Int. J. Adv. Appl. Math. and Mech. 7(4) (2020) 20 – 33 (ISSN: 2347-2529)

Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

On Generalized 4-primes Numbers

Research Article

Yüksel Soykan *

Department of Mathematics, Art and Science Faculty, Zonguldak Bülent Ecevit University, 67100, Zonguldak, Turkey

Received 26 March 2020; accepted (in revised version) 06 June 2020

Abstract: In this paper, we introduce the generalized 4-primes numbers sequences and we deal with, in detail, three special cases which we call them 4-primes, Lucas 4-primes and modified 4-primes sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

MSC: 11B39 • 11B83

Keywords: 4-primes numbers • Lucas 4-primes numbers • modified 4-primes numbers • Tetranacci numbers

1. Introduction

In this paper, we investigate the generalized 4-primes sequences and we investigate, in detail, three special cases which we call them 4-primes, Lucas-4-primes and modified 4-primes sequences.

The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \quad F_0 = 0, \; F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \quad L_0 = 2, \; L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences. See [1],[2],[13] for some work on second-order generalization of Fibonacci numbers.

The generalized Tetranacci sequence $\{W_n(W_0, W_1, W_2, W_3; r, s, t, u)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined as follows:

$$W_n = rW_{n-1} + sW_{n-2} + tW_{n-3} + uW_{n-4}, \quad W_0 = c_0, W_1 = c_1, W_2 = c_2, W_3 = c_3, \quad n \geq 4 \tag{1}$$

where W_0, W_1, W_2, W_3 are arbitrary complex (or real) numbers and r, s, t, u are real numbers.

This sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example [[3]], [7], [8], [9], [14], [15]. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{t}{u}W_{-(n-1)} - \frac{s}{u}W_{-(n-2)} - \frac{r}{u}W_{-(n-3)} + \frac{1}{u}W_{-(n-4)}$$

* Corresponding author.

E-mail address(es): yuksel_soykan@hotmail.com

for $n = 1,2,3,...$ when $t \neq 0$. Therefore, recurrence (1) holds for all integer n.

As $\{W_n\}$ is a fourth order recurrence sequence (difference equation), it's characteristic equation is

$$x^4 - rx^3 - sx^2 - tx - u = 0 \tag{2}$$

whose roots are $\alpha, \beta, \gamma, \delta$. Note that we have the following identities

$$\begin{aligned} \alpha+\beta+\gamma+\delta &= r, \\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -s, \\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= t, \\ \alpha\beta\gamma\delta &= -u. \end{aligned}$$

Generalized Tetranacci numbers can be expressed, for all integers n, using Binet's formula

$$W_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{b_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{3}$$

where

$$\begin{aligned} b_1 &= W_3 - (\beta+\gamma+\delta)W_2 + (\beta\gamma+\beta\delta+\gamma\delta)W_1 - \beta\gamma\delta W_0, \\ b_2 &= W_3 - (\alpha+\gamma+\delta)W_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)W_1 - \alpha\gamma\delta W_0, \\ b_3 &= W_3 - (\alpha+\beta+\delta)W_2 + (\alpha\beta+\alpha\delta+\beta\delta)W_1 - \alpha\beta\delta W_0, \\ b_4 &= W_3 - (\alpha+\beta+\gamma)W_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)W_1 - \alpha\beta\gamma W_0. \end{aligned}$$

Usually, it is customary to choose $\alpha, \beta, \gamma, \delta$ so that the Equ. (2) has at least one real (say α) solutions. Note that the Binet form of a sequence satisfying (2) for non-negative integers is valid for all integers n, for a proof of this result see [**?**].

In this paper we consider the case $r = 2$, $s = 3$, $t = 5$, $u = 7$ and in this case we write $V_n = W_n$. A generalized 4-primes sequence $\{V_n\}_{n\geq 0} = \{V_n(V_0, V_1, V_2, V_3)\}_{n\geq 0}$ is defined by the fourth-order recurrence relations

$$V_n = 2V_{n-1} + 3V_{n-2} + 5V_{n-3} + 7V_{n-4} \tag{4}$$

with the initial values $V_0 = c_0, V_1 = c_1, V_2 = c_2, V_3 = c_3$ not all being zero.

The sequence $\{V_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$V_{-n} = -\frac{5}{7}V_{-(n-1)} - \frac{3}{7}V_{-(n-2)} - \frac{2}{7}V_{-(n-3)} + \frac{1}{7}V_{-(n-4)}$$

for $n = 1,2,3,....$ Therefore, recurrence (4) holds for all integer n.

(3) can be used to obtain Binet formula of generalized 4-primes numbers. Binet formula of generalized 4-primes numbers can be given as

$$V_n = \frac{b_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{b_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{b_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{b_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}$$

where

$$\begin{aligned} b_1 &= V_3 - (\beta+\gamma+\delta)V_2 + (\beta\gamma+\beta\delta+\gamma\delta)V_1 - \beta\gamma\delta V_0, \\ b_2 &= V_3 - (\alpha+\gamma+\delta)V_2 + (\alpha\gamma+\alpha\delta+\gamma\delta)V_1 - \alpha\gamma\delta V_0, \\ b_3 &= V_3 - (\alpha+\beta+\delta)V_2 + (\alpha\beta+\alpha\delta+\beta\delta)V_1 - \alpha\beta\delta V_0, \\ b_4 &= V_3 - (\alpha+\beta+\gamma)V_2 + (\alpha\beta+\alpha\gamma+\beta\gamma)V_1 - \alpha\beta\gamma V_0. \end{aligned} \tag{5}$$

Here, α, β, γ and δ are the roots of the equation $x^4 - 2x^3 - 3x^2 - 5x - 7 = 0$. Moreover

$$\begin{aligned} \alpha &= 3.456157801461113 \\ \beta &= -1.184059685093579 \\ \gamma &= -0.1360490581837671 + 1.300777225148450i \\ \delta &= -0.1360490581837671 - 1.300777225148450i \end{aligned}$$

Note that

$$\begin{aligned} \alpha+\beta+\gamma+\delta &= 2, \\ \alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta &= -3, \\ \alpha\beta\gamma+\alpha\beta\delta+\alpha\gamma\delta+\beta\gamma\delta &= 5, \\ \alpha\beta\gamma\delta &= -7. \end{aligned}$$

