

## On a fractional order generalized thermo-elastic diffusion theorem

Research Article

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**Abstract:** In this work, a new theory of thermo-diffusion in elastic solids is derived using the methodology of fractional calculus. The theories of coupled thermo-elastic diffusion and of generalized thermo-elastic diffusion problem with one relaxation time and also of generalized thermo-elastic diffusion in GN models follow as limit cases. A uniqueness and reciprocity theorem for these equations are derived using Laplace transform technique on the assumption of symmetry of stress tensor and positivity of associated parameters. Finally, a variational theorem is obtained for the governing equations.

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**Keywords:** Fractional calculus • Generalized thermoelastic-diffusion theory • Uniqueness theorem • Reciprocity theorem • Variational principle

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### 1. Introduction

In recent years increasing attention is directed towards the generalized theories of thermo-elasticity considering the second sound effect, which permits the propagation of waves at a finite speed. Thus, the generalized theories were found to give more realistic results than the classical (coupled) theory [1] of thermo-elasticity, which predicts infinite speeds of propagation. Lord and Shulman [2] obtained a wave-type heat equation by postulating a new law of heat conduction instead of the classical Fourier's law. Sherief and Ezzat [3] obtained the fundamental solution for this theory. The uniqueness of the solution of these equations was proved by Ignaczak [4]. Sherief and Dhaliwal [5] and Ignaczak [6] established the uniqueness by means of an associated conservation law involving a higher-order time derivative. The theory of generalized thermo-elasticity with two relaxation times was first introduced by Muller [7]. More explicit version was then introduced by Green and Laws [8], Green and Lindsay [9] and independently by Suhubi [10] by considering temperature rates among the constitutive variables. This theory also predicts finite speeds of propagation as in Lord and Shulman's theory. The low temperature thermo-elasticity is developed by Hetnarski and Ignaczak [11]. Green and Nagdhi [12] have posulated a new concept of thermo-elasticity which is called the thermo-elasticity without energy dissipation. In this theory, the classical Fourier law is replaced by a heat flux rate-temperature gradient relation. The general idea is posulated by Green and Nagdhi [13] in making use of the general entropy balance by suggesting three types of constitutive response functions. Type I, after linearization of the theory, is the same as the classical heat conduction theory (based on Fourier's law), while the types II and III permit propagation of thermo-elastic disturbances with a finite speed, only type II without energy dissipation. Also GN model III

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[14] of thermo-elasticity theory involves a heat conduction law and one that involves the thermal displacement gradient among the constitutive variables. Chandrasekharaiah [15-17] represents uniqueness theorem, variational and reciprocal principles of thermo-elasticity without energy dissipation and also represent a review work on generalized thermoelasticity.

Diffusion in elastic solids can be defined as the transport of matter from one point to another by thermal motion of atoms or molecules until goes to equilibrium is fundamental in the art and science of materials in solid-state physics, physical chemistry and material science. The characteristic of thermo-diffusions in deformable bodies with its historical development and importance commencing from the fundamental Fourier's and Frick's law is referred to the work of Olesiak [18], Podstrigac and Pavlina [19] and then of course of Nowacki [20-23]. Nowacki developed the theory of thermo-elastic diffusion within the context of classical coupled thermo-elasticity (CTE) and studied some dynamical problems of diffusion in solids using Frick's law. It has a major drawback that it predicts an infinite speed of wave propagation. Sherief et al. [24] developed the theory of generalized thermo-elastic diffusion in an elastic solid, which allows finite speeds of propagation of thermal disturbances. In this theory, Fick's law was modified to include the time derivative of the flux of the diffusive mass. A uniqueness and reciprocity theorems of generalized thermo-elastic diffusion theory is represented by Aouadi [25]. This theory is applied in the current work. Elhagary [26] has discussed the one-dimensional problem of generalized thermo-elastic diffusion for a long hollow cylinder. Recently, Tripathi et al. [27] discussed a two-dimensional dynamic problem of generalized thermo-elasticity in Lord-Shulman theory for a thick circular cylinder with heat sources.

