by

$$D^{\alpha}(f)(x) = \lim_{\delta \to 0} \frac{f(xe^{\delta x^{-\alpha}}) - f(x)}{\delta}; \ \alpha \in (0,1] \tag{1}$$

provided limit exists.

Also, if f is differentiable,then

$$D^{\alpha}[f(x)] = x^{1-\alpha} f'(x). \tag{2}$$

2. The methodology

If the partial differential equation is linear, associated coefficients in the equation are separable and Boundary conditions are homogeneous and bounded then this method is a powerful technique to solve initial-boundary value problem which converts a partial differential equation involving m number of independent variables into a system of m ordinary differential equations as we assume that it's solution is in separable form $f(x,t) = X(x).T(t)$. It reduces each independent variable into an eigenvalue problem like $X'' + \lambda X = 0$ corresponding to some suitable constantλ. We apply the boundary conditions and find the eigenvalueλ and the corresponding eigen functions X. By the principle of superposition we obtain the general solution$\sum X_m$ and using the initial condition get the complete solution as $f(x,t) = \sum X_m T_m$.

The separation of variables technique transforms the respective boundary value problem into a number of ordinary differential equations.

Consider the solution function as $z(x) = e^{mx^{\alpha}}$ for each dependent variable $z(x)$ and from Katugampola's derivative

$$\frac{d^{\alpha}}{dx^{\alpha}} z(x) = x^{1-\alpha} e^{mx^{\alpha}} m\alpha x^{\alpha-1} = m\alpha e^{mx^{\alpha}} \tag{3}$$

and

$$\frac{d^{\alpha}}{dx^{\alpha}} \frac{d^{\alpha}}{dx^{\alpha}} z(x) = \frac{d^{\alpha}}{dx^{\alpha}} (m\alpha e^{mx^{\alpha}}) = m^2\alpha^2 e^{mx^{\alpha}}. \tag{4}$$

Provided all the independent variables are positive and bounded.

The notations

$$X^{\alpha}(x) \equiv \frac{d^{\alpha}X}{dx^{\alpha}} \ \text{ and } \ X^{2\alpha}(x) \equiv \frac{d^{\alpha}}{dx^{\alpha}}\left(\frac{d^{\alpha}X}{dx^{\alpha}}\right)$$

are used to represent one and two times $\alpha-$ order differentiation of X respectively.

3. Solved problems

Here we assume that the domain of each variable is positive and fractional order α, β lies in $(0,1]$.

Example 3.1.

Consider the following partial differential equation

$$\frac{\partial^{\beta}}{\partial x^{\beta}}\left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right) = e^{-t^{\alpha}} \cos x^{\beta}; \quad x, \ t > 0 \tag{5}$$

subject to conditions, $u(x,0) = 0$ and $u(0,t) = 0$.

Consider $u(x,t) = T(t)X(x)$, so that we have

$$T^{\alpha}X^{\beta} = e^{-t^{\alpha}} \cos x^{\beta} \implies \frac{T^{\alpha}}{e^{-t^{\alpha}}} = \frac{\cos x^{\beta}}{X^{\beta}} = K \text{ (Constant)} \tag{6}$$

after solving we get,

$$T^{\alpha} = Ke^{-t^{\alpha}}, \;\; X^{\beta} = \frac{1}{K}\cos x^{\beta}$$

$$\implies T = K\int e^{-t^{\alpha}}\frac{dt}{t^{1-\alpha}}, X = \frac{1}{K}\int \cos x^{\beta}\frac{dx}{x^{1-\beta}}$$

which gives

$$T = \frac{-K}{\alpha}e^{-t^{\alpha}} + c_1, \;\; X = \frac{1}{\beta K}\sin x^{\beta} + c_2$$

so that

$$u(x,t) = \left(\frac{-K}{\alpha}e^{-t^{\alpha}} + c_1\right)\left(\frac{1}{\beta K}\sin x^{\beta} + c_2\right) \tag{7}$$

applying the given conditions, we get $c_1 = \frac{K}{\alpha}$ and $c_2 = 0$.
Hence

$$u(x,t) = \frac{1}{\alpha\beta}\left(1 - e^{-t^{\alpha}}\right)\sin x^{\beta}$$

.