Table 1. A few generalized 4-primes numbers

n	V_n	V_{-n}
0	V_0	
1	V_1	$\frac{1}{7}V_3-\frac{3}{7}V_1-\frac{2}{7}V_2-\frac{5}{7}V_0$
2	V_2	$\frac{4}{49}V_0+\frac{1}{49}V_1+\frac{17}{49}V_2-\frac{5}{49}V_3$
3	V_3	$\frac{107}{343}V_1-\frac{13}{343}V_0-\frac{43}{343}V_2+\frac{4}{343}V_3$
4	$7V_0+5V_1+3V_2+2V_3$	$\frac{814}{2401}V_0-\frac{262}{2401}V_1+\frac{54}{2401}V_2-\frac{13}{2401}V_3$
5	$14V_0+17V_1+11V_2+7V_3$	$\frac{814}{16807}V_3-\frac{2064}{16807}V_1-\frac{1719}{16807}V_2-\frac{5904}{16807}V_0$
6	$49V_0+49V_1+38V_2+25V_3$	$\frac{15072}{117649}V_0+\frac{5679}{117649}V_1+\frac{17506}{117649}V_2-\frac{5904}{117649}V_3$
7	$175V_0+174V_1+124V_2+88V_3$	$\frac{77326}{823543}V_1-\frac{35607}{823543}V_0-\frac{71472}{823543}V_2+\frac{15072}{823543}V_3$
8	$616V_0+615V_1+438V_2+300V_3$	$\frac{719317}{5764801}V_0-\frac{393483}{5764801}V_1+\frac{176718}{5764801}V_2-\frac{35607}{5764801}V_3$

The first few generalized 4-primes numbers with positive subscript and negative subscript are given in the following Table 1.

Now we define three special cases of the sequence $\{V_n\}$. 4-primes sequence$\{G_n\}_{n\geq 0}$, Lucas 4-primes sequence $\{H_n\}_{n\geq 0}$ and modified 4-primes sequence $\{E_n\}_{n\geq 0}$ are defined, respectively, by the third-order recurrence relations

$$G_{n+4}=2G_{n+3}+3G_{n+2}+5G_{n+1}+7G_n, \quad G_0=0, G_1=0, G_2=1, G_3=2,$$

$$H_{n+4}=2H_{n+3}+3H_{n+2}+5H_{n+1}+7H_n, \quad H_0=4, H_1=2, H_2=10, H_3=41, \tag{6}$$

and

$$E_{n+4}=2E_{n+3}+3E_{n+2}+5E_{n+1}+7E_n, \quad E_0=0, E_1=0, E_2=1, E_3=1, \tag{7}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n}=-\frac{5}{7}G_{-(n-1)}-\frac{3}{7}G_{-(n-2)}-\frac{2}{7}G_{-(n-3)}+\frac{1}{7}G_{-(n-4)}, \tag{8}$$

$$H_{-n}=-\frac{5}{7}H_{-(n-1)}-\frac{3}{7}H_{-(n-2)}-\frac{2}{7}H_{-(n-3)}+\frac{1}{7}H_{-(n-4)} \tag{9}$$

and

$$E_{-n}=-\frac{5}{7}E_{-(n-1)}-\frac{3}{7}E_{-(n-2)}-\frac{2}{7}E_{-(n-3)}+\frac{1}{7}E_{-(n-4)} \tag{10}$$

for $n=1,2,3,...$ respectively. Therefore, recurrences (8), (9) and (10) hold for all integer n.

Note that the sequences G_n, H_n and E_n are not indexed in [**?**] yet. Next, we present the first few values of the 4-primes, Lucas 4-primes and modified 4-primes numbers with positive and negative subscripts:

Table 2. The first few values of the special fourth-order numbers with positive and negative subscripts.

n	0	1	2	3	4	5	6	7	8	9	10
G_n	0	0	1	2	7	25	88	300	1038	3591	12412
G_{-n}		0	$\frac{1}{7}$	$-\frac{5}{49}$	$\frac{4}{343}$	$-\frac{13}{2401}$	$\frac{814}{16807}$	$-\frac{5904}{117649}$	$\frac{15072}{823543}$	$-\frac{35607}{5764801}$	$\frac{719317}{40353607}$
H_n	4	2	10	41	150	487	1699	5896	20374	70340	243175
H_{-n}		$-\frac{5}{7}$	$-\frac{17}{49}$	$-\frac{104}{343}$	$\frac{2739}{2401}$	$-\frac{11560}{16807}$	$\frac{4642}{117649}$	$-\frac{84544}{823543}$	$\frac{2397595}{5764801}$	$-\frac{14632547}{40353607}$	$\frac{32690758}{282475249}$
E_n	0	0	1	1	5	18	63	212	738	2553	8821
E_{-n}		$-\frac{1}{7}$	$\frac{12}{49}$	$-\frac{39}{343}$	$\frac{41}{2401}$	$-\frac{905}{16807}$	$\frac{11602}{117649}$	$-\frac{56400}{823543}$	$\frac{141111}{5764801}$	$-\frac{968566}{40353607}$	$\frac{11386185}{282475249}$

For all integers n, 4-primes, Lucas 4-primes and modified 4-primes numbers (using initial conditions in (5)) can be expressed using Binet's formulas as

$$\begin{aligned}
G_n &= \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{\delta^{n+1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}, \\
H_n &= \alpha^n+\beta^n+\gamma^n+\delta^n, \\
E_n &= \frac{(\alpha-1)\alpha^{n}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{(\beta-1)\beta^{n}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{(\gamma-1)\gamma^{n}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{(\delta-1)\delta^{n}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},
\end{aligned}$$

respectively.

2. Generating Functions

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty} V_n x^n$ of the sequence V_n.