The theory of fractional calculus has been used successfully to model polymers materials. The first application of fractional derivatives was given by Abel who applied fractional calculus in the solution of an integral equation that arises in the formulation of the tautochrone problem. A quasi-static uncoupled theory of thermo-elasticity based on fractional heat conduction equation was put forward by Povstenko [28-29]. The theory of thermal stresses based on the heat conduction equation with the Caputo time-fractional derivative, is used by Povstenko [30] to investigate thermal stresses in an infinite body with a circular cylindrical hole. Sherief et al. [31] introduced a new model of thermo-elasticity using fractional calculus, proved a uniqueness theorem, and derived a reciprocity relation and a variational principle. Youssef [32] introduced another new model of fractional heat conduction equation, proved a uniqueness theorem and presented one-dimensional application. Also, Youssef [33] represent also a thermo-elastic model with fractional order strain. Ezzat [34] established a new model of fractional heat conduction equation by using the new Taylor series expansion of time-fractional order which developed by Jumarie [35]. El-Karamany and Ezzat [36] introduced two models where the fractional derivatives and integrals are used to modify the Cattaneo heat conduction law [37] and in the context of the two-temperature thermo-elasticity theory, uniqueness and reciprocal theorems are proved, the convolutional variational principle is given and used to prove a uniqueness theorem with no restrictions imposed on the elasticity or thermal conductivity tensors except symmetry conditions. A fractional order theory of thermo-elastic diffusion is represented by Ezzat and Fayik[38]. Hamza et al.[39] represents generalized fractional thermo-elasticity associated with two relaxation times. Also, Shaw and Mukhopadhyay [40] represent a theory of fractional ordered thermo-elastic diffusion. Recently, Kanoria et al. [41-42] represent some thermo-elastic problem on this field. Also, Sheoran et al. [43] have studied a fractional order magneto thermo-elastic problem with three phase lag effect. One can follow the review work on fractional order thermo-elasticity theories by Sheoran and Kundu [44] to know more details.

The importance of reciprocity relations and the variational principles stems from the fact that these principles provide a theoretical basis for the modern numerical techniques such as finite element method [45] and boundary element methods [46]. In this concern, Chirita and Ciarletta [47] established the reciprocal and variational principle in linear therm-oelasticity without energy dissipation.

In the present work, we represent a new model of thermo-elasticity using fractional order diffusion theory in which every governing equation such as stress, heat conduction and diffusion equation etc. contain time fractional derivative of temperature of order up to developed by Jumarie. A uniqueness and reciprocity theorem is established. Also, a variational principle is derived. Classical couple theory of thermo-elasticity, thermo-elasticity with one relaxation time and GN models are derived as a limiting case.

## 2. Revised basic equations

The governing equations for an isotropic, homogeneous elastic solid with generalized thermo-elastic diffusion theory at constant temperature  $T_0$  are taken to establish the revised basic equations of our new proposed model given below:

(i) Equation of motion

$$\sigma_{ij,j} + \rho F_i = \rho \frac{\partial^2 \mathbf{u}_i}{\partial t^2} \quad (1)$$

(ii) Strain-displacement relation

$$e_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \quad (2)$$

(iii) Entropy balance equation

$$\rho Q - q_{ii} = \rho T \frac{\partial S}{\partial t} \tag{3}$$

(iv) Mass conservation equation

$$\eta_{i,i} + \frac{\partial C}{\partial t} = 0 \tag{4}$$

(v) Modified Fourier law

$$k\theta_{,i} + k^* \xi_{,i} = -q_i \tag{5}$$

(vi) Diffusion equation

$$DP_{,i} = -\eta_i \tag{6}$$

Here, we deal with the memory effect on an isotropic, homogeneous thermo-elastic solid using time fractional derivative of temperature of order  $\alpha$  in Caputo sense. A new model on fractional order thermo-elastic diffusion with one relaxation time is considered in which displacement components, temperature increment, stresses components, chemical potential and mass concentration of the material satisfy the following equations:

$$\sigma_{ij} = 2\mu e_{ij} + \delta_{ij}(\lambda_0 e - \gamma_2 P) - \delta_{ij} \gamma_1 \left( 1 + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \theta \tag{7}$$

$$C = \gamma_2 P + nP + d \left( 1 + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \theta \tag{8}$$

$$P = -\beta_2 + bC - a \left( 1 + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \theta \tag{9}$$

$$\mu u_{i,jj} + (\lambda_0 + \mu) u_{i,ji} - \gamma_1 \left( 1 + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \theta_{,i} - \gamma_2 P_{,i} + \rho F_i = \rho \frac{\partial^2 u_i}{\partial t^2} \tag{10}$$

$$\rho S = h_1 \left( 1 + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \theta + \gamma_1 e + dP \tag{11}$$

$$\left( k \frac{\partial}{\partial t} + k^* \right) \theta_{,ii} + \rho \frac{\partial Q}{\partial t} = \gamma_1 T_0 \frac{\partial^2 \mathbf{e}}{\partial t^2} + dT_0 \frac{\partial^2 \mathbf{P}}{\partial t^2} + l_1 T_0 \left( \frac{\partial^2}{\partial t^2} + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}} \right) \theta \tag{12}$$

$$DP_{,ii} = \gamma_2 \frac{\partial \mathbf{e}}{\partial t} + n \frac{\partial \mathbf{P}}{\partial t} + d \left( \frac{\partial}{\partial t} + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) \theta \tag{13}$$

