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Horadam Numbers: Sum of the Squares of Terms of Sequence

Research Article

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Abstract: In this paper, closed forms of the sum formulas for the squares of generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers.

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1. Introduction

The Fibonacci and Lucas sequences are well-known examples of second order recurrence sequences. The sequence of Fibonacci numbers $\{F_n\}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \; n \geq 2, \quad F_0 = 0, \; F_1 = 1.$$

and the sequence of Lucas numbers $\{L_n\}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \; n \geq 2, \quad L_0 = 2, \; L_1 = 1.$$

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field. Horadam [?] defined a generalization of Fibonacci sequence, that is, he defined a second-order linear recurrence sequence $\{W_n(W_0, W_1; r, s)\}$, or simply $\{W_n\}$, as follows:

$$W_n = rW_{n-1} + sW_{n-2}; \; W_0 = a, \; W_1 = b, \quad (n \geq 2) \tag{1}$$

where W_0, W_1 are arbitrary complex numbers and r, s are real numbers, see also Horadam [?], [11] and [12]. Now these generalized Fibonacci numbers $\{W_n(a, b; r, s)\}$ are also called Horadam numbers. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s} W_{-(n-1)} + \frac{1}{s} W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1) holds for all integer n.

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Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	Notation: $W_n(a,b;r,s)$	OEIS: [19]
Fibonacci	$F_n = W_n(0,1;1,1)$	A000045
Lucas	$L_n = W_n(2,1;1,1)$	A000032
Pell	$P_n = W_n(0,1;2,1)$	A000129
Pell-Lucas	$Q_n = W_n(2,2;2,1)$	A002203
Jacobsthal	$J_n = W_n(0,1;1,2)$	A001045
Jacobsthal-Lucas	$j_n = W_n(2,1;1,2)$	A014551

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values. See [7],[8],[21] for some work on second-order generalization of Fibonacci numbers. The evaluation of sums of powers of these sequences is a challenging issue. Two pretty examples are

$$\sum_{k=0}^{n} P_k^2 = \frac{1}{2}(-2P_{n+1}^2 + P_{n+2}P_{n+1})$$

and

$$\sum_{k=1}^{n} J_{-k}^2 = \frac{1}{9}(nJ_{-n+1}^2 + (4n+5)J_{-n}^2 - 4(n+1)J_{-n+1}J_{-n}).$$

In this work, we derive expressions for sums of second powers of generalized Fibonacci numbers. We present some works on sum formulas of powers of the numbers in the following Table 2.

Table 2. A few special study on sum formulas of second, third and arbitrary powers.

Name of sequence	sums of second powers	sums of third powers	sums of powers
Generalized Fibonacci	[1],[2],[6],[13],[14],[20]	[5],[22]	[3],[4],[15]
Generalized Tribonacci	[17]		
Generalized Tetranacci	[16],[18]		

2. Summing Formulas of Generalized Fibonacci Numbers with Positive Subscripts

The following Theorem presents some summing formulas of generalized Fibonacci numbers with positive subscripts.

Theorem 2.1.
Let x be a complex number. If $(sx+1)(r^2x - s^2x^2 + 2sx - 1) \neq 0$ then

(a)

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta_1}{(sx+1)(r^2x - s^2x^2 + 2sx - 1)}$$

where

$$\begin{aligned}\Delta_1 = {} & (-(sx-1)x^{n+2}W_{n+2}^2 - (r^2x + sx + r^2sx^2 - 1)x^{n+1}W_{n+1}^2 \\ & +2rsx^{n+3}W_{n+2}W_{n+1} + x(sx-1)W_1^2 + (r^2x + sx + r^2sx^2 - 1)W_0^2 - 2rsx^2W_1W_0).\end{aligned}$$

(b)

$$\sum_{k=0}^{n} x^k W_{k+1}W_k = \frac{\Delta_2}{(sx+1)(r^2x - s^2x^2 + 2sx - 1)}.$$

where

$$\begin{aligned}\Delta_2 = {} & (rx^{n+2}W_{n+2}^2 + rs^2x^{n+3}W_{n+1}^2 - (r^2x + s^2x^2 - 1)x^{n+1}W_{n+2}W_{n+1} \\ & -rxW_1^2 - rs^2x^2W_0^2 + (r^2x + s^2x^2 - 1)W_1W_0).\end{aligned}$$

Proof. Using the recurrence relation

$$W_{n+2} = rW_{n+1} + sW_n$$

i.e.

$$sW_n = W_{n+2} - rW_{n+1}$$

we obtain

$$\begin{aligned}
s^2x^nW_n^2 &= x^nW_{n+2}^2 + r^2x^nW_{n+1}^2 - 2rx^nW_{n+2}W_{n+1} \\
s^2x^{n-1}W_{n-1}^2 &= x^{n-1}W_{n+1}^2 + r^2x^{n-1}W_n^2 - 2rx^{n-1}W_{n+1}W_n \\
s^2x^{n-2}W_{n-2}^2 &= x^{n-2}W_n^2 + r^2x^{n-2}W_{n-1}^2 - 2rx^{n-2}W_nW_{n-1} \\
s^2x^{n-3}W_{n-3}^2 &= x^{n-3}W_{n-1}^2 + r^2x^{n-3}W_{n-2}^2 - 2rx^{n-3}W_{n-1}W_{n-2} \\
&\vdots \\
s^2x^2W_2^2 &= x^2W_4^2 + r^2x^2W_3^2 - 2rx^2W_4W_3 \\
s^2x^1W_1^2 &= x^1W_3^2 + r^2x^1W_2^2 - 2rx^1W_3W_2 \\
s^2x^0W_0^2 &= x^0W_2^2 + r^2x^0W_1^2 - 2rx^0W_2W_{12}.
\end{aligned}$$

If we add the above equations by side by, we get

$$s^2\sum_{k=0}^{n}x^kW_k^2 = \sum_{k=2}^{n+2}x^{k-2}W_k^2 + r^2\sum_{k=1}^{n+1}x^{k-1}W_k^2 - 2r\sum_{k=1}^{n+1}x^{k-1}W_{k+1}W_k. \tag{2}$$

Note that

$$\begin{aligned}
\sum_{k=2}^{n+2}x^{k-2}W_k^2 &= -x^{-2}W_0^2 - x^{-1}W_1^2 + x^{n-1}W_{n+1}^2 + x^nW_{n+2}^2 + x^{-2}\sum_{k=0}^{n}x^kW_k^2 \\
\sum_{k=1}^{n+1}x^{k-1}W_k^2 &= -x^{-1}W_0^2 + x^nW_{n+1}^2 + x^{-1}\sum_{k=0}^{n}x^kW_k^2 \\
\sum_{k=1}^{n+1}x^{k-1}W_{k+1}W_k &= -x^{-1}W_1W_0 + x^nW_{n+2}W_{n+1} + x^{-1}\sum_{k=0}^{n}x^kW_{k+1}W_k.
\end{aligned}$$