Lemma 2.1.
Suppose that $f_{V_n}(x) = \sum\limits_{n=0}^{\infty} V_n x^n$ *is the ordinary generating function of the generalized 4-primes sequence* $\{V_n\}_{n\geq 0}$. *Then,* $\sum\limits_{n=0}^{\infty} V_n x^n$ *is given by*

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)x^3}{1 - 2x - 3x^2 - 5x^3 - 7x^4}. \tag{11}$$

Proof. Using the definition of generalized 4-primes numbers, and substracting $2x\sum_{n=0}^{\infty} V_n x^n$, $3x^2\sum_{n=0}^{\infty} V_n x^n$, $5x^3\sum_{n=0}^{\infty} V_n x^n$ and $7x^4\sum_{n=0}^{\infty} V_n x^n$ from $\sum_{n=0}^{\infty} V_n x^n$ we obtain

$$\begin{aligned}
(1 - 2x - 3x^2 - 5x^3 - 7x^4)\sum_{n=0}^{\infty} V_n x^n &= \sum_{n=0}^{\infty} V_n x^n - 2x\sum_{n=0}^{\infty} V_n x^n - 3x^2\sum_{n=0}^{\infty} V_n x^n - 5x^3\sum_{n=0}^{\infty} V_n x^n - 7x^4\sum_{n=0}^{\infty} V_n x^n \\
&= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=0}^{\infty} V_n x^{n+1} - 3\sum_{n=0}^{\infty} V_n x^{n+2} - 5\sum_{n=0}^{\infty} V_n x^{n+3} - 7\sum_{n=0}^{\infty} V_n x^{n+4} \\
&= \sum_{n=0}^{\infty} V_n x^n - 2\sum_{n=1}^{\infty} V_{n-1} x^n - 3\sum_{n=2}^{\infty} V_{n-2} x^n - 5\sum_{n=3}^{\infty} V_{n-3} x^n - 7\sum_{n=4}^{\infty} V_{n-4} x^n \\
&= (V_0 + V_1 x + V_2 x^2 + V_3 x^3) - 2(V_0 x + V_1 x^2 + V_2 x^3) \\
&\quad -3(V_0 x^2 + V_1 x^3) - 5V_0 x^3 \\
&\quad + \sum_{n=4}^{\infty} (V_n - 2V_{n-1} - 3V_{n-2} - 5V_{n-3} - 7V_{n-4}) x^n \\
&= V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)x^3.
\end{aligned}$$

Rearranging above equation, we obtain

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1 - 2V_0)x + (V_2 - 2V_1 - 3V_0)x^2 + (V_3 - 2V_2 - 3V_1 - 5V_0)x^3}{1 - 2x - 3x^2 - 5x^3 - 7x^4}.$$

The previous lemma gives the following results as particular examples.

Corollary 2.1.
Generated functions of 4-primes, Lucas 4-primes and modified 4-primes numbers are

$$\sum_{n=0}^{\infty} G_n x^n = \frac{x^2}{1 - 2x - 3x^2 - 5x^3 - 7x^4},$$

and

$$\sum_{n=0}^{\infty} H_n x^n = \frac{4 - 6x - 6x^2 - 5x^3}{1 - 2x - 3x^2 - 5x^3 - 7x^4},$$

and

$$\sum_{n=0}^{\infty} E_n x^n = \frac{x^2 - x^3}{1 - 2x - 3x^2 - 5x^3 - 7x^4},$$

respectively.

3. Obtaining Binet Formula From Generating Function

We next find Binet formula of generalized 4-primes numbers $\{V_n\}$ by the use of generating function for V_n.

Theorem 3.1.
(Binet formula of generalized 4-primes numbers)

$$V_n = \frac{d_1\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{d_2\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{d_3\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{d_4\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} \tag{12}$$

where

$$\begin{aligned} d_1 &= V_0\alpha^3 + (V_1-2V_0)\alpha^2 + (V_2-2V_1-3V_0)\alpha + (V_3-2V_2-3V_1-5V_0), \\ d_2 &= V_0\beta^3 + (V_1-2V_0)\beta^2 + (V_2-2V_1-3V_0)\beta + (V_3-2V_2-3V_1-5V_0), \\ d_3 &= V_0\gamma^3 + (V_1-2V_0)\gamma^2 + (V_2-2V_1-3V_0)\gamma + (V_3-2V_2-3V_1-5V_0), \\ d_4 &= V_0\delta^3 + (V_1-2V_0)\delta^2 + (V_2-2V_1-3V_0)\delta + (V_3-2V_2-3V_1-5V_0). \end{aligned}$$

Proof. Let

$$h(x) = 1-2x-3x^2-5x^3-7x^4.$$

Then for some α,β,γ and δ we write

$$h(x) = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)$$

i.e.,

$$1-2x-3x^2-5x^3-7x^4 = (1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x) \tag{13}$$

Hence $\frac{1}{\alpha},\frac{1}{\beta},\frac{1}{\gamma}$ and $\frac{1}{\delta}$ are the roots of $h(x)$. This gives α,β,γ and δ as the roots of

$$h(\frac{1}{x}) = 1-\frac{2}{x}-\frac{3}{x^2}-\frac{5}{x^3}-\frac{7}{x^4} = 0.$$

This implies $x^4-2x^3-3x^2-5x-7=0$. Now, by (11) and (13), it follows that

$$\sum_{n=0}^{\infty} V_n x^n = \frac{V_0 + (V_1-2V_0)x + (V_2-2V_1-3V_0)x^2 + (V_3-2V_2-3V_1-5V_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)}.$$

Then we write

$$\frac{V_0 + (V_1-2V_0)x + (V_2-2V_1-3V_0)x^2 + (V_3-2V_2-3V_1-5V_0)x^3}{(1-\alpha x)(1-\beta x)(1-\gamma x)(1-\delta x)} = \frac{A_1}{(1-\alpha x)} + \frac{A_2}{(1-\beta x)} + \frac{A_3}{(1-\gamma x)} + \frac{A_4}{(1-\delta x)}. \tag{14}$$

So

$$\begin{aligned} & V_0 + (V_1-2V_0)x + (V_2-2V_1-3V_0)x^2 + (V_3-2V_2-3V_1-5V_0)x^3 \\ = \; & A_1(1-\beta x)(1-\gamma x)(1-\delta x) + A_2(1-\alpha x)(1-\gamma x)(1-\delta x) \\ & + A_3(1-\alpha x)(1-\beta x)(1-\delta x) + A_3(1-\alpha x)(1-\beta x)(1-\gamma x). \end{aligned}$$