Here, it is very important to note that

(A) the physical positive constants associated with diffusion satisfying the condition[23]

$$nh_1 - d^2 > 0. \tag{14}$$

(B) the mass diffusion wave exists[48-49] if

$$\beta_2^2 < b(\lambda + 2\mu). \tag{15}$$

In the above equations we have considered the following:

$$\theta = T - T_0, \lambda_0 = \lambda - \frac{\beta_2^2}{b}, \gamma_1 = \beta_1 + d\beta_2, d = \frac{a}{b}, \gamma_2 = \frac{\beta_2^2}{b}, n = \frac{1}{b}, \dot{\xi} = T, h_1 = \frac{\rho E}{T_0} + \frac{a^2}{b}, \beta_1 = (3\lambda + 2\mu) \alpha_t, \beta_2 = (3\lambda + 2\mu) \alpha_c.$$

Here,  $a$  is measure of thermo-diffusion,  $b$  is measure of diffusive effect,  $k$  is thermal conduction,  $k^*$  is material characteristic of G-N model,  $D$  is diffusion coefficient,  $\lambda, \mu$ , Lamé's constants,  $C_E$ , specific heat at constant strain,  $\alpha_t$ , the coefficient of linear thermal expansion,  $\alpha_c$  is linear diffusion expansion,  $T_0$  is initial reference temperature,  $\rho$  is mass density assumed independent of time,  $T$  is absolute temperature,  $\theta$  is the temperature above the reference temperature,  $\tau_1$  is the relaxation time,  $\xi$  is the temperature difference between two considerable state,  $\sigma_{ij}$  ( $i, j = x, y, z$ ) is the component of stress tensor,  $C$  is the mass concentration,  $P$  is the chemical potential per unit mass,  $\alpha$  ( $0 < \alpha \leq 1$ ) is the order of the fractional order derivative,  $S$  is the entropy per unit mass,  $q_i$  is the components of heat flux vector,  $\eta_i$  is the components of mass flow vector,  $Q$  is the intensity of the heat source per unit mass,  $F_i$  are the components of the body force per unit mass.

### 3. Derivation of the fundamental equations

The law of conservation of energy in an arbitrary volume  $V$  bounded by the surface  $A$  can be written as

$$\frac{d}{dt} \int_V \left[ \frac{1}{2} \frac{\partial \mathbf{u}_i}{\partial t} \frac{\partial \mathbf{u}_i}{\partial t} + U \right] \rho dV = \int_V \rho F_i \frac{\partial \mathbf{u}_i}{\partial t} dV + \int_A \left[ \sigma_{ij} \frac{\partial \mathbf{u}_i}{\partial t} - q_i \right] n_i dA - \int_V P \eta_{i,i} dV + \int_V \rho Q dV \quad (16)$$

where  $U$  is the internal energy per unit mass and  $n_i$  are the components of the unit outward normal vector to the surface  $A$ .

Using Gauss divergent theorem, dynamical equation of motion (1) and symmetry of stress tensor, we obtain the point-wise form of (16) in the form

$$\rho \frac{\partial U}{\partial t} = \rho Q - q_{i,i} + \sigma_{ij} \frac{\partial e_{ij}}{\partial t} + P \frac{\partial C}{\partial t} \quad (17)$$

Now, using entropy balance equation (3) and mass conservation equation (4) in equation (17) can be represented as

$$\rho dU = \sigma_{ij} de_{ij} + \rho T dS + P dC \quad (18)$$

We now introduce the Helmholtz free energy function, defined by

$$\Psi = U - TS \quad (19)$$

Using (19) in (18) we get,

$$\rho d\Psi = \sigma_{ij} de_{ij} - \rho S dT + P dC \quad (20)$$

Using chain rule  $\Psi$  can be represented in the form

$$\rho d\Psi(e_{ij}, T, C) = \frac{\partial \Psi}{\partial e_{ij}} de_{ij} + \rho \frac{\partial \Psi}{\partial T} dT + \rho \frac{\partial \Psi}{\partial C} dC \quad (21)$$

Comparison between (20) and (21) yields

$$\begin{aligned} \sigma_{ij} &= \rho \frac{\partial \Psi}{\partial e_{ij}} \\ S &= -\frac{\partial \Psi}{\partial T} = -\frac{\partial \Psi}{\partial \theta} \\ P &= \rho \frac{\partial \Psi}{\partial C} \end{aligned} \quad (22)$$

Now expanding the function  $\Psi = \Psi(e_{ij}, \theta, \omega, C)$  in a power series [35] in the independent variable of the form

$$\begin{aligned} \rho \Psi &= \rho \Psi_0 + a_0 \theta + c_{ij} e_{ij} + b_0 C + a_1 \omega - \frac{\rho C_E}{2T_0} \theta^2 + \frac{1}{2} C_{ijkl} e_{ij} e_{kl} \\ &+ \frac{1}{2} b C^2 + M_{ij} e_{ij} \left( \theta + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \omega \right) - \frac{\rho C_E \tau_1^\alpha}{T_0 \Gamma(\alpha+1)} \theta \omega + b_{ij} e_{ij} C - a \left( \theta + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \omega \right) C + \dots \end{aligned} \quad (23)$$

where  $\omega = \frac{\partial^\alpha \theta}{\partial t^\alpha}$ .