If we put them into the (2) we get

$$\begin{aligned}
s^2\sum_{k=0}^{n}x^kW_k^2 &= (-x^{-2}W_0^2 - x^{-1}W_1^2 + x^{n-1}W_{n+1}^2 + x^nW_{n+2}^2 + x^{-2}\sum_{k=0}^{n}x^kW_k^2) \\
&\quad + r^2(-x^{-1}W_0^2 + x^nW_{n+1}^2 + x^{-1}\sum_{k=0}^{n}x^kW_k^2) \\
&\quad - 2r(-x^{-1}W_1W_0 + x^nW_{n+2}W_{n+1} + x^{-1}\sum_{k=0}^{n}x^kW_{k+1}W_k).
\end{aligned} \tag{3}$$

Next we obtain $\sum_{k=0}^{n}W_{k+1}W_k$. Multiplying the both side of the recurrence relation

$$sW_n = W_{n+2} - rW_{n+1}$$

by W_{n+1} we get

$$sW_{n+1}W_n = W_{n+2}W_{n+1} - rW_{n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{aligned}
sx^nW_{n+1}W_n &= x^nW_{n+2}W_{n+1} - rx^nW_{n+1}^2 \\
sx^{n-1}W_nW_{n-1} &= x^{n-1}W_{n+1}W_n - rx^{n-1}W_n^2 \\
sx^{n-2}W_{n-1}W_{n-2} &= x^{n-2}W_nW_{n-1} - rx^{n-2}W_{n-1}^2 \\
&\vdots \\
sx^2W_3W_2 &= x^2W_4W_3 - rx^2W_3^2 \\
sx^1W_2W_1 &= x^1W_3W_2 - rx^1W_2^2 \\
sx^0W_1W_0 &= x^0W_2W_1 - rx^0W_1^2
\end{aligned}$$

If we add the above equations by side by, we get

$$s\sum_{k=0}^{n}x^{k}W_{k+1}W_{k}=\sum_{k=1}^{n+1}x^{k-1}W_{k+1}W_{k}-r\sum_{k=1}^{n+1}x^{k-1}W_{k}^{2}. \tag{4}$$

Note that

$$\begin{aligned}\sum_{k=1}^{n+1}x^{k-1}W_{k+1}W_{k} &= -x^{-1}W_{1}W_{0}+x^{n}W_{n+2}W_{n+1}+x^{-1}\sum_{k=0}^{n}x^{k}W_{k+1}W_{k}\\ \sum_{k=1}^{n+1}x^{k-1}W_{k}^{2} &= -x^{-1}W_{0}^{2}+x^{n}W_{n+1}^{2}+x^{-1}\sum_{k=0}^{n}x^{k}W_{k}^{2}\end{aligned}$$

If we put them into the (4) then we obtain

$$\begin{aligned}s\sum_{k=0}^{n}x^{k}W_{k+1}W_{k} &= (-x^{-1}W_{1}W_{0}+x^{n}W_{n+2}W_{n+1}+x^{-1}\sum_{k=0}^{n}x^{k}W_{k+1}W_{k})\\ &\quad -r(-x^{-1}W_{0}^{2}+x^{n}W_{n+1}^{2}+x^{-1}\sum_{k=0}^{n}x^{k}W_{k}^{2}).\end{aligned} \tag{5}$$

Then, solving the system (3)-(5), the required results of (a) and (b) follow.

2.1. The case $x = 1$

In this subsection we consider the special case $x = 1$.
Taking $x = 1, r = s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.1.
If $r = s = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = -W_{n+1}^2 + W_{n+2}W_{n+1} + W_0^2 - W_1W_0$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{2}(W_{n+2}^2 + W_{n+1}^2 - W_{n+1}W_{n+2} - W_1^2 - W_0^2 + W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.1.
For $n \geq 0$, Fibonacci numbers have the following properties:

(a) $\sum_{k=0}^{n} F_k^2 = -F_{n+1}^2 + F_{n+2}F_{n+1}$.

(b) $\sum_{k=0}^{n} F_{k+1}F_k = \frac{1}{2}(F_{n+2}^2 + F_{n+1}^2 - F_{n+1}F_{n+2} - 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.2.
For $n \geq 0$, Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} L_k^2 = -L_{n+1}^2 + L_{n+2}L_{n+1} + 2$.

(b) $\sum_{k=0}^{n} L_{k+1}L_k = \frac{1}{2}(L_{n+2}^2 + L_{n+1}^2 - L_{n+1}L_{n+2} - 3)$.

Taking $x = 1, r = 2, s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.2.
If $r = 2, s = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{2}(-2W_{n+1}^2 + W_{n+2}W_{n+1} + 2W_0^2 - W_1W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{4}(W_{n+2}^2 + W_{n+1}^2 - 2W_{n+2}W_{n+1} - W_1^2 - W_0^2 + 2W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.3.
For $n \geq 0$, Pell numbers have the following properties:

(a) $\sum_{k=0}^{n} P_k^2 = \frac{1}{2}(-2P_{n+1}^2 + P_{n+2}P_{n+1})$.

(b) $\sum_{k=0}^{n} P_{k+1}P_k = \frac{1}{4}(P_{n+2}^2 + P_{n+1}^2 - 2P_{n+2}P_{n+1} - 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.4.
For $n \geq 0$, Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n} Q_k^2 = \frac{1}{2}(-2Q_{n+1}^2 + Q_{n+2}Q_{n+1} + 4)$.

(b) $\sum_{k=0}^{n} Q_{k+1}Q_k = \frac{1}{4}(Q_{n+2}^2 + Q_{n+1}^2 - 2Q_{n+2}Q_{n+1})$.

If $x = 1, r = 1, s = 2$ then $(sx+1)(r^2x - s^2x^2 + 2sx - 1) = 0$ so we can't use Theorem 2.1, directly. But we can give another method to find $\sum_{k=0}^{n} W_k^2$ and $\sum_{k=0}^{n} W_{k+1}W_k$.

Theorem 2.2.
If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n} W_k^2 = \frac{1}{9}((n+4)W_{n+2}^2 + (4n+11)W_{n+1}^2 - 4(n+3)W_{n+2}W_{n+1} - 3W_1^2 - 7W_0^2 + 8W_1W_0)$.

(b) $\sum_{k=0}^{n} W_{k+1}W_k = \frac{1}{9}(-(n+2)W_{n+2}^2 - 4(n+3)W_{n+1}^2 + (4n+13)W_{n+2}W_{n+1} + W_1^2 + 8W_0^2 - 9W_1W_0)$.

Proof.