If we consider $x=\frac{1}{\alpha}$, we get $V_0 + (V_1-2V_0)\frac{1}{\alpha} + (V_2-2V_1-3V_0)\frac{1}{\alpha^2} + (V_3-2V_2-3V_1-5V_0)\frac{1}{\alpha^3} = A_1(1-\frac{\beta}{\alpha})(1-\frac{\gamma}{\alpha})(1-\frac{\delta}{\alpha})$.This gives

$$\begin{aligned} A_1 &= \frac{\alpha^3(V_0 + (V_1-2V_0)\frac{1}{\alpha} + (V_2-2V_1-3V_0)\frac{1}{\alpha^2} + (V_3-2V_2-3V_1-5V_0)\frac{1}{\alpha^3})}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \\ &= \frac{V_0\alpha^3 + (V_1-2V_0)\alpha^2 + (V_2-2V_1-3V_0)\alpha + (V_3-2V_2-3V_1-5V_0)}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} A_2 &= \frac{V_0\beta^3 + (V_1-2V_0)\beta^2 + (V_2-2V_1-3V_0)\beta + (V_3-2V_2-3V_1-5V_0)}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}, \\ A_3 &= \frac{V_0\gamma^3 + (V_1-2V_0)\gamma^2 + (V_2-2V_1-3V_0)\gamma + (V_3-2V_2-3V_1-5V_0)}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}, \\ A_4 &= \frac{V_0\delta^3 + (V_1-2V_0)\delta^2 + (V_2-2V_1-3V_0)\delta + (V_3-2V_2-3V_1-5V_0)}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}. \end{aligned}$$

Thus (14) can be written as

$$\sum_{n=0}^{\infty} V_n x^n = A_1(1-\alpha x)^{-1} + A_2(1-\beta x)^{-1} + A_3(1-\gamma x)^{-1} + A_4(1-\delta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} V_n x^n = A_1 \sum_{n=0}^{\infty} \alpha^n x^n + A_2 \sum_{n=0}^{\infty} \beta^n x^n + A_3 \sum_{n=0}^{\infty} \gamma^n x^n + A_4 \sum_{n=0}^{\infty} \delta^n x^n = \sum_{n=0}^{\infty} (A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n) x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$V_n = A_1\alpha^n + A_2\beta^n + A_3\gamma^n + A_4\delta^n$$

and then we get (12).

Note that from (5) and (12) we have

$$\begin{array}{rcl} V_3-(\beta+\gamma+\delta)V_2+(\beta\gamma+\beta\delta+\gamma\delta)V_1-\beta\gamma\delta V_0 &=& V_0\alpha^3+(V_1-2V_0)\alpha^2+(V_2-2V_1-3V_0)\alpha \\ && +(V_3-2V_2-3V_1-5V_0), \\ V_3-(\alpha+\gamma+\delta)V_2+(\alpha\gamma+\alpha\delta+\gamma\delta)V_1-\alpha\gamma\delta V_0 &=& V_0\beta^3+(V_1-2V_0)\beta^2+(V_2-2V_1-3V_0)\beta \\ && +(V_3-2V_2-3V_1-5V_0), \\ V_3-(\alpha+\beta+\delta)V_2+(\alpha\beta+\alpha\delta+\beta\delta)V_1-\alpha\beta\delta V_0 &=& V_0\gamma^3+(V_1-2V_0)\gamma^2+(V_2-2V_1-3V_0)\gamma \\ && +(V_3-2V_2-3V_1-5V_0), \\ V_3-(\alpha+\beta+\gamma)V_2+(\alpha\beta+\alpha\gamma+\beta\gamma)V_1-\alpha\beta\gamma V_0 &=& V_0\delta^3+(V_1-2V_0)\delta^2+(V_2-2V_1-3V_0)\delta \\ && +(V_3-2V_2-3V_1-5V_0). \end{array}$$

Next, using Theorem 3.1, we present the Binet formulas of 4-primes, Lucas 4-primes and modified 4-primes sequences.

Corollary 3.1.
Binet formulas of 4-primes, Lucas 4-primes and modified 4-primes sequences are

$$G_n = \frac{\alpha^{n+1}}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{\beta^{n+1}}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{\gamma^{n+1}}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{\delta^{n+1}}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$

and

$$H_n = \alpha^n + \beta^n + \gamma^n + \delta^n,$$

and

$$E_n = \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{(\gamma-1)\gamma^n}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{(\delta-1)\delta^n}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)},$$

respectively.

We can find Binet formulas by using matrix method with a similar technique which is given in [6]. Take $k = i = 4$ in Corollary 3.1 in [6]. Let

$$\Lambda = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{pmatrix}, \Lambda_1 = \begin{pmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{pmatrix}, \Lambda_2 = \begin{pmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{pmatrix},$$

$$\Lambda_3 = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{pmatrix}, \Lambda_4 = \begin{pmatrix} \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^3 & \delta^2 & \delta & \delta^{n-1} \end{pmatrix}$$

Then the Binet formula for 4-primes numbers is

$$
\begin{aligned}
G_n &= \frac{1}{\det(\Lambda)}\sum_{j=1}^{4} G_{5-j}\det(\Lambda_j) = \frac{1}{\Lambda}(G_4\det(\Lambda_1)+G_3\det(\Lambda_2)+G_2\det(\Lambda_3)+G_1\det(\Lambda_4)) \\
&= \frac{1}{\det(\Lambda)}(7\det(\Lambda_1)+2\det(\Lambda_2)+\det(\Lambda_3)) \\
&= \left(7\begin{vmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{vmatrix} + 2\begin{vmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{vmatrix} + \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{vmatrix}\right) / \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{vmatrix}.
\end{aligned}
$$