Example 3.2.
Consider another problem

$$\frac{\partial^{\alpha} z}{\partial x^{\alpha}} + \frac{\partial^{\beta}}{\partial y^{\beta}}\left(\frac{\partial^{\beta} z}{\partial y^{\beta}}\right) = 0, \;\; x, \; y > 0 \tag{8}$$

with the following conditions

$$z(x,0) = z(x,\pi) = 0, \;\; z(0,y) = \eta \sin 2\pi^{1-\beta} y^{\beta}.$$

Considering $z = Y(y)X(x)$ to get $X^{\alpha}Y + XY^{2\beta} = 0$ and with some non zero constant p^2 so that

$$\frac{X^{\alpha}}{X} = \frac{-Y^{2\beta}}{Y} = p^2$$

we have

$$X^{\alpha} - p^2 X = 0, \;\; Y^{2\beta} + p^2 Y = 0.$$

Solving we get

$$z(x,y) = XY = Ae^{p^2\frac{x^{\alpha}}{\alpha}}\left(B\sin p\frac{y^{\beta}}{\beta} + C\cos p\frac{y^{\beta}}{\beta}\right) \tag{9}$$

and from the boundary conditions

$$z(x,0) = 0, \;\; z(x,\pi) = 0, \;\; z(0,y) = \eta\sin 2\pi^{1-\beta}y^{\beta}$$

we get $C = 0, \;\; p = n\pi^{1-\beta}\beta, \;\; n = 2$ and $AB = \eta$ respectively. Hence,

$$z(x,y) = \eta e^{4\pi^{2-2\beta}\beta^2\frac{x^{\alpha}}{\alpha}}\sin 2\pi^{1-\beta}y^{\beta}. \tag{10}$$

Example 3.3.
Consider one dimensional fractional Heat equation

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = c^2\frac{\partial^{\beta}}{\partial x^{\beta}}\frac{\partial^{\beta} u}{\partial x^{\beta}}; \;\; x, t > 0 \tag{11}$$

subject to conditions

$$u(0,t) = 0, \;\; u(L,t) = 0 \; t \geq 0; \;\; u(x,0) = f(x), \;\; 0 \leq x \leq L.$$

We assume that it's solution is of the form $u = X(x)T(t)$. Then we may write the equation as

$$XT^{\alpha} = c^2 X^{2\beta} T, \text{ where, } T^{\alpha} = \frac{d^{\alpha} T}{dt^{\alpha}}, \ X^{2\beta} = \frac{d^{\beta}}{dx^{\beta}}\left(\frac{d^{\beta} X}{dx^{\beta}}\right).$$

For any constant $\mu \in \mathbb{R}$, we have

$$\frac{T^{\alpha}}{c^2 T} = \frac{X^{2\beta}}{X} = \mu \Longrightarrow T^{\alpha} - c^2 \mu T = 0 \Longrightarrow \alpha m = \mu c^2 \Longrightarrow m = c^2 \frac{\mu}{\alpha}$$

and

$$\frac{X^{2\beta}}{X} = \mu \Longrightarrow X^{2\beta} - \mu X = 0 \Longrightarrow \beta^2 m^2 - \mu = 0 \Longrightarrow m = \pm \frac{\sqrt{\mu}}{\beta}.$$

Now we have the following possible solutions:

1. $\mu = 0 \Longrightarrow u = XT = c_1(c_2 + \frac{x^{\beta}}{\beta} c_3)$
2. $\mu = \lambda^2 \Longrightarrow u = XT = c_4 e^{\frac{c^2\lambda^2}{\alpha} t^{\alpha}} \left(c_5 e^{\frac{\lambda}{\beta} x^{\beta}} + c_6 e^{-\frac{\lambda}{\beta} x^{\beta}}\right)$
3. $\mu = -\lambda^2 \Longrightarrow u = XT = c_7 e^{-\frac{c^2\lambda^2}{\alpha} t^{\alpha}} \left(c_8 cos\left(\frac{\lambda}{\beta} x^{\beta}\right) + c_9 sin\left(\frac{\lambda}{\beta} x^{\beta}\right)\right).$

From the given conditions, (3) gives the non trivial solution as

$$u(0,t) = 0 \Rightarrow c_8 = 0, u(L,t) = 0 \Rightarrow \lambda = \frac{n\pi\beta}{L^{\beta}}.$$

So that general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x^{\beta}}{L^{\beta}}\right) exp\left(-\frac{n^2\pi^2 c^2 \beta^2 t^{\alpha}}{\alpha L^{2\beta}}\right). \tag{12}$$

In which coefficient A_n is determined by fractional Fourier series.