(a) We use Theorem 2.1 (a). If we set $r = 1, s = 2$ in Theorem 2.1 (a) then we have

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{f_1(x)}{-(2x+1)(4x^2-5x+1)}.$$

where

$$\begin{aligned} f_1(x) &= (2x^2+3x-1)W_0^2 - (2x^2+3x-1)x^{n+1}W_{n+1}^2 - (2x-1)x^{n+2}W_{n+2}^2 \\ &\quad + x(2x-1)W_1^2 - 4x^2W_1W_0 + 4x^{n+3}W_{n+2}W_{n+1} \end{aligned}$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n} W_k^2 &= \left. \frac{\frac{d}{dx}(f_1(x))}{\frac{d}{dx}(-(2x+1)(4x^2-5x+1))} \right|_{x=1} \\ &= \frac{1}{9}((n+4)W_{n+2}^2 + (4n+11)W_{n+1}^2 - 4(n+3)W_{n+2}W_{n+1} - 3W_1^2 - 7W_0^2 + 8W_1W_0). \end{aligned}$$

(b) We use Theorem 2.1 (b). If we set $r = 1, s = 2$ in Theorem 2.1 (b) then we have

$$\sum_{k=0}^{n} x^k W_{k+1}W_k = \frac{f_2(x)}{-(2x+1)(4x^2-5x+1)}$$

where

$$f_2(x) = x^{n+2}W_{n+2}^2 + 4x^{n+3}W_{n+1}^2 - x^{n+1}(4x^2+x-1)W_{n+2}W_{n+1} - xW_1^2 - 4x^2W_0^2 + (4x^2+x-1)W_1W_0$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n} W_k^2 &= \left. \frac{\frac{d}{dx}(f_2(x))}{\frac{d}{dx}(-(2x+1)(4x^2-5x+1))} \right|_{x=1} \\ &= \frac{1}{9}(-(n+2)W_{n+2}^2 - 4(n+3)W_{n+1}^2 + (4n+13)W_{n+2}W_{n+1} + W_1^2 + 8W_0^2 - 9W_1W_0). \end{aligned}$$

From the last theorem we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.5.
For $n \geq 0$, Jacobsthal numbers have the following property:

(a) $\sum_{k=0}^{n} J_k^2 = \frac{1}{9}((n+4) J_{n+2}^2 + (4n+11) J_{n+1}^2 - 4(n+3) J_{n+2}J_{n+1} - 3)$.

(b) $\sum_{k=0}^{n} J_{k+1}J_k = \frac{1}{9}(-(n+2) J_{n+2}^2 - 4(n+3) J_{n+1}^2 + (4n+13) J_{n+2}J_{n+1} + 1)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.6.
For $n \geq 0$, Jacobsthal-Lucas numbers have the following property:

(a) $\sum_{k=0}^{n} j_k^2 = \frac{1}{9}((n+4) j_{n+2}^2 + (4n+11) j_{n+1}^2 - 4(n+3) j_{n+2}j_{n+1} - 15)$.

(b) $\sum_{k=0}^{n} j_{k+1}j_k = \frac{1}{9}(-(n+2) j_{n+2}^2 - 4(n+3) j_{n+1}^2 + (4n+13) j_{n+2}j_{n+1} + 15)$.

2.2. The case $x = -1$

In this subsection we consider the special case $x = -1$.

If $x = -1, r = 1, s = 1$ then $(sx+1)(r^2x - s^2x^2 + 2sx - 1) = 0$ so we can't use Theorem 2.1, directly. But we can give another method to find $\sum_{k=0}^{n} W_k^2$ and $\sum_{k=0}^{n} W_{k+1}W_k$.

Theorem 2.3.
If $r = 1, s = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{5}((-1)^n((2n+5) W_{n+2}^2 - (2n+2) W_{n+1}^2 - 2(n+3) W_{n+2}W_{n+1}) + 3W_1^2 - 4W_1W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{5}((-1)^n((n+2) W_{n+2}^2 - (n+3) W_{n+1}^2 - nW_{n+2}W_{n+1}) + W_1^2 - 2W_0^2 + W_1W_0)$.

Proof.

(a) We use Theorem 2.1 (a). If we set $r = 1, s = 1$ in Theorem 2.1 (a) then we have

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{f_3(x)}{-(x+1)(x^2-3x+1)}.$$

where

$$\begin{aligned} f_3(x) &= -x^{n+2}(x-1) W_{n+2}^2 - x^{n+1}(x^2+2x-1) W_{n+1}^2 + 2x^{n+3} W_{n+2}W_{n+1} \\ &\quad + x(x-1) W_1^2 + (x^2+2x-1) W_0^2 - 2x^2 W_1 W_0 \end{aligned}$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n}(-1)^k W_k^2 &= \left. \frac{\frac{d}{dx}(f_3(x))}{\frac{d}{dx}(-(x+1)(x^2-3x+1))} \right|_{x=-1} \\ &= \frac{1}{5}((-1)^n((2n+5) W_{n+2}^2 - (2n+2) W_{n+1}^2 - 2(n+3) W_{n+2}W_{n+1}) + 3W_1^2 - 4W_1W_0). \end{aligned}$$

(b) We use Theorem 2.1 (b). If we set $r = 1, s = 1$ in Theorem 2.1 (b) then we have

$$\sum_{k=0}^{n} x^k W_{k+1}W_k = \frac{f_4(x)}{-(x+1)(x^2-3x+1)}$$

where

$$f_4(x) = x^{n+2} W_{n+2}^2 + x^{n+3} W_{n+1}^2 - x^{n+1}(x^2+x-1) W_{n+2}W_{n+1} - xW_1^2 - x^2 W_0^2 + (x^2+x-1) W_1 W_0$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k W_k^2 &= \left.\frac{\frac{d}{dx}(f_4(x))}{\frac{d}{dx}(-(x+1)(x^2-3x+1))}\right|_{x=-1} \\ &= \frac{1}{5}((-1)^n((n+2)W_{n+2}^2-(n+3)W_{n+1}^2-nW_{n+2}W_{n+1})+W_1^2-2W_0^2+W_1W_0).\end{aligned}$$

From the above theorem, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0, F_1=1$).

Corollary 2.7.
For $n\geq 0$, Fibonacci numbers have the following properties:

(a) $\sum_{k=0}^{n}(-1)^k F_k^2=\frac{1}{5}((-1)^n((2n+5)F_{n+2}^2-(2n+2)F_{n+1}^2-2(n+3)F_{n+2}F_{n+1})+3)$.

(b) $\sum_{k=0}^{n}(-1)^k F_{k+1}F_k=\frac{1}{5}((-1)^n((n+2)F_{n+2}^2-(n+3)F_{n+1}^2-nF_{n+2}F_{n+1})+1)$.

Taking $W_n=L_n$ with $L_0=2, L_1=1$ in the last theorem, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.8.
For $n\geq 0$, Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n}(-1)^k L_k^2=\frac{1}{5}((-1)^n((2n+5)L_{n+2}^2-(2n+2)L_{n+1}^2-2(n+3)L_{n+2}L_{n+1})-5)$.

(b) $\sum_{k=0}^{n}(-1)^k L_{k+1}L_k=\frac{1}{5}((-1)^n((n+2)L_{n+2}^2-(n+3)L_{n+1}^2-nL_{n+2}L_{n+1})-5)$.

If $x=-1, r=2, s=1$ then $(sx+1)(r^2x-s^2x^2+2sx-1)=0$ so we can't use Theorem 2.1, directly. But we can give another method to find $\sum_{k=0}^{n}W_k^2$ and $\sum_{k=0}^{n}W_{k+1}W_k$.

Theorem 2.4.
If $r=2, s=1$ then for $n\geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k^2=\frac{1}{8}((-1)^n((2n+5)W_{n+2}^2-(2n-1)W_{n+1}^2-4(n+3)W_{n+2}W_{n+1})+3W_1^2+3W_0^2-8W_1W_0)$

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k=\frac{1}{4}((-1)^n((n+2)W_{n+2}^2-(n+3)W_{n+1}^2-(2n+3)W_{n+2}W_{n+1})+W_1^2-2W_0^2-W_1W_0)$

Proof.