Similarly, we obtain the Binet formula for Lucas 4-primes and modified 4-primes numbers as

$$
\begin{aligned}
H_n &= \frac{1}{\det(\Lambda)}(H_4\det(\Lambda_1)+H_3\det(\Lambda_2)+H_2\det(\Lambda_3)+H_1\det(\Lambda_4)) \\
&= (150\begin{vmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{vmatrix} + 41\begin{vmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{vmatrix} \\
&\quad +10\begin{vmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{vmatrix} + 2\begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & \alpha^{n-1} \\ \beta^3 & \beta^2 & \beta & \beta^{n-1} \\ \gamma^3 & \gamma^2 & \gamma & \gamma^{n-1} \\ \delta^3 & \delta^2 & \delta & \delta^{n-1} \end{vmatrix}) / \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{vmatrix}.
\end{aligned}
$$

and

$$
\begin{aligned}
E_n &= \frac{1}{\det(\Lambda)}(E_4\det(\Lambda_1)+E_3\det(\Lambda_2)+E_2\det(\Lambda_3)+E_1\det(\Lambda_4)) \\
&= \left(5\begin{vmatrix} \alpha^{n-1} & \alpha^2 & \alpha & 1 \\ \beta^{n-1} & \beta^2 & \beta & 1 \\ \gamma^{n-1} & \gamma^2 & \gamma & 1 \\ \delta^{n-1} & \delta^2 & \delta & 1 \end{vmatrix} + \begin{vmatrix} \alpha^3 & \alpha^{n-1} & \alpha & 1 \\ \beta^3 & \beta^{n-1} & \beta & 1 \\ \gamma^3 & \gamma^{n-1} & \gamma & 1 \\ \delta^3 & \delta^{n-1} & \delta & 1 \end{vmatrix} + \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha^{n-1} & 1 \\ \beta^3 & \beta^2 & \beta^{n-1} & 1 \\ \gamma^3 & \gamma^2 & \gamma^{n-1} & 1 \\ \delta^3 & \delta^2 & \delta^{n-1} & 1 \end{vmatrix}\right) / \begin{vmatrix} \alpha^3 & \alpha^2 & \alpha & 1 \\ \beta^3 & \beta^2 & \beta & 1 \\ \gamma^3 & \gamma^2 & \gamma & 1 \\ \delta^3 & \delta^2 & \delta & 1 \end{vmatrix}.
\end{aligned}
$$

respectively.

4. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized 4-primes sequence $\{V_n\}_{n\geq 0}$.

Theorem 4.1 (Simson Formula of Generalized 4-primes Numbers).
For all integers n, we have

$$\begin{vmatrix} V_{n+3} & V_{n+2} & V_{n+1} & V_n \\ V_{n+2} & V_{n+1} & V_n & V_{n-1} \\ V_{n+1} & V_n & V_{n-1} & V_{n-2} \\ V_n & V_{n-1} & V_{n-2} & V_{n-3} \end{vmatrix} = (-1)^n 7^n \begin{vmatrix} V_3 & V_2 & V_1 & V_0 \\ V_2 & V_1 & V_0 & V_{-1} \\ V_1 & V_0 & V_{-1} & V_{-2} \\ V_0 & V_{-1} & V_{-2} & V_{-3} \end{vmatrix}. \tag{15}$$

Proof. (15) is given in Soykan [11].
The previous theorem gives the following results as particular examples.

Corollary 4.1.
For all integers n, Simson formula of 4-primes, Lucas 4-primes and modified 4-primes numbers are given as

$$\begin{vmatrix} G_{n+3} & G_{n+2} & G_{n+1} & G_n \\ G_{n+2} & G_{n+1} & G_n & G_{n-1} \\ G_{n+1} & G_n & G_{n-1} & G_{n-2} \\ G_n & G_{n-1} & G_{n-2} & G_{n-3} \end{vmatrix} = (-1)^n 7^{n-2}$$

: and

$$\begin{vmatrix} H_{n+3} & H_{n+2} & H_{n+1} & H_n \\ H_{n+2} & H_{n+1} & H_n & H_{n-1} \\ H_{n+1} & H_n & H_{n-1} & H_{n-2} \\ H_n & H_{n-1} & H_{n-2} & H_{n-3} \end{vmatrix} = 241727(-1)^n 7^{n-3}$$

and

$$\begin{vmatrix} E_{n+3} & E_{n+2} & E_{n+1} & E_n \\ E_{n+2} & E_{n+1} & E_n & E_{n-1} \\ E_{n+1} & E_n & E_{n-1} & E_{n-2} \\ E_n & E_{n-1} & E_{n-2} & E_{n-3} \end{vmatrix} = 16(-1)^n 7^{n-3}$$

respectively.

5. Some Identities

In this section, we obtain some identities of 4-primes, Lucas 4-primes and modified 4-primes numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 5.1.
The following equalities are true:

$$\begin{aligned} 49H_n &= -17G_{n+4} - G_{n+3} + 317G_{n+2} - 104G_{n+1} \quad (16) \\ 7H_n &= -5G_{n+3} + 38G_{n+2} - 27G_{n+1} - 17G_n \\ H_n &= 4G_{n+2} - 6G_{n+1} - 6G_n - 5G_{n-1} \\ H_n &= 2G_{n+1} + 6G_n + 15G_{n-1} + 28G_{n-2} \\ H_n &= 10G_n + 21G_{n-1} + 38G_{n-2} + 14G_{n-3} \end{aligned}$$

and

$$\begin{aligned} 241727G_n &= 684H_{n+4} - 4806H_{n+3} + 9382H_{n+2} + 313H_{n+1} \\ 241727G_n &= -3438H_{n+3} + 11434H_{n+2} + 3733H_{n+1} + 4788H_n \\ 241727G_n &= 4558H_{n+2} - 6581H_{n+1} - 12402H_n - 24066H_{n-1} \\ 241727G_n &= 2535H_{n+1} + 1272H_n - 1276H_{n-1} + 31906H_{n-2} \\ 241727G_n &= 6342H_n + 6329H_{n-1} + 44581H_{n-2} + 17745H_{n-3} \end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (16). To show (16), writing

$$H_n = a \times G_{n+4} + b \times G_{n+3} + c \times G_{n+2} + d \times G_{n+1}$$

and solving the system of equations

$$\begin{aligned} H_0 &= a \times G_4 + b \times G_3 + c \times G_2 + d \times G_1 \\ H_1 &= a \times G_5 + b \times G_4 + c \times G_3 + d \times G_2 \\ H_2 &= a \times G_6 + b \times G_5 + c \times G_4 + d \times G_3 \\ H_3 &= a \times G_7 + b \times G_6 + c \times G_5 + d \times G_4 \end{aligned}$$

we find that $a = -\frac{17}{49}, b = -\frac{1}{49}, c = \frac{317}{49}, d = -\frac{104}{49}$. The other equalities can be proved similarly.