#### 3.1 Initial conditions

It is assumed that the system considered in the model is at natural state. So, the initial conditions can be written as

$$\Psi = 0, \theta = 0, \omega = 0, C = 0, \sigma_{ij} = 0, e_{ij} = 0, at t = 0. \quad (24)$$

Now, one can get easily from (23) after using initial conditions (24) as

$$b_0 = 0, \Psi_0 = 0, a_0 = 0, a_1 = 0, c_{ij} = 0. \quad (25)$$

We now use the relation (25) in the expression (23) and remaining the terms up to second order only which is consistent with linear theory and finally obtain the following relation:

$$\begin{aligned} \rho \Psi &= \frac{1}{2} C_{ijkl} e_{ij} e_{kl} - \frac{\rho C_E}{2T_0} \theta^2 + \frac{1}{2} b C^2 + M_{ij} e_{ij} \left( \theta + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \omega \right) \\ &- \frac{\rho C_E \tau_1^\alpha}{T_0 \Gamma(\alpha+1)} \theta \omega + b_{ij} e_{ij} C - a \left( \theta + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \omega \right) C \end{aligned} \quad (26)$$

It is very easy to use relation (22) in the expression (26) to obtain the following expression of stress, entropy and chemical potential in the form

$$\sigma_{ij} = C_{ijkl}e_{kl} + M_{ij} \left( \theta + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \omega \right) + b_{ij}C, \tag{27}$$

$$\rho S = \frac{\rho C_E}{T_0} \left( \theta + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \omega \right) - M_{ij}e_{ij} + aC, \tag{28}$$

$$P = bC + b_{ij}e_{ij} - a \left( \theta + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \omega \right). \tag{29}$$

For isotropic media,

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu \delta_{ik} \delta_{jl} + \mu \delta_{il} \delta_{jk}, M_{ij} = -\beta_1 \delta_{ij}, b_{ij} = -\beta_2 \delta_{ij}. \tag{30}$$

Using the relation (30) in (27-29) and arranging one can easily obtain the expression of stress, entropy, mass concentration and chemical potential in the form of governing equations (7), (8), (9) and (11) of the proposed model respectively.

Now our aim is to represent the governing equation of motion of our model. In this purpose we use relation (2) and (7) in equation (1) and obtain the required governing equation of motion in the form of (10).

To establish equation of heat conduction law we use energy balance equation in linearized form  $\rho Q - q_{i,i} = \rho T_0 \frac{\partial S}{\partial t}$  and entropy balance relation (11) in modified Fourier law (5) and arranging we get our governing equation of heat conduction in the form (12).

Finally, we establish the governing equation of diffusion in the form (13) with the help of equations (4), (6) and (8).

## 4. Uniqueness theorem

### 4.1 Statement of the theorem:

Supposing that a linear isotropic thermo-elastic diffusive material occupies a regular region V with boundary surface A in the 3-dimensional space, there is only one solution of the functions:  $u_i(x, t), \theta(x, t), P(x, t)$  of class  $C^m (m \geq 2)$  and  $\sigma_{ij}(x, t), e_{ij}(x, t)$  of class  $C^1$  at the point  $x \in (V + A)$  having co-ordinate  $x = (x_1, x_2, x_3)$  at  $t \geq 0$  which satisfies equations (7-13) subject to the boundary conditions

$$\theta = \phi^{(1)}(x, t), u_i = \vartheta_i^{(1)}(x, t), P = \chi^{(1)}(x, t), x \in A, t \geq 0 \tag{31}$$

and initial conditions

$$\begin{aligned} \theta &= \phi^{(2)}(x, 0), u_i = \vartheta_i^{(2)}(x, 0), P = \chi^{(2)}(x, 0), x \in V, t = 0 \\ \frac{\partial \theta}{\partial t} &= \frac{\partial \phi^{(3)}(x, t)}{\partial t}, \frac{\partial u_i}{\partial t} = \frac{\partial \vartheta_i^{(3)}(x, 0)}{\partial t}, \frac{\partial P}{\partial t} = \frac{\partial \chi^{(3)}(x, t)}{\partial t}, x \in V, t = 0. \end{aligned} \tag{32}$$

We assume that the Laplace transform of all the variables exists and the material parameters satisfy the inequalities  $T_0 > 0, \mu > 0, \lambda_0 > 0, l_1 > 0, d > 0, n > 0, k > 0, k^* > 0, D > 0$ .

### 4.2 Proof of the theorem:

Let,  $u_i^{(1)}, \theta^{(1)}, P^{(1)}, \dots$  and  $u_i^{(2)}, \theta^{(2)}, P^{(2)}, \dots$  are the two solution sets of the equations (7-13) with same body force, heat source, same boundary conditions (31) and same initial conditions (32).