(a) We use Theorem 2.1 (a). If we set $r=2, s=1$ in Theorem 2.1 (a) then we have

$$\sum_{k=0}^{n}x^k W_k^2=\frac{f_5(x)}{(x+1)\left(x^2-6x+1\right)}$$

where

$$\begin{aligned}f_5(x) &= (x-1)x^{n+2}W_{n+2}^2+x^{n+1}\left(4x^2+5x-1\right)W_{n+1}^2-4x^{n+3}W_{n+2}W_{n+1} \\ &\quad -x(x-1)W_1^2-\left(4x^2+5x-1\right)W_0^2+4x^2W_1W_0.\end{aligned}$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}\sum_{k=0}^{n}(-1)^k W_k^2 &= \left.\frac{\frac{d}{dx}(f_5(x))}{\frac{d}{dx}((x+1)\left(x^2-6x+1\right))}\right|_{x=-1} \\ &= \frac{1}{8}((-1)^n((2n+5)W_{n+2}^2-(2n-1)W_{n+1}^2-4(n+3)W_{n+2}W_{n+1})+3W_1^2+3W_0^2-8W_1W_0).\end{aligned}$$

(b) We use Theorem 2.1 (b). If we set $r = 2, s = 1$ in Theorem 2.1 (b) then we have

$$\sum_{k=0}^{n} x^k W_{k+1} W_k = \frac{f_6(x)}{(x+1)\left(x^2-6x+1\right)}$$

where

$$f_6(x) = -2x^{n+2}W_{n+2}^2 - 2x^{n+3}W_{n+1}^2 + \left(x^2+4x-1\right)x^{n+1}W_{n+2}W_{n+1} + 2xW_1^2 + 2x^2W_0^2 - \left(x^2+4x-1\right)W_1W_0$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=0}^{n}(-1)^k W_k^2 &= \left.\frac{\frac{d}{dx}(f_6(x))}{\frac{d}{dx}((x+1)\left(x^2-6x+1\right))}\right|_{x=-1} \\ &= \frac{1}{4}((-1)^n((n+2)W_{n+2}^2 - (n+3)W_{n+1}^2 - (2n+3)W_{n+2}W_{n+1}) + W_1^2 - 2W_0^2 - W_1W_0). \end{aligned}$$

From the last theorem, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.9.
For $n \geq 0$, Pell numbers have the following properties:

(a) $\sum_{k=0}^{n}(-1)^k P_k^2 = \frac{1}{8}((-1)^n((2n+5)P_{n+2}^2 - (2n-1)P_{n+1}^2 - 4(n+3)P_{n+2}P_{n+1}) + 3)$.

(b) $\sum_{k=0}^{n}(-1)^k P_{k+1}P_k = \frac{1}{4}((-1)^n((n+2)P_{n+2}^2 - (n+3)P_{n+1}^2 - (2n+3)P_{n+2}P_{n+1}) + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.10.
For $n \geq 0$, Pell-Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n}(-1)^k Q_k^2 = \frac{1}{8}((-1)^n((2n+5)Q_{n+2}^2 - (2n-1)Q_{n+1}^2 - 4(n+3)Q_{n+2}Q_{n+1}) - 8)$.

(b) $\sum_{k=0}^{n}(-1)^k Q_{k+1}Q_k = \frac{1}{4}((-1)^n((n+2)Q_{n+2}^2 - (n+3)Q_{n+1}^2 - (2n+3)Q_{n+2}Q_{n+1}) - 8)$.

Taking $x = -1, r = 1, s = 2$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.3.
If $r = 1, s = 2$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(-1)^k W_k^2 = \frac{1}{10}((-1)^n(3W_{n+2}^2 - 2W_{n+1}^2 - 4W_{n+2}W_{n+1}) + 3W_1^2 - 2W_0^2 - 4W_1W_0)$.

(b) $\sum_{k=0}^{n}(-1)^k W_{k+1}W_k = \frac{1}{10}((-1)^n(W_{n+2}^2 - 4W_{n+1}^2 + 2W_{n+2}W_{n+1}) + W_1^2 - 4W_0^2 + 2W_1W_0)$.

From the last proposition we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.11.
For $n \geq 1$, Jacobsthal numbers have the following property:

(a) $\sum_{k=0}^{n}(-1)^k J_k^2 = \frac{1}{10}((-1)^n(3J_{n+2}^2 - 2J_{n+1}^2 - 4J_{n+2}J_{n+1}) + 3)$.

(b) $\sum_{k=0}^{n}(-1)^k J_{k+1}J_k = \frac{1}{10}((-1)^n(J_{n+2}^2 - 4J_{n+1}^2 + 2J_{n+2}J_{n+1}) + 1)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.12.
For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:

(a) $\sum_{k=0}^{n}(-1)^k j_k^2 = \frac{1}{10}((-1)^n(3j_{n+2}^2 - 2j_{n+1}^2 - 4j_{n+2}j_{n+1}) - 13)$.

(b) $\sum_{k=0}^{n}(-1)^k j_{k+1}j_k = \frac{1}{10}((-1)^n(j_{n+2}^2 - 4j_{n+1}^2 + 2j_{n+2}j_{n+1}) - 11)$.

2.3. The case $x = 1 + i$

In this subsection we consider the special case $x = 1 + i$.
Taking $x = 1 + i, r = s = 1$ in Theorem 2.1 (a) and (b), we obtain the following proposition.

Proposition 2.4.
If $r = s = 1$ then for $n \geq 0$ we have the following formulas:

(a) $\sum_{k=0}^{n}(1+i)^k W_k^2 = \frac{1}{3+4i}((1+i)^n(2W_{n+2}^2 + (3-5i)W_{n+1}^2 + 4(-1+i)W_{n+2}W_{n+1}) - (1-i)W_1^2 + (1+4i)W_0^2 - 4iW_1W_0)$.

(b) $\sum_{k=0}^{n}(1+i)^k W_{k+1}W_k = \frac{1}{3+4i}((1+i)^n(2iW_{n+2}^2 + (-2+2i)W_{n+1}^2 + (3-3i)W_{n+2}W_{n+1}) - (1+i)W_1^2 - 2iW_0^2 + 3iW_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 2.13.
For $n \geq 0$, Fibonacci numbers have the following properties:

(a) $\sum_{k=0}^{n}(1+i)^k F_k^2 = \frac{1}{3+4i}((1+i)^n(2F_{n+2}^2 + (3-5i)F_{n+1}^2 + 4(-1+i)F_{n+2}F_{n+1}) + (-1+i))$.

(b) $\sum_{k=0}^{n}(1+i)^k F_{k+1}F_k = \frac{1}{3+4i}((1+i)^n(2iF_{n+2}^2 + (-2+2i)F_{n+1}^2 + (3-3i)F_{n+2}F_{n+1}) - (1+i))$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.14.
For $n \geq 0$, Lucas numbers have the following properties:

(a) $\sum_{k=0}^{n}(1+i)^k L_k^2 = \frac{1}{3+4i}((1+i)^n(2L_{n+2}^2 + (3-5i)L_{n+1}^2 + 4(-1+i)L_{n+2}L_{n+1}) + 3(1+3i))$.