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 5.2.
The following equalities are true:

$$\begin{aligned} 49E_n &= 12G_{n+4} - 31G_{n+3} - 22G_{n+2} - 39G_{n+1}, \\ 7E_n &= -G_{n+3} + 2G_{n+2} + 3G_{n+1} + 12G_n, \\ E_n &= G_n - G_{n-1}, \end{aligned}$$

and

$$\begin{aligned} 16G_n &= E_{n+4} - E_{n+3} - 4E_{n+2} - 9E_{n+1}, \\ 16G_n &= E_{n+3} - E_{n+2} - 4E_{n+1} + 7E_n, \\ 16G_n &= E_{n+2} - E_{n+1} + 12E_n + 7E_{n-1}, \\ 16G_n &= E_{n+1} + 15E_n + 12E_{n-1} + 7E_{n-2}, \\ 16G_n &= 17E_n + 15E_{n-1} + 12E_{n-2} + 7E_{n-3}. \end{aligned}$$

Note that all the identities in the above Lemma can be proved by induction as well.
Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

Lemma 5.3.
The following equalities are true:

$$\begin{aligned} 112H_n &= -11E_{n+4} - 69E_{n+3} + 572E_{n+2} + 195E_{n+1}, \\ 16H_n &= -13E_{n+3} + 77E_{n+2} + 20E_{n+1} - 11E_n, \\ 16H_n &= 51E_{n+2} - 19E_{n+1} - 76E_n - 91E_{n-1}, \\ 16H_n &= 83E_{n+1} + 77E_n + 164E_{n-1} + 357E_{n-2}, \\ 16H_n &= 243E_n + 413E_{n-1} + 772E_{n-2} + 581E_{n-3}, \end{aligned}$$

and

$$\begin{aligned} 241727E_n &= -4122H_{n+3} + 16240H_{n+2} - 5649H_{n+1} + 4475H_n, \\ 241727E_n &= 7996H_{n+2} - 18015H_{n+1} - 16135H_n - 28854H_{n-1}, \\ 241727E_n &= -2023H_{n+1} + 7853H_n + 11126H_{n-1} + 55972H_{n-2}, \\ 241727E_n &= 3807H_n + 5057H_{n-1} + 45857H_{n-2} - 14161H_{n-3}. \end{aligned}$$

We now present a few special identities for the modified 4-primes sequence $\{E_n\}$.

Theorem 5.1.
(Catalan's identity) For all integers n and m, the following identity holds

$$\begin{aligned} E_{n+m}E_{n-m} - E_n^2 &= (G_{n+m} - G_{n+m-1})(G_{n-m} - G_{n-m-1}) - (G_n - G_{n-1})^2 \\ &= (G_n(G_m - G_{m+1}) + G_{n-1}(-G_m + G_{m-2}) + G_{n-2}(-G_m + G_{m-1})) \\ &\quad (G_n(G_{-m} - G_{1-m}) + G_{n-1}(-G_{-m} + G_{-m-2}) + G_{n-2}(-G_{-m} + G_{-m-1})) \\ &\quad -(G_n - G_{n-1})^2 \end{aligned}$$

Proof. We use the identity

$$E_n = G_n - G_{n-1}.$$

Note that for $m = 1$ in Catalan's identity, we get the Cassini identity for the modified 4-primes sequnce

Corollary 5.1.
(Cassini's identity) For all integers numbers n and m, the following identity holds

$$E_{n+1}E_{n-1} - E_n^2 = (G_{n+1} - G_n)(G_{n-1} - G_{n-2}) - (G_n - G_{n-1})^2.$$

The d'Ocagne's, Gelin-Cesàro's and Melham' identities can also be obtained by using $E_n = G_n - G_{n-1}$.The next theorem presents d'Ocagne's, Gelin-Cesàro's and Melham' identities of modified 4-primes sequence $\{E_n\}$.

Theorem 5.2.
Let n and m be any integers. Then the following identities are true:

(a) *(d'Ocagne's identity)*

$$E_{m+1}E_n - E_mE_{n+1} = (G_{m+1} - G_m)(G_n - G_{n-1}) - (G_m - G_{m-1})(G_{n+1} - G_n).$$

(b) *(Gelin-Cesàro's identity)*

$$E_{n+2}E_{n+1}E_{n-1}E_{n-2} - E_n^4 = (G_{n+2} - G_{n+1})(G_{n+1} - G_n)(G_{n-1} - G_{n-2})(G_{n-2} - G_{n-3}) - (G_n - G_{n-1})^4.$$

(c) *(Melham's identity)*

$$E_{n+1}E_{n+2}E_{n+6} - E_{n+3}^3 = (G_{n+1} - G_n)(G_{n+2} - G_{n+1})(G_{n+6} - G_{n+5}) - (G_{n+3} - G_{n+2})^3.$$

Proof. Use the identity $E_n = G_n - G_{n-1}$.

6. Linear Sums

The following proposition presents some formulas of generalized 4-primes numbers with positive subscripts.

Proposition 6.1.
If $r = 2, s = 3, t = 5, u = 7$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} V_k = \frac{1}{16}(V_{n+4} - V_{n+3} - 4V_{n+2} - 9V_{n+1} - V_3 + V_2 + 4V_1 + 9V_0)$.

(b) $\sum_{k=0}^{n} V_{2k} = \frac{1}{32}(9V_{2n+2} - 25V_{2n+1} + 28V_{2n} - 49V_{2n-1} + 7V_3 - 23V_2 + 4V_1 - 31V_0)$.

(c) $\sum_{k=0}^{n} V_{2k+1} = \frac{1}{32}(-7V_{2n+2} + 55V_{2n+1} - 4V_{2n} + 63V_{2n-1} - 9V_3 + 25V_2 + 4V_1 + 49V_0)$.

Proof. Take $r = 2, s = 3, t = 5, u = 7$ in Theorem 2.1 in [12].
As special cases of above proposition, we have the following three corollaries. First one presents some summing formulas of 4-primes numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 6.1.
For $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} G_k = \frac{1}{16}(G_{n+4} - G_{n+3} - 4G_{n+2} - 9G_{n+1} - 1)$.