Now, consider the difference function

$$u_i^* = u_i^{(1)} - u_i^{(2)}, \theta^* = \theta^{(1)} - \theta^{(2)}, P^* = P^{(1)} - P^{(2)}, \dots \tag{33}$$

which satisfy the governing equations(7-13) and can be represented as

$$\sigma_{ij}^* = 2\mu e_{ij}^* + \delta_{ij} (\lambda_0 e^* - \gamma_2 P^*) - \delta_{ij} \gamma_1 \left( 1 + \frac{\tau_1^\alpha}{\Gamma(\alpha + 1)} \frac{\partial^\alpha}{\partial t^\alpha} \right) \theta^* \tag{34}$$

$$\sigma_{ij,j}^* = \rho \frac{\partial^2 u_i^*}{\partial t^2} \tag{35}$$

$$\left(k \frac{\partial}{\partial t} + k^*\right) \theta_{,ii}^* = \gamma_1 T_0 \frac{\partial^2 \mathbf{e}^*}{\partial t^2} + d T_0 \frac{\partial^2 \mathbf{P}^*}{\partial t^2} + l_1 T_0 \left( \frac{\partial^2}{\partial t^2} + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \frac{\partial^{\alpha+2}}{\partial t^{\alpha+2}} \right) \theta^* \quad (36)$$

$$D P_{,ii}^* = \gamma_2 \frac{\partial \mathbf{e}^*}{\partial t} + n \frac{\partial \mathbf{P}^*}{\partial t} + d \left( \frac{\partial}{\partial t} + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \frac{\partial^{\alpha+1}}{\partial t^{\alpha+1}} \right) \theta^* \quad (37)$$

The difference functions (33) also satisfy homogeneous boundary conditions (31) and initial conditions (32) and can be written as

$$\theta^*(x, t) = 0, u_i^*(x, t) = 0, P^*(x, t) = 0, t > 0, x \in A \quad (38)$$

$$\begin{aligned} \theta^*(x, 0) = 0, u_i^*(x, 0) = 0, P^*(x, 0) = 0, t = 0, x \in V \\ \frac{\partial \theta^*(x, 0)}{\partial t} = 0, \frac{\partial u_i^*(x, 0)}{\partial t} = 0, \frac{\partial P^*(x, 0)}{\partial t} = 0, t = 0, x \in V \end{aligned} \quad (39)$$

### Definition of Laplace transform of a function $g(x, t)$ containing Caputo fractional derivative

$$\begin{aligned} L\{D_C^\alpha g(x, t)\} = \{s^\alpha \bar{g}(x, s)\} - \sum_{\zeta=0}^{n-1} g^\zeta(x, 0^+) s^{\alpha-1-\zeta}, n-1 < \alpha \leq n \\ \bar{g}(x, s) = L\{g(x, t)\} = \int_0^\infty g(x, t) e^{-st} dt, Res > 0 \end{aligned} \quad (40)$$

where,  $D_C^\alpha$  is the Caputo fractional derivative.

Now, we apply the relations (40) in the equations (34-39) and arranging we obtain the following set of equations and boundary conditions:

$$\bar{\sigma}_{ij}^* = 2\mu \bar{e}_{ij}^* + \delta_{ij} (\lambda_0 \bar{e}^* - \gamma_2 \bar{P}^*) - \delta_{ij} \gamma_1 L_1 \bar{\theta}^* \quad (41)$$

$$\bar{\sigma}_{ij,j}^* = \rho s^2 \bar{u}_i^* \quad (42)$$

$$(ks + k^*) \bar{\theta}_{,ii}^* = \gamma_1 T_0 s^2 \bar{e}^* + d T_0 s^2 \bar{P}^* + l_1 T_0 s^2 L_1 \bar{\theta}^* \quad (43)$$

$$D \bar{P}_{,ii}^* = \gamma_2 s \bar{e}^* + n s \bar{P}^* + d s L_1 \bar{\theta}^* \quad (44)$$

$$\bar{\theta}^*(x, s) = 0, \bar{u}_i^*(x, s) = 0, \bar{P}^*(x, s) = 0, x \in A \quad (45)$$

where, we take  $L_1 = \left(1 + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} s^\alpha\right) > 0$ .