Note 1. The same solution can also be obtained for fractional Telegraph equation of electrical LCR circuit, putting

$$c^2 = \frac{1}{RC} \text{ in } \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = c^2 \frac{\partial^{\beta}}{\partial x^{\beta}} \frac{\partial^{\beta} u}{\partial x^{\beta}}.$$

Example 3.4.

The fractional wave equation

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}} \frac{\partial^{\alpha} u}{\partial t^{\alpha}} = c^2 \frac{\partial^{\beta}}{\partial x^{\beta}} \frac{\partial^{\beta} u}{\partial x^{\beta}}; \quad x,t > 0 \tag{13}$$

subject to conditions

$$u(0,t) = 0, \ \ u(L,t) = 0 \ \ \left(\frac{\partial^{\alpha} u}{\partial t^{\alpha}}\right)_{t=0} = 0, \ u(x,0) = \phi \ \sin\left(\frac{2\pi x^{\beta}}{L^{\beta}}\right)$$

Consider the separable solution $u = X(x)T(t)$, so that we have

$$XT^{2\alpha} = c^2 X^{2\beta} T \text{ where } T^{2\alpha} = \frac{d^{\alpha}}{dt^{\alpha}} \frac{d^{\alpha} T}{dt^{\alpha}}, \ X^{2\beta} = \frac{d^{\beta}}{dx^{\beta}}\left(\frac{d^{\beta} X}{dx^{\beta}}\right).$$

We separate the variables with the help of some constant $\mu \in \mathbb{R}$ as

$$\frac{T^{2\alpha}}{c^2 T} = \frac{X^{2\beta}}{X} = \mu.$$

Here

$$\frac{T^{2\alpha}}{c^2 T} = \mu \Longrightarrow T^{2\alpha} - c^2 \mu T = 0 \Longrightarrow \alpha^2 m^2 = \mu c^2 \Longrightarrow m = \pm c \frac{\sqrt{\mu}}{\alpha}.$$

and

$$\frac{X^{2\beta}}{X} = \mu \Longrightarrow X^{2\beta} - \mu X = 0 \Longrightarrow \beta^2 m^2 - \mu = 0 \Longrightarrow m = \pm \frac{\sqrt{\mu}}{\beta}.$$

Now consider the following cases:

1. $\mu = 0 \Longrightarrow u = XT = (c_1 + c_2 \frac{x^\beta}{\beta})(c_3 + \frac{t^\alpha}{\alpha} c_4)$
2. $\mu = \lambda^2 \Longrightarrow u = XT = \left(c_5 e^{\frac{\lambda}{\beta} x^\beta} + c_6 e^{-\frac{\lambda}{\beta} x^\beta}\right)\left(c_7 e^{\frac{c\lambda}{\alpha} t^\alpha} + c_8 e^{\frac{c\lambda}{\alpha} t^\alpha}\right)$
3. $\mu = -\lambda^2 \Longrightarrow u = XT = \left(c_9 \cos\left(\frac{c\lambda}{\alpha} t^\alpha\right) + c_{10} \sin\left(\frac{c\lambda}{\alpha} t^\alpha\right)\right)$ $\left(c_{11} \cos\left(\frac{\lambda}{\beta} x^\beta\right) + c_{12} \sin\left(\frac{\lambda}{\beta} x^\beta\right)\right).$

Out of these solutions the most consistent solution is (3) and so

$$u(0,t) = 0 \Rightarrow c_{11} = 0, u(L,t) = 0 \Rightarrow \lambda = \frac{n\pi\beta}{L^\beta}$$

and

$$\left(\frac{\partial^\alpha u}{\partial t^\alpha}\right)_{t=0} = 0 \Rightarrow c_{10} = 0.$$

So that general solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x^\beta}{L^\beta}\right)\cos\left(\frac{n\pi\beta c t^\alpha}{L^\beta \alpha}\right). \tag{14}$$

In which coefficient A_n is ϕ for $n = 2$ and hence complete solution is

$$u(x,t) = \phi \sin\left(\frac{2\pi x^\beta}{L^\beta}\right)\cos\left(\frac{2\pi\beta c t^\alpha}{L^\beta \alpha}\right).$$