(b) $\sum_{k=0}^{n} G_{2k} = \frac{1}{32}(9G_{2n+2} - 25G_{2n+1} + 28G_{2n} - 49G_{2n-1} - 9)$.

(c) $\sum_{k=0}^{n} G_{2k+1} = \frac{1}{32}(-7G_{2n+2} + 55G_{2n+1} - 4G_{2n} + 63G_{2n-1} + 7)$.

Second one presents some summing formulas of Lucas 4-primes numbers (take $V_n = H_n$ with $H_0 = 4$, $H_1 = 2, H_2 = 10, H_3 = 41$).

Corollary 6.2.
For $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} H_k = \frac{1}{16}(H_{n+4} - H_{n+3} - 4H_{n+2} - 9H_{n+1} + 13)$.

(b) $\sum_{k=0}^{n} H_{2k} = \frac{1}{32}(9H_{2n+2} - 25H_{2n+1} + 28H_{2n} - 49H_{2n-1} - 59)$.

(c) $\sum_{k=0}^{n} H_{2k+1} = \frac{1}{32}(-7H_{2n+2} + 55H_{2n+1} - 4H_{2n} + 63H_{2n-1} + 85)$.

Third one presents some summing formulas of modified 4-primes numbers (take $V_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 6.3.
For $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} E_k = \frac{1}{16}(E_{n+4} - E_{n+3} - 4E_{n+2} - 9E_{n+1})$.

(b) $\sum_{k=0}^{n} E_{2k} = \frac{1}{32}(9E_{2n+2} - 25E_{2n+1} + 28E_{2n} - 49E_{2n-1} - 16)$.

(c) $\sum_{k=0}^{n} E_{2k+1} = \frac{1}{32}(-7E_{2n+2} + 55E_{2n+1} - 4E_{2n} + 63E_{2n-1} + 16)$.

The following proposition presents some formulas of generalized 4-primes numbers with negative subscripts.

Proposition 6.2.
If $r = 2, s = 3, t = 5, u = 7$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} V_{-k} = \frac{1}{16}(-V_{-n+3} + V_{-n+2} + 4V_{-n+1} + 9V_{-n} + V_3 - V_2 - 4V_1 - 9V_0)$.

(b) $\sum_{k=1}^{n} V_{-2k} = \frac{1}{32}(-9V_{-2n+2} + 25V_{-2n+1} + 4V_{-2n} + 49V_{-2n-1} - 7V_3 + 23V_2 - 4V_1 + 31V_0)$.

(c) $\sum_{k=1}^{n} V_{-2k+1} = \frac{1}{32}(7V_{-2n+2} - 23V_{-2n+1} + 4V_{-2n} - 63V_{-2n-1} + 9V_3 - 25V_2 - 4V_1 - 49V_0)$.

Proof. Take $r = 2, s = 3, t = 5, u = 7$ in Theorem 3.1 in [12].
From the above proposition, we have the following corollary which gives sum formulas of 4-primes numbers (take $V_n = G_n$ with $G_0 = 0, G_1 = 0, G_2 = 1, G_3 = 2$).

Corollary 6.4.
For $n \geq 1$, 4-primes numbers have the following properties.

(a) $\sum_{k=1}^{n} G_{-k} = \frac{1}{16}(-G_{-n+3} + G_{-n+2} + 4G_{-n+1} + 9G_{-n} + 1)$.

(b) $\sum_{k=1}^{n} G_{-2k} = \frac{1}{32}(-9G_{-2n+2} + 25G_{-2n+1} + 4G_{-2n} + 49G_{-2n-1} + 9)$.

(c) $\sum_{k=1}^{n} G_{-2k+1} = \frac{1}{32}(7G_{-2n+2} - 23G_{-2n+1} + 4G_{-2n} - 63G_{-2n-1} - 7)$.

Taking $V_n = H_n$ with $H_0 = 4$, $H_1 = 2, H_2 = 10, H_3 = 41$ in the last proposition, we have the following corollary which presents sum formulas of 4-primes -Lucas numbers.

Corollary 6.5.
For $n \geq 1$, 4-primes -Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} H_{-k} = \frac{1}{16}(-H_{-n+3} + H_{-n+2} + 4H_{-n+1} + 9H_{-n} - 13)$.

(b) $\sum_{k=1}^{n} H_{-2k} = \frac{1}{32}(-9H_{-2n+2} + 25H_{-2n+1} + 4H_{-2n} + 49H_{-2n-1} + 59$.

(c) $\sum_{k=1}^{n} H_{-2k+1} = \frac{1}{32}(7H_{-2n+2} - 23H_{-2n+1} + 4H_{-2n} - 63H_{-2n-1} - 85)$.

From the above proposition, we have the following corollary which gives sum formulas of modified 4-primes numbers (take $V_n = E_n$ with $E_0 = 0, E_1 = 0, E_2 = 1, E_3 = 1$).

Corollary 6.6.
For $n \geq 1$, modified 4-primes numbers have the following properties.

(a) $\sum_{k=1}^{n} E_{-k} = \frac{1}{16}(-E_{-n+3} + E_{-n+2} + 4E_{-n+1} + 9E_{-n})$.

(b) $\sum_{k=1}^{n} E_{-2k} = \frac{1}{32}(-9E_{-2n+2} + 25E_{-2n+1} + 4E_{-2n} + 49E_{-2n-1} + 16)$.

(c) $\sum_{k=1}^{n} E_{-2k+1} = \frac{1}{32}(7E_{-2n+2} - 23E_{-2n+1} + 4E_{-2n} - 63E_{-2n-1} - 16)$.