(b) $\sum_{k=0}^{n}(1+i)^k L_{k+1}L_k = \frac{1}{3+4i}((1+i)^n(2iL_{n+2}^2 + (-2+2i)L_{n+1}^2 + (3-3i)L_{n+2}L_{n+1}) - (1+3i))$.

Corresponding sums of Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers can be calculated similarly.

3. Summing Formulas of Generalized Fibonacci Numbers with Negative Subscripts

The following theorem presents some linear summing formulas of generalized Fibonacci numbers with negative subscripts.

Theorem 3.1.
Let x be a complex number. If $(s+x)(r^2x + 2sx - s^2 - x^2) \neq 0$ then

(a)

$$\sum_{k=1}^{n} x^k W_{-k}^2 = \frac{\Delta_3}{(s+x)(r^2x + 2sx - s^2 - x^2)}$$

where

$$\begin{aligned}\Delta_3 &= x^{n+1}(s-x)W_{-n+1}^2 + x^{n+1}(r^2s + r^2x + sx - x^2)W_{-n}^2 - 2rsx^{n+1}W_{-n+1}W_{-n} \\ &\quad + x(x-s)W_1^2 + x(-r^2s - r^2x - sx + x^2)W_0^2 + 2rsxW_1W_0.\end{aligned}$$

(b)

$$\sum_{k=1}^{n} x^k W_{-k+1}W_{-k} = \frac{\Delta_4}{(s+x)(r^2x + 2sx - s^2 - x^2)}$$

where

$$\begin{aligned}\Delta_4 &= (-rx^{n+2}W_{-n+1}^2 - rs^2x^{n+1}W_{-n}^2 + x^{n+1}(r^2x + s^2 - x^2)W_{-n+1}W_{-n} \\ &\quad + rx^2W_1^2 + rs^2xW_0^2 - x(r^2x + s^2 - x^2)W_1W_0).\end{aligned}$$

Proof. Using the recurrence relation

$$W_{-n+2} = rW_{-n+1} + sW_{-n} \Rightarrow W_{-n} = -\frac{r}{s}W_{-n+1} + \frac{1}{s}W_{-n+2}$$

i.e.

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

we obtain

$$\begin{array}{rcl} s^2x^nW_{-n}^2 &=& x^nW_{-n+2}^2 + r^2x^nW_{-n+1}^2 - 2rx^nW_{-n+2}W_{-n+1} \\ s^2x^{n-1}W_{-n+1}^2 &=& x^{n-1}W_{-n+3}^2 + r^2x^{n-1}W_{-n+2}^2 - 2rx^{n-1}W_{-n+3}W_{-n+2} \\ s^2x^{n-2}W_{-n+2}^2 &=& x^{n-2}W_{-n+4}^2 + r^2x^{n-2}W_{-n+3}^2 - 2rx^{n-2}W_{-n+4}W_{-n+3} \\ && \vdots \\ s^2x^3W_{-3}^2 &=& x^3W_{-1}^2 + r^2x^3W_{-2}^2 - 2rx^3W_{-1}W_{-2} \\ s^2x^2W_{-2}^2 &=& x^2W_0^2 + r^2x^2W_{-1}^2 - 2rx^2W_0W_{-1} \\ s^2x^1W_{-1}^2 &=& x^1W_1^2 + r^2x^1W_0^2 - 2rx^1W_1W_0 \end{array}$$

If we add the above equations by side by, we get

$$\begin{array}{rcl} s^2\sum_{k=1}^{n}x^kW_{-k}^2 &=& (x^1W_1^2 + x^2W_0^2 + \sum_{k=1}^{n-2}x^{k+2}W_{-k}^2) + r^2(x^1W_0^2 \\ && +\sum_{k=1}^{n-1}x^{k+1}W_{-k}^2) - 2r(x^1W_1W_0 + \sum_{k=1}^{n-1}x^{k+1}W_{-k+1}W_{-k}) \end{array} \quad (6)$$

Note that

$$\begin{array}{rcl} \sum_{k=1}^{n-2}x^{k+2}W_{-k}^2 &=& -x^{n+1}W_{-n+1}^2 - x^{n+2}W_{-n}^2 + x^2\sum_{k=1}^{n}x^kW_{-k}^2 \\ \sum_{k=1}^{n-1}x^{k+1}W_{-k}^2 &=& -x^{n+1}W_{-n}^2 + x\sum_{k=1}^{n}x^kW_{-k}^2 \\ \sum_{k=1}^{n-1}x^{k+1}W_{-k+1}W_{-k} &=& -x^{n+1}W_{-n+1}W_{-n} + x\sum_{k=1}^{n}x^kW_{-k+1}W_{-k}. \end{array}$$

If we put them into the (6) then we obtain

$$\begin{array}{rcl} s^2\sum_{k=1}^{n}x^kW_{-k}^2 &=& (x^1W_1^2 + x^2W_0^2 - x^{n+1}W_{-n+1}^2 - x^{n+2}W_{-n}^2 + x^2\sum_{k=1}^{n}x^kW_{-k}^2) \\ && +r^2(x^1W_0^2 - x^{n+1}W_{-n}^2 + x\sum_{k=1}^{n}x^kW_{-k}^2) - 2r(x^1W_1W_0 - x^{n+1}W_{-n+1}W_{-n} \\ && +x\sum_{k=1}^{n}x^kW_{-k+1}W_{-k}) \end{array} \quad (7)$$

Next we calculate $\sum_{k=1}^{n}W_{-k+1}W_{-k}$. Multiplying the both side of the recurrence relation

$$sW_{-n} = W_{-n+2} - rW_{-n+1}$$

by W_{-n+1} we get

$$sW_{-n+1}W_{-n} = W_{-n+2}W_{-n+1} - rW_{-n+1}^2.$$

Then using last recurrence relation, we obtain

$$\begin{array}{rcl} sx^nW_{-n+1}W_{-n} &=& x^nW_{-n+2}W_{-n+1} - rx^nW_{-n+1}^2 \\ sx^{n-1}W_{-n+2}W_{-n+1} &=& x^{n-1}W_{-n+3}W_{-n+2} - rx^{n-1}W_{-n+2}^2 \\ sx^{n-2}W_{-n+3}W_{-n+2} &=& x^{n-2}W_{-n+4}W_{-n+3} - rx^{n-2}W_{-n+3}^2 \\ && \vdots \\ sx^3W_{-2}W_{-3} &=& x^3W_{-1}W_{-2} - rx^3W_{-2}^2 \\ sx^2W_{-1}W_{-2} &=& x^2W_0W_{-1} - rx^2W_{-1}^2 \\ sxW_0W_{-1} &=& xW_1W_0 - rxW_0^2 \end{array}$$

If we add the above equations by side by, we get

$$s\sum_{k=1}^{n}x^{k}W_{-k+1}W_{-k}=(xW_{1}W_{0}+\sum_{k=1}^{n-1}x^{k+1}W_{-k+1}W_{-k})-r(xW_{0}^{2}+\sum_{k=1}^{n-1}x^{k+1}W_{-k}^{2}) \tag{8}$$

Note that

$$\begin{aligned}\sum_{k=1}^{n-1}x^{k+1}W_{-k+1}W_{-k} &= -x^{n+1}W_{-n+1}W_{-n}+x\sum_{k=1}^{n}x^{k}W_{-k+1}W_{-k}\\ \sum_{k=1}^{n-1}x^{k+1}W_{-k}^{2} &= -x^{n+1}W_{-n}^{2}+x\sum_{k=1}^{n}x^{k}W_{-k}^{2}.\end{aligned}$$

If we put them into the (8) then we obtain

$$s\sum_{k=1}^{n}x^{k}W_{-k+1}W_{-k}=(xW_{1}W_{0}-x^{n+1}W_{-n+1}W_{-n}+x\sum_{k=1}^{n}x^{k}W_{-k+1}W_{-k})-r(xW_{0}^{2}-x^{n+1}W_{-n}^{2}+x\sum_{k=1}^{n}x^{k}W_{-k}^{2}) \tag{9}$$

Then, solving the system (7)-(9), the required results of (a) and (b) follow.