Now,  $\int_V \bar{\sigma}_{ij}^* \bar{e}_{ij}^* dV = \int_V \bar{\sigma}_{ij}^* \bar{u}_{i,j}^* dV = \int_V (\bar{\sigma}_{ij}^* \bar{u}_{i,j}^*)_{,j} dV - \int_V \bar{\sigma}_{ij,j}^* \bar{u}_i^* dV = \int_A \bar{\sigma}_{ij}^* \bar{u}_i^* n_j dA - \int_V \bar{\sigma}_{ij,j}^* \bar{u}_i^* dV = - \int_V \bar{\sigma}_{ij,j}^* \bar{u}_i^* dV$  (using divergence theorem and relation (45))

$$\text{Or, } \int_V \bar{\sigma}_{ij}^* \bar{e}_{ij}^* dV + \int_V \bar{\sigma}_{ij,j}^* \bar{u}_i^* dV = 0. \quad (46)$$

Using (41) and (42) in the relation (46) and arranging we obtain the following relations

$$\int_V \left[ 2\mu \bar{e}_{ij}^* \bar{e}_{ij}^* + \lambda_0 \bar{e}^* \bar{e}^* - \gamma_2 \bar{e}^* \bar{P}^* - \gamma_1 L_1 \bar{e}^* \bar{\theta}^* + \rho s^2 \bar{u}_i^* \bar{u}_i^* \right] dV = 0. \quad (47)$$

Since,  $\bar{\theta}^* \bar{\theta}_{,ii}^* = (\bar{\theta}^* \bar{\theta}_{,i}^*)_{,i} - \bar{\theta}_{,i}^* \bar{\theta}_{,i}^*$  and  $\int_V \bar{\theta}^* \bar{\theta}_{,i,i}^* dV = \int_A \bar{\theta}^* \bar{\theta}_{,i}^* n_i dA = 0$ , (using (45))

$$\text{We have, } \int_V \bar{\theta}^* \left[ (ks + k^*) \bar{\theta}_{,ii}^* \right] dV = - (ks + k^*) \int_V \bar{\theta}_{,i}^* \bar{\theta}_{,i}^* dV. \quad (48)$$

$$\text{Similarly, } \int_V \bar{P}^* \left[ D\bar{P}^*_{,ii} \right] dV = -D \int_V \bar{P}^*_{,i} \bar{P}^*_{,i} dV. \tag{49}$$

Now, we use (43) in (48) and arranging to get the following relation

$$\gamma_1 L_1 \int_V \bar{\theta}^* \bar{e}^* dV = -dL_1 \int_V \bar{\theta}^* \bar{P}^* dV - l_1 L_1^2 \int_V \bar{\theta}^* \bar{\theta}^* dV - \frac{L_1 (ks + k^*)}{T_0 s^2} \int_V \bar{\theta}^*_{,i} \bar{\theta}^*_{,i} dV \tag{50}$$

Also, using (44) in (49) and arranging we obtain the following relation

$$\gamma_2 \int_V \bar{P}^* \bar{e}^* dV = -dL_1 \int_V \bar{\theta}^* \bar{P}^* dV - n \int_V \bar{P}^* \bar{P}^* dV - \frac{D}{s} \int_V \bar{P}^*_{,i} \bar{P}^*_{,i} dV. \tag{51}$$

We now arrange the relations (47), (50) and (51) and finally obtain the equation

$$\int_V \left[ 2\mu \bar{e}^*_{ij} \bar{e}^*_{ij} + \lambda_0 \bar{e}^* \bar{e}^* + \rho s^2 \bar{u}^*_i \bar{u}^*_i + l_1 L_1^2 \bar{\theta}^{*2} + 2dL_1 \bar{\theta}^* \bar{P}^* + n \bar{P}^{*2} + v_2 \bar{P}^*_{,i} \bar{P}^*_{,i} + v_1 \bar{\theta}^*_{,i} \bar{\theta}^*_{,i} \right] dV = 0, \tag{52}$$

where,  $v_1 = \frac{(ks+k^*)}{T_0 s^2}$ ,  $v_2 = \frac{D}{s}$ .

Under the condition (14) the quadratic form  $f = l_1 L_1^2 \bar{\theta}^{*2} + 2dL_1 \bar{\theta}^* \bar{P}^* + n \bar{P}^{*2}$  is positive [19]. Since the material parameters are positive and the integrated function in (52) is positive and sum of squares, so we conclude that

$$\bar{P}^* = \bar{\theta}^* = \bar{u}^*_i = \bar{e}^*_{ij} = \bar{\sigma}^* = 0. \tag{53}$$

Thus, the Laplace transformation of all the difference functions in the equation (33) are zeros and according to Learch's theorem the inverse Laplace transformation of each is unique, consequently,

$$u_i^{(1)} = u_i^{(2)}, \theta^{(1)} = \theta^{(2)}, P^{(1)} = P^{(2)}, \sigma_{ij}^{(1)} = \sigma_{ij}^{(2)}. \tag{54}$$

This proves the uniqueness of the theorem under the aforementioned assumptions.