7. Matrices Related with Generalized 4-primes Numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+3} \\ W_{n+2} \\ W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_3 \\ W_2 \\ W_1 \\ W_0 \end{pmatrix}. \tag{17}$$

For matrix formulation (17), see [5]. In fact, Kalman give the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \\ W_{n+2} \\ W_{n+3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ r & s & t & u \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \\ W_2 \\ W_3 \end{pmatrix}.$$

We define the square matrix A of order 4 as:

$$A = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = -7$. From (4) we have

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} V_{n+2} \\ V_{n+1} \\ V_n \\ V_{n-1} \end{pmatrix}. \tag{18}$$

and from (17) (or using (18) and induction) we have

$$\begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}.$$

If we take $V_n = G_n$ in (18) we have

$$\begin{pmatrix} G_{n+3} \\ G_{n+2} \\ G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} 2 & 3 & 5 & 7 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} G_{n+2} \\ G_{n+1} \\ G_n \\ G_{n-1} \end{pmatrix}. \tag{19}$$

We also define

$$B_n = \begin{pmatrix} G_{n+2} & 3G_{n+1}+5G_n+7G_{n-1} & 5G_{n+1}+7G_n & 7G_{n+1} \\ G_{n+1} & 3G_n+5G_{n-1}+7G_{n-2} & 5G_n+7G_{n-1} & 7G_n \\ G_n & 3G_{n-1}+5G_{n-2}+7G_{n-3} & 5G_{n-1}+7G_{n-2} & 7G_{n-1} \\ G_{n-1} & 3G_{n-2}+5G_{n-3}+7G_{n-4} & 5G_{n-2}+7G_{n-3} & 7G_{n-2} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} V_{n+2} & 3V_{n+1}+5V_n+7V_{n-1} & 5V_{n+1}+7V_n & 7V_{n+1} \\ V_{n+1} & 3V_n+5V_{n-1}+7V_{n-2} & 5V_n+7V_{n-1} & 7V_n \\ V_n & 3V_{n-1}+5V_{n-2}+7V_{n-3} & 5V_{n-1}+7V_{n-2} & 7V_{n-1} \\ V_{n-1} & 3V_{n-2}+5V_{n-3}+7V_{n-4} & 5V_{n-2}+7V_{n-3} & 7V_{n-2} \end{pmatrix}$$

Theorem 7.1.
For all integer $m, n \geq 0$, we have

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (19) to 4-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1}B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1}C_1$. Now

$$C_{n+m} = A^{n+m-1}C_1 = A^{n-1}A^m C_1 = A^{n-1}C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

Some properties of matrix A^n can be given as

$$A^n = 2A^{n-1} + 3A^{n-2} + 5A^{n-3} + 7A^{n-4}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-7)^n$$

for all integer m and n.

Theorem 7.2.
For $m, n \geq 0$ we have

$$V_{n+m} = V_n G_{m+2} + V_{n-1}(3G_{m+1} + 5G_m + 7G_{m-1}) + V_{n-2}(5G_{m+1} + 7G_m) + 7V_{n-3}G_{m+1} \tag{20}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof.

Remark 7.1.
By induction, it can be proved that for all integers $m, n \leq 0$, (20) holds. So for all integers m, n, (20) is true.

Corollary 7.1.
For all integers m, n, we have

$$G_{n+m} = G_n G_{m+2} + G_{n-1}(3G_{m+1} + 5G_m + 7G_{m-1}) + G_{n-2}(5G_{m+1} + 7G_m) + 7G_{n-3}G_{m+1}, \tag{21}$$

$$H_{n+m} = H_n G_{m+2} + H_{n-1}(3G_{m+1} + 5G_m + 7G_{m-1}) + H_{n-2}(5G_{m+1} + 7G_m) + 7H_{n-3}G_{m+1}, \tag{22}$$

$$E_{n+m} = E_n G_{m+2} + E_{n-1}(3G_{m+1} + 5G_m + 7G_{m-1}) + E_{n-2}(5G_{m+1} + 7G_m) + 7E_{n-3}G_{m+1}. \tag{23}$$

References

[1] A. D. Godase, M. B. Dhakne, On the properties of k-Fibonacci and k-Lucas numbers, Int. J. Adv. Appl. Math. and Mech. 2(1) (2014) 100-106.

[2] H. Gökbaş, H. Köse, Some sum formulas for products of Pell and Pell-Lucas numbers, Int. J. Adv. Appl. Math. and Mech. 4(4) (2017) 1–4.

[3] G. S. Hathiwala, D. V. Shah, Binet–Type Formula For The Sequence of Tetranacci Numbers by Alternate Methods, Mathematical Journal of Interdisciplinary Sciences 6(1) (2017), 37–48.

[4] F.T. Howard, F. Saidak, Zhou's Theory of Constructing Identities, Congress Numer. 200 (2010), 225-237.

[5] D. Kalman, Generalized Fibonacci Numbers By Matrix Methods, Fibonacci Quarterly, 20(1) (1982), 73-76.

[6] E. Kiliç, P. Stanica, A Matrix Approach for General Higher Order Linear Recurrences, Bulletin of the Malaysian Mathematical Sciences Society, (2) 34(1) (2011), 51–67.

[7] R. S. Melham, Some Analogs of the Identity $F_n^2 + F_{n+1}^2 = F_{2n+1}^2$, Fibonacci Quarterly, (1999), 305-311.

[8] L. R. Natividad, On Solving Fibonacci-Like Sequences of Fourth, Fifth and Sixth Order, International Journal of Mathematics and Computing, 3(2) (2013), 38-40.

[9] B. Singh, P. Bhadouria, O. Sikhwal, K. Sisodiya, A Formula for Tetranacci-Like Sequence, Gen. Math. Notes, 20(2) (2014), 136-141.

[10] N.J.A. Sloane, The on-line encyclopedia of integer sequences, http://oeis.org/

[11] Y. Soykan, Simson Identity of Generalized m-step Fibonacci Numbers, Int. J. Adv. Appl. Math. and Mech. 7(2) (2019), 45-56.

[12] Y. Soykan, Summation Formulas For Generalized Tetranacci Numbers, Asian Journal of Advanced Research and Reports, 7(2) (2019), 1-12. doi.org/10.9734/ajarr/2019/v7i230170.

[13] S. Uygun, The binomial transforms of the generalized (s,t)-Jacobsthal matrix sequence, Int. J. Adv. Appl. Math. and Mech. 6(3) (2019) 14–20.

[14] M. E. Waddill, Another Generalized Fibonacci Sequence, M. E., Fibonacci Quarterly, 5(3) (1967), 209-227.

[15] M. E. Waddill, The Tetranacci Sequence and Generalizations, Fibonacci Quarterly, (1992), 9-20.