3.1. The case $x = 1$

In this subsection we consider the special case $x = 1$.
Taking $x = 1, r = s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.1.
If $r = s = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} W_{-k}^{2} = W_{-n}^{2} - W_{-n+1}W_{-n} - W_{0}^{2} + W_{1}W_{0}$.

(b) $\sum_{k=1}^{n} W_{-k+1}W_{-k} = \frac{1}{2}(-W_{-n+1}^{2} - W_{-n}^{2} + W_{-n+1}W_{-n} + W_{1}^{2} + W_{0}^{2} - W_{1}W_{0})$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.1.
For $n \geq 1$, Fibonacci numbers have the following properties.

(a) $\sum_{k=1}^{n} F_{-k}^{2} = F_{-n}^{2} - F_{-n+1}F_{-n}$.

(b) $\sum_{k=1}^{n} F_{-k+1}F_{-k} = \frac{1}{2}(-F_{-n+1}^{2} - F_{-n}^{2} + F_{-n+1}F_{-n} + 1)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.2.
For $n \geq 1$, Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} L_{-k}^{2} = L_{-n}^{2} - L_{-n+1}L_{-n} - 2$.

(b) $\sum_{k=1}^{n} L_{-k+1}L_{-k} = \frac{1}{2}(-L_{-n+1}^{2} - L_{-n}^{2} + L_{-n+1}L_{-n} + 3)$.

Taking $x = 1, r = 2, s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.2.
If $r = 2, s = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} W_{-k}^{2} = \frac{1}{2}(2W_{-n}^{2} - W_{-n+1}W_{-n} - 2W_{0}^{2} + W_{1}W_{0})$.

(b) $\sum_{k=1}^{n} W_{-k+1}W_{-k} = \frac{1}{4}(-W_{-n+1}^{2} - W_{-n}^{2} + 2W_{-n+1}W_{-n} + W_{1}^{2} + W_{0}^{2} - 2W_{1}W_{0})$.

From the last proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 3.3.
For $n \geq 1$, Pell numbers have the following properties.

(a) $\sum_{k=1}^{n} P_{-k}^2 = \frac{1}{2}(2P_{-n}^2 - P_{-n+1}P_{-n})$.

(b) $\sum_{k=1}^{n} P_{-k+1}P_{-k} = \frac{1}{4}(-P_{-n+1}^2 - P_{-n}^2 + 2P_{-n+1}P_{-n} + 1)$.

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.4.
For $n \geq 1$, Pell-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n} Q_{-k}^2 = \frac{1}{2}(2Q_{-n}^2 - Q_{-n+1}Q_{-n} - 4)$.

(b) $\sum_{k=1}^{n} Q_{-k+1}Q_{-k} = \frac{1}{4}(-Q_{-n+1}^2 - Q_{-n}^2 + 2Q_{-n+1}Q_{-n})$.

If $x = 1, r = 1, s = 2$ then $(s + x)(r^2x + 2sx - s^2 - x^2) = 0$ so we can't use Theorem 3.1, directly. But we can give another method to find $\sum_{k=1}^{n} W_{-k}^2$ and $\sum_{k=1}^{n} W_{-k+1}W_{-k}$.

Theorem 3.2.
If $r = 1, s = 2$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n} W_{-k}^2 = \frac{1}{9}(nW_{-n+1}^2 + (4n+5)W_{-n}^2 - 4(n+1)W_{-n+1}W_{-n} - 5W_0^2 + 4W_1W_0)$.

(b) $\sum_{k=1}^{n} W_{-k+1}W_{-k} = \frac{1}{9}(-(n+2)W_{-n+1}^2 - 4(n+1)W_{-n}^2 + (4n+3)W_{-n+1}W_{-n} + 2W_1^2 + 4W_0^2 - 3W_1W_0)$.

Proof.

(a) We use Theorem 3.1 (a). If we set $r = 1, s = 2$ in Theorem 3.1 (a) then we have

$$\sum_{k=1}^{n} x^k W_{-k}^2 = \frac{g_1(x)}{-(x+2)(x^2-5x+4)}$$

where

$$\begin{aligned} g_1(x) &= -(x-2)x^{n+1}W_{-n+1}^2 + (-x^2+3x+2)x^{n+1}W_{-n}^2 - 4x^{n+1}W_{-n+1}W_{-n} \\ &\quad + x(x-2)W_1^2 - x\left(-x^2+3x+2\right)W_0^2 + 4xW_1W_0. \end{aligned}$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} \sum_{k=1}^{n} W_{-k}^2 &= \left.\frac{\frac{d}{dx}(g_1(x))}{\frac{d}{dx}(-(x+2)(x^2-5x+4))}\right|_{x=1} \\ &= \frac{1}{9}(nW_{-n+1}^2 + (4n+5)W_{-n}^2 - 4(n+1)W_{-n+1}W_{-n} - 5W_0^2 + 4W_1W_0). \end{aligned}$$

(b) We use Theorem 3.1 (b). If we set $r = 1, s = 2$ in Theorem 3.1 (b) then we have

$$\sum_{k=1}^{n} x^k W_{-k+1}W_{-k} = \frac{g_2(x)}{-(x+2)(x^2-5x+4)}$$

where

$$\begin{aligned} g_2(x) &= -x^{n+2}W_{-n+1}^2 - 4x^{n+1}W_{-n}^2 + (-x^2+x+4)x^{n+1}W_{-n+1}W_{-n} \\ &\quad + x^2W_1^2 + 4xW_0^2 - x\left(-x^2+x+4\right)W_1W_0 \end{aligned}$$

For $x = 1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule.

$$\begin{aligned} \sum_{k=1}^{n} W_{-k+1}W_{-k} &= \left.\frac{\frac{d}{dx}(g_2(x))}{\frac{d}{dx}(-(x+2)(x^2-5x+4))}\right|_{x=1} \\ &= \frac{1}{9}(-(n+2)W_{-n+1}^2 - 4(n+1)W_{-n}^2 + (4n+3)W_{-n+1}W_{-n} + 2W_1^2 + 4W_0^2 - 3W_1W_0). \end{aligned}$$

From the last theorem, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.5.
For $n \geq 1$, Jacobsthal numbers have the following property:

(a) $\sum_{k=1}^{n} J_{-k}^2 = \frac{1}{9}(nJ_{-n+1}^2 + (4n+5)J_{-n}^2 - 4(n+1)J_{-n+1}J_{-n})$.