## 5. Reciprocity Theorem

### 5.1 Statement of Reciprocity theorem

We consider, a homogeneous isotropic, perfectly diffusive elastic material occupies a regular region  $V$ , bounded by the surface  $A$  subject to the action of the body force  $F_i$ , surface traction  $p_i$ , heat source  $Q$ , heating of the surface to the temperature  $\vartheta$  and the chemical potential  $P_0$  under homogeneous initial conditions  $\theta(x,0) = u_i(x,0) = P(x,0) = \frac{\partial \theta(x,0)}{\partial t} = \frac{\partial u_i(x,0)}{\partial t} = \frac{\partial P(x,0)}{\partial t} = 0, t = 0, x \in V$  and boundary conditions  $\sigma_{ji} n_j = p_i(x,t), \theta(x,t) = \vartheta, t, P(x,t) = P_0(x,t), x \in A, t > 0$ , symbolically as causes,  $\Xi = \{F_i, p_i, Q, \vartheta, P_0\}$ , these causes produce in the body the displacement  $u_i$ , temperature increment  $\theta$  and chemical potential increment  $P$ , symbolically as  $R = \{u_i, \theta, P\}$ .

We assume that stress  $\sigma_{ij}$  and strain  $e_{ij}$  are continuous together with their first derivative whereas the displacement components  $u_i$ , temperature  $\theta$ , concentration  $C$  and chemical potential  $P$  are continuous together with their second derivative for  $x \in (V + A), t > 0$ .

Now we assume that there exists another system of causes and effects namely,

$\Xi' = \{F'_i, p'_i, Q', \vartheta', P'_0\}, R' = \{u'_i, \theta', P'\}$ , and this two system of causes and effects are connected by the relation

$$\begin{aligned} & T_0 \int_A \int_0^t \left[ p_i(x, t-\tau) \frac{\partial^2 u'_i(x,t)}{\partial \tau^2} - p'_i(x, t-\tau) \frac{\partial^2 u_i(x,t)}{\partial \tau^2} \right] d\tau dA \\ & + \rho T_0 \int_V \int_0^t \left[ F_i(x, t-\tau) \frac{\partial^2 u'_i(x,t)}{\partial \tau^2} - F'_i(x, t-\tau) \frac{\partial^2 u_i(x,t)}{\partial \tau^2} \right] d\tau dV \\ & - k^* l_1 \int_A \int_0^t \left[ \vartheta'(x, t-\tau) \theta_{,n}(x,t) - \vartheta(x, t-\tau) \theta'_{,n}(x,t) \right] d\tau dA \\ & + k l_1 \int_A \int_0^t \left[ \vartheta'(x, t-\tau) \frac{\partial \theta_{,n}(x,t)}{\partial \tau} - \vartheta(x, t-\tau) \frac{\partial \theta'_{,n}(x,t)}{\partial \tau} \right] d\tau dA \\ & + k l_1 \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \int_A \int_0^t \left[ \vartheta'(x, t-\tau) \frac{\partial^{\alpha+1} \theta_{,n}(x,t)}{\partial \tau^{\alpha+1}} - \vartheta(x, t-\tau) \frac{\partial^{\alpha+1} \theta'_{,n}(x,t)}{\partial \tau^{\alpha+1}} \right] d\tau dA \\ & + k^* l_1 \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \int_A \int_0^t \left[ \vartheta'(x, t-\tau) \frac{\partial^\alpha \theta_{,n}(x,t)}{\partial \tau^\alpha} - \vartheta(x, t-\tau) \frac{\partial^\alpha \theta'_{,n}(x,t)}{\partial \tau^\alpha} \right] d\tau dA \\ & + \rho l_1 \int_V \int_0^t \left[ \theta(x, t-\tau) \frac{\partial Q'(x,t)}{\partial \tau} - \theta'(x, t-\tau) \frac{\partial Q(x,t)}{\partial \tau} \right] d\tau dV \end{aligned}$$

$$\begin{aligned}
& + \rho l_1 \int_V \int_0^t \left[ \theta(x, t - \tau) \frac{\partial^{\alpha+1} Q'(x, t)}{\partial \tau^{\alpha+1}} - \theta'(x, t - \tau) \frac{\partial^{\alpha+1} Q(x, t)}{\partial \tau^{\alpha+1}} \right] d\tau dV \\
& = DT_0 \int_A \int_0^t \left[ P'_0(x, t - \tau) \frac{\partial P_{,n}(x, t)}{\partial \tau} - P_0(x, t - \tau) \frac{\partial P'_{,n}(x, t)}{\partial \tau} \right] d\tau dA.
\end{aligned}$$