Note 2. The same solution can also be obtained for fractional Radio equation of electrical LCR circuit, putting

$$c^2 = \frac{1}{LC} \text{ in } \frac{\partial^\alpha}{\partial t^\alpha}\frac{\partial^\alpha u}{\partial t^\alpha} = c^2 \frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\beta u}{\partial x^\beta}.$$

Example 3.5.
Laplace equation involving fractional derivatives in both the variables is given as

$$\frac{\partial^\alpha}{\partial y^\alpha}\frac{\partial^\alpha z}{\partial y^\alpha} + \frac{\partial^\beta}{\partial x^\beta}\frac{\partial^\beta z}{\partial x^\beta} = 0 \tag{15}$$

with the following conditions

$$z(0,y) = 0,\ z(a,y) = 0;\ 0 < x < a,\ \ z(x,b) = 0,\ \ z(x,0) = f(x);\ 0 < y < b.$$

Consider $z(x,y) = X(x)Y(y)$, so we get

$$XY^{2\alpha} + X^{2\beta}Y = 0; \text{ where } Y^{2\alpha} \equiv \frac{d^\alpha}{dy^\alpha}\frac{d^\alpha Y}{dy^\alpha},\quad X^{2\beta} \equiv \frac{d^\beta}{dx^\beta}\frac{d^\beta X}{dx^\beta}.$$

So we get,

$$\frac{Y^{2\alpha}}{Y} = -\frac{X^{2\beta}}{X} = \psi \text{ (constant)}$$

which gives

$$\frac{Y^{2\alpha}}{Y} = \psi \Longrightarrow Y^{2\alpha} - \psi Y = 0$$

and

$$-\frac{X^{2\beta}}{X} = \psi \Longrightarrow X^{2\beta} + \psi X = 0.$$

There are three possible solution based on the value of constant :

1. $\psi = 0 \Longrightarrow z = XY = (c_1 + c_2 \frac{x^\beta}{\beta})(c_3 + \frac{y^\alpha}{\alpha} c_4),$
2. $\psi = \eta^2 \Longrightarrow z = XY = \left(c_5 \cos\left(\frac{\eta}{\beta} x^\beta\right) + c_6 \sin\left(\frac{\eta}{\beta} x^\beta\right)\right)\left(c_7 e^{\frac{\eta}{\alpha} y^\alpha} + c_8 e^{-\frac{\eta}{\alpha} y^\alpha}\right),$

3. $\psi = -\eta^2 \Longrightarrow z = XY = \left(c_9 \cos\left(\frac{\eta}{\alpha} y^\alpha\right) + c_{10} \sin\left(\frac{\eta}{\alpha} y^\alpha\right)\right)$
$\left(c_{11} e^{\left(\frac{\eta}{\beta} x^\beta\right)} + c_{12} e^{-\left(\frac{\eta}{\beta} x^\beta\right)}\right).$

From the available conditions one may get that solution (2) is the only suitable solution in which

$$z(0, y) = 0 \Rightarrow c_5 = 0,\ \ z(a, y) = 0 \Rightarrow \eta = \frac{n\pi\beta}{a^\beta}$$

and

$$z(x, b) = 0 \Rightarrow (c_7 e^{\frac{\eta}{\alpha} b^\alpha} + c_8 e^{-\frac{\eta}{\alpha} b^\alpha}) = 0.$$

Set $c_8 = Ce^{\frac{\eta}{\alpha} b^\alpha}$, $c_7 = -Ce^{-\frac{\eta}{\alpha} b^\alpha}$, so that general solution is

$$z(x, y) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x^\beta}{a^\beta}\right) \sin h\left(\frac{n\pi\beta(b^\alpha - y^\alpha)}{a^\beta \alpha}\right). \tag{16}$$

In which coefficient A_n is determined using $f(x)$ by fractional Fourier series.

4. Conclusion

We have discussed the solution of fractional boundary value problems involving fractional-order derivative concerning time or space or both. Same problems can also be solved using the fractional Taylor series method but it gives approximate solution instead of that we get a general solution using the proposed method. It is clear that the method is convenient and there is no complexity in the computation to get the solution. Here we have considered only two variable problems and within a certain range of fractional order but in the future, it may also be extended for the higher dimensional problems and the higher range.

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