(b) $\sum_{k=1}^{n} J_{-k+1}J_{-k} = \frac{1}{9}(-(n+2)J_{-n+1}^2 - 4(n+1)J_{-n}^2 + (4n+3)J_{-n+1}J_{-n} + 2)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.6.
For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:

(a) $\sum_{k=1}^{n} j_{-k}^2 = \frac{1}{9}(nj_{-n+1}^2 + (4n+5)j_{-n}^2 - 4(n+1)j_{-n+1}j_{-n} - 12)$.

(b) $\sum_{k=1}^{n} j_{-k+1}j_{-k} = \frac{1}{9}(-(n+2)j_{-n+1}^2 - 4(n+1)j_{-n}^2 + (4n+3)j_{-n+1}j_{-n} + 12)$.

3.2. The case $x = -1$

In this subsection we consider the special case $x = -1$.

If $x = -1, r = 1, s = 1$ then $(s+x)(r^2x + 2sx - s^2 - x^2) = 0$ so we can't use Theorem 3.1, directly. But we can give another method to find $\sum_{k=1}^{n} W_{-k}^2$ and $\sum_{k=1}^{n} W_{-k+1}W_{-k}$.

Theorem 3.3.
If $r = 1, s = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^k W_{-k}^2 = \frac{1}{5}((-1)^n(-(2n+3)W_{-n+1}^2 + (2n+6)W_{-n}^2 + 2(n+1)W_{-n+1}W_{-n}) + 3W_1^2 - 6W_0^2 - 2W_1W_0)$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-k+1}W_{-k} = \frac{1}{5}((-1)^n(-(n+2)W_{-n+1}^2 + (n+1)W_{-n}^2 + (n+4)W_{-n+1}W_{-n}) + 2W_1^2 - W_0^2 - 4W_1W_0)$.

Proof.

(a) We use Theorem 3.1 (a). If we set $r = 1, s = 1$ in Theorem 3.1 (a) then we have

$$\sum_{k=1}^{n} x^k W_{-k}^2 = \frac{g_3(x)}{-(x+1)(x^2-3x+1)}$$

where

$$\begin{aligned} g_3(x) &= -(x-1)x^{n+1}W_{-n+1}^2 + x^{n+1}(-x^2+2x+1)W_{-n}^2 - 2x^{n+1}W_{-n+1}W_{-n} \\ &\quad + x(x-1)W_1^2 - x\left(-x^2+2x+1\right)W_0^2 + 2xW_1W_0 \end{aligned}$$

For $x = -1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} &\sum_{k=1}^{n}(-1)^k W_{-k}^2 \\ &= \left.\frac{\frac{d}{dx}(g_3(x))}{\frac{d}{dx}(-(x+1)(x^2-3x+1))}\right|_{x=-1} \\ &= \frac{1}{5}((-1)^n(-(2n+3)W_{-n+1}^2 + (2n+6)W_{-n}^2 + 2(n+1)W_{-n+1}W_{-n}) + 3W_1^2 - 6W_0^2 - 2W_1W_0). \end{aligned}$$

(b) We use Theorem 3.1 (b). If we set $r = 1, s = 1$ in Theorem 3.1 (b) then we have

$$\sum_{k=1}^{n} x^k W_{-k+1}W_{-k} = \frac{g_4(x)}{-(x+1)(x^2-3x+1)}$$

where

$$\begin{aligned} g_4(x) &= -x^{n+2}W_{-n+1}^2 - x^{n+1}W_{-n}^2 + (-x^2+x+1)x^{n+1}W_{-n+1}W_{-n} \\ &\quad +x^2W_1^2 + xW_0^2 - x\left(-x^2+x+1\right)W_1W_0 \end{aligned}$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned} &\sum_{k=1}^{n}(-1)^kW_{-k+1}W_{-k} \\ &= \left.\frac{\frac{d}{dx}(g_4(x))}{\frac{d}{dx}(-(x+1)(x^2-3x+1))}\right|_{x=-1} \\ &= \frac{1}{5}((-1)^n(-(n+2)W_{-n+1}^2 + (n+1)W_{-n}^2 + (n+4)W_{-n+1}W_{-n}) + 2W_1^2 - W_0^2 - 4W_1W_0). \end{aligned}$$

From the above theorem, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.7.
For $n \geq 1$, Fibonacci numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^kF_{-k}^2 = \frac{1}{5}((-1)^n(-(2n+3)F_{-n+1}^2 + (2n+6)F_{-n}^2 + 2(n+1)F_{-n+1}F_{-n}) + 3)$.

(b) $\sum_{k=1}^{n}(-1)^kF_{-k+1}F_{-k} = \frac{1}{5}((-1)^n(-(n+2)F_{-n+1}^2 + (n+1)F_{-n}^2 + (n+4)F_{-n+1}F_{-n}) + 2)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last theorem, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.8.
For $n \geq 1$, Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n}(-1)^kL_{-k}^2 = \frac{1}{5}((-1)^n(-(2n+3)L_{-n+1}^2 + (2n+6)L_{-n}^2 + 2(n+1)L_{-n+1}L_{-n}) - 25)$.

(b) $\sum_{k=1}^{n}(-1)^kL_{-k+1}L_{-k} = \frac{1}{5}((-1)^n(-(n+2)L_{-n+1}^2 + (n+1)L_{-n}^2 + (n+4)L_{-n+1}L_{-n}) - 10)$.

If $x=-1, r=2, s=1$ then $(s+x)(r^2x+2sx-s^2-x^2)=0$ so we can't use Theorem 3.1, directly. But we can give another method to find $\sum_{k=1}^{n}W_{-k}^2$ and $\sum_{k=1}^{n}W_{-k+1}W_{-k}$.

Theorem 3.4.
If $r=2, s=1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^kW_{-k}^2 = \frac{1}{8}((-1)^n(-(2n+3)W_{-n+1}^2 + (2n+9)W_{-n}^2 + 4(n+1)W_{-n+1}W_{-n}) + 3W_1^2 - 9W_0^2 - 4W_1W_0)$.

(b) $\sum_{k=1}^{n}(-1)^kW_{-k+1}W_{-k} = \frac{1}{4}((-1)^n(-(n+2)W_{-n+1}^2 + (n+1)W_{-n}^2 + (2n+5)W_{-n+1}W_{-n}) + 2W_1^2 - W_0^2 - 5W_1W_0)$.

Proof.

(a) We use Theorem 3.1 (a). If we set $r=2, s=1$ in Theorem 3.1 (a) then we have

$$\sum_{k=1}^{n}x^kW_{-k}^2 = \frac{g_5(x)}{(x+1)\left(x^2-6x+1\right)}$$

where

$$\begin{aligned} g_5(x) &= x^{n+1}(x-1)W_{-n+1}^2 - x^{n+1}\left(-x^2+5x+4\right)W_{-n}^2 + 4x^{n+1}W_{-n+1}W_{-n} \\ &\quad -x(x-1)W_1^2 + x\left(-x^2+5x+4\right)W_0^2 - 4xW_1W_0 \end{aligned}$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^k W_{-k}^2\\ =&\left.\frac{\frac{d}{dx}(g_5(x))}{\frac{d}{dx}((x+1)\left(x^2-6x+1\right))}\right|_{x=-1}\\ =&\frac{1}{8}((-1)^n(-(2n+3)W_{-n+1}^2+(2n+9)W_{-n}^2+4(n+1)W_{-n+1}W_{-n})+3W_1^2-9W_0^2-4W_1W_0)\end{aligned}$$

(b) We use Theorem 3.1 (b). If we set $r=2, s=1$ in Theorem 3.1 (b) then we have

$$\sum_{k=1}^{n}x^k W_{-k+1}W_{-k}=\frac{g_6(x)}{(x+1)\left(x^2-6x+1\right)}$$

where

$$\begin{aligned}g_6(x) &= 2x^{n+2}W_{-n+1}^2+2x^{n+1}W_{-n}^2-x^{n+1}\left(-x^2+4x+1\right)W_{-n+1}W_{-n}\\ &\quad -2x^2W_1^2-2xW_0^2+x\left(-x^2+4x+1\right)W_1W_0\end{aligned}$$

For $x=-1$, the right hand side of the above sum formula is an indeterminate form. Now, we can use L'Hospital rule. Then we get

$$\begin{aligned}&\sum_{k=1}^{n}(-1)^k W_{-k+1}W_{-k}\\ =&\left.\frac{\frac{d}{dx}(g_6(x))}{\frac{d}{dx}((x+1)\left(x^2-6x+1\right))}\right|_{x=-1}\\ =&\frac{1}{4}((-1)^n(-(n+2)W_{-n+1}^2+(n+1)W_{-n}^2+(2n+5)W_{-n+1}W_{-n})+2W_1^2-W_0^2-5W_1W_0)\end{aligned}$$

From the last theorem, we have the following corollary which gives sum formulas of Pell numbers (take $W_n=P_n$ with $P_0=0, P_1=1$).

Corollary 3.9.
For $n\geq 1$, Pell numbers have the following properties.

(a) $\sum_{k=1}^{n}P_{-k}^2=\frac{1}{8}((-1)^n(-(2n+3)P_{-n+1}^2+(2n+9)P_{-n}^2+4(n+1)P_{-n+1}P_{-n})+3)$.

(b) $\sum_{k=1}^{n}P_{-k+1}P_{-k}=\frac{1}{4}((-1)^n(-(n+2)P_{-n+1}^2+(n+1)P_{-n}^2+(2n+5)P_{-n+1}P_{-n})+2)$.

Taking $W_n=Q_n$ with $Q_0=2, Q_1=2$ in the last theorem, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 3.10.
For $n\geq 1$, Pell-Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n}Q_{-k}^2=\frac{1}{8}((-1)^n(-(2n+3)Q_{-n+1}^2+(2n+9)Q_{-n}^2+4(n+1)Q_{-n+1}Q_{-n})-40)$.

(b) $\sum_{k=1}^{n}Q_{-k+1}Q_{-k}=\frac{1}{4}((-1)^n(-(n+2)Q_{-n+1}^2+(n+1)Q_{-n}^2+(2n+5)Q_{-n+1}Q_{-n})-16)$.

Taking $x=-1, r=1, s=2$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.3.
If $r=1, s=2$ then for $n\geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(-1)^k W_{-k}^2=\frac{1}{10}((-1)^n(3W_{-n+1}^2-2W_{-n}^2-4W_{-n+1}W_{-n})-3W_1^2+2W_0^2+4W_0W_1)$.

(b) $\sum_{k=1}^{n}(-1)^k W_{-k+1}W_{-k}=\frac{1}{10}((-1)^n(W_{-n+1}^2-4W_{-n}^2+2W_{-n+1}W_{-n})-W_1^2+4W_0^2-2W_1W_0)$.

From the last proposition, we have the following corollary which gives sum formula of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 3.11.
For $n \geq 1$, Jacobsthal numbers have the following property:

(a) $\sum_{k=1}^{n}(-1)^k J_{-k}^2 = \frac{1}{10}((-1)^n(3J_{-n+1}^2 - 2J_{-n}^2 - 4J_{-n+1}J_{-n}) - 3)$.

(b) $\sum_{k=1}^{n}(-1)^k J_{-k+1}J_{-k} = \frac{1}{10}((-1)^n(J_{-n+1}^2 - 4J_{-n}^2 + 2J_{-n+1}J_{-n}) - 1)$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 3.12.
For $n \geq 1$, Jacobsthal-Lucas numbers have the following property:

(a) $\sum_{k=1}^{n}(-1)^k j_{-k}^2 = \frac{1}{10}((-1)^n(3j_{-n+1}^2 - 2j_{-n}^2 - 4j_{-n+1}j_{-n}) + 13)$.

(b) $\sum_{k=1}^{n}(-1)^k j_{-k+1}j_{-k} = \frac{1}{10}((-1)^n(j_{-n+1}^2 - 4j_{-n}^2 + 2j_{-n+1}j_{-n}) + 11)$.

3.3. The case $x = 1 + i$

In this subsection we consider the special case $x = 1 + i$.
Taking $x = 1 + i, r = s = 1$ in Theorem 3.1 (a) and (b), we obtain the following proposition.

Proposition 3.4.
If $r = s = 1$ then for $n \geq 1$ we have the following formulas:

(a) $\sum_{k=1}^{n}(1+i)^k W_{-k}^2 = \frac{1}{3+4i}((1+i)^n((1-i)W_{-n+1}^2 + 3(1+i)W_{-n}^2 - 2(1+i)W_{-n+1}W_{-n}) - (1-i)W_1^2 - (2+3i)W_0^2 + 2iW_1W_0)$.

(b) $\sum_{k=1}^{n}(1+i)^k W_{-k+1}W_{-k} = \frac{1}{3+4i}((1+i)^n(-2iW_{-n+1}^2 - (1+i)W_{-n}^2 + (3+i)W_{-n+1}W_{-n}) + 2iW_1^2 + 2W_0^2 - (5-i)W_1W_0)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n = F_n$ with $F_0 = 0, F_1 = 1$).

Corollary 3.13.
For $n \geq 1$, Fibonacci numbers have the following properties.

(a) $\sum_{k=1}^{n}(1+i)^k F_{-k}^2 = \frac{1}{3+4i}((1+i)^n((1-i)F_{-n+1}^2 + 3(1+i)F_{-n}^2 - 2(1+i)F_{-n+1}F_{-n}) + (-1+i))$.

(b) $\sum_{k=1}^{n}(1+i)^k F_{-k+1}F_{-k} = \frac{1}{3+4i}((1+i)^n(-2iF_{-n+1}^2 - (1+i)F_{-n}^2 + (3+i)F_{-n+1}F_{-n}) + 2i)$.

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 3.14.
For $n \geq 1$, Lucas numbers have the following properties.

(a) $\sum_{k=1}^{n}(1+i)^k L_{-k}^2 = \frac{1}{3+4i}((1+i)^n((1-i)L_{-n+1}^2 + 3(1+i)L_{-n}^2 - 2(1+i)L_{-n+1}L_{-n}) - (9+7i))$.

(b) $\sum_{k=1}^{n}(1+i)^k L_{-k+1}L_{-k} = \frac{1}{3+4i}((1+i)^n(-2iL_{-n+1}^2 - (1+i)L_{-n}^2 + (3+i)L_{-n+1}L_{-n}) + (-2+4i))$.

Corresponding sums of Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas numbers can be calculated similarly.

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