## 5.2 Proof of the theorem

Stress components (7) and equation of motion (1) can be represented in Laplace transform domain for the first system as

$$\bar{\sigma}_{ij} = 2\mu\bar{e}_{ij} + \delta_{ij}(\lambda_0\bar{e} - \gamma_2\bar{P}) - \delta_{ij}\gamma_1 L_1\bar{\theta} \quad (55)$$

$$\bar{\sigma}_{ij,j} + \rho\bar{F}_i = \rho s^2\bar{u}_i \quad (56)$$

Similarly, for second system (55) and (56) can be represented as

$$\bar{\sigma}'_{ij} = 2\mu\bar{e}'_{ij} + \delta_{ij}(\lambda_0\bar{e}' - \gamma_2\bar{P}') - \delta_{ij}\gamma_1 L_1\bar{\theta}' \quad (57)$$

$$\bar{\sigma}'_{ij,j} + \rho\bar{F}'_i = \rho s^2\bar{u}'_i \quad (58)$$

From (55) and (57) one can be written

$$\int_V (\bar{\sigma}_{ij}\bar{e}'_{ij} - \bar{\sigma}'_{ij}\bar{e}_{ij}) dV = l_1\gamma_1 L_1 \int_V (\bar{\theta}'_{ij} - \bar{e}'\bar{\theta}) dV + \gamma_2 \int_V (\bar{e}\bar{P}'_{ij} - \bar{e}'\bar{P}) dV. \quad (59)$$

Now, by Gauss's divergence theorem, we write

$$\int_V \bar{\sigma}_{ij}\bar{e}'_{ij} dV = \int_V \bar{\sigma}_{ij}\bar{u}'_{i,j} dV = \int_A \bar{\sigma}_{ij}\bar{u}'_i n_j dA - \int_V \bar{\sigma}_{ij}\bar{u}'_i dV \quad (60)$$

Combining (56) and (60), we obtain the following integral equation

$$\int_V \bar{\sigma}_{ij}\bar{e}'_{ij} dV = \rho \int_V \bar{F}_i\bar{u}'_i dV + \int_A \bar{p}_i\bar{u}'_i dA - \rho s^2 \int_V \bar{\sigma}_{ij}\bar{u}_i\bar{u}'_i dV \quad (61)$$

$$\text{Similarly, } \int_V \bar{\sigma}'_{ij}\bar{e}_{ij} dV = \rho \int_V \bar{F}'_i\bar{u}_i dV + \int_A \bar{p}'_i\bar{u}_i dA - \rho s^2 \int_V \bar{\sigma}'_{ij}\bar{u}_i\bar{u}'_i dV \quad (62)$$

Now, using (61) and (62) in (59) and arranging one can get the following integral equation

$$\begin{aligned}
\rho \int_V (\bar{F}_i\bar{u}'_i - \bar{F}'_i\bar{u}_i) dV + \int_A (\bar{p}_i\bar{u}'_i - \bar{p}'_i\bar{u}_i) dA + \gamma_1 l_1 L_1 \int_V (\bar{\theta}_i\bar{e}'_i - \bar{\theta}'_i\bar{e}_i) dV \\
+ \gamma_2 \int_V (\bar{P}_i\bar{e}'_i - \bar{P}'_i\bar{e}_i) dV = 0.
\end{aligned} \quad (63)$$

First two terms of (63) contains only causes of a mechanical nature, namely, the mechanical forces and surface traction. So, we have to eliminate last two terms of (63) to establish reciprocity theorem.

The heat conduction equation (12), in Laplace transform domain, for both the systems takes the form

$$(ks + k^*)\bar{\theta}_{,ii} + \rho s\bar{Q} = \gamma_1 T_0 s^2\bar{e} + dT_0 s^2\bar{P} + l_1 T_0 s^2 L_1\bar{\theta} \quad (64)$$

$$(ks + k^*)\bar{\theta}'_{,ii} + \rho s\bar{Q}' = \gamma_1 T_0 s^2\bar{e}' + dT_0 s^2\bar{P}' + l_1 T_0 s^2 L_1\bar{\theta}' \quad (65)$$

Now, using (64) and (65) one can be written easily as

$$\begin{aligned}
(ks + k^*) \int_V (\bar{\theta}_{,ii}\bar{\theta}' - \bar{\theta}'_{,ii}\bar{\theta}) dV + \rho s \int_V (\bar{Q}\bar{\theta}' - \bar{Q}'\bar{\theta}) dV \\
= \gamma_1 T_0 s^2 \int_V (\bar{e}\bar{\theta}' - \bar{e}'\bar{\theta}) dV + dT_0 s^2 \int_V (\bar{P}\bar{\theta}' - \bar{P}'\bar{\theta}) dV.
\end{aligned} \quad (66)$$



Consider,  $\int_V \bar{\theta}_{,ii} \bar{\theta}' dV = \int_A \bar{\theta}_{,i} \bar{\theta}' n_i dA - \int_V \bar{\theta}_{,i} \bar{\theta}'_{,i} dV = \int_A \bar{\theta}_{,i} \bar{\theta}' dA - \int_V \bar{\theta}_{,i} \bar{\theta}'_{,i} dV$ .

Using the notation,  $\bar{\theta}_{,n} = \bar{\theta}_{,i} n_i$ , and boundary condition we can write the above equation for both the system as

$$\int_V \bar{\theta}_{,ii} \bar{\theta}' dV = \int_A \bar{\theta}'_{,n} \bar{\theta} dA - \int_V \bar{\theta}_{,i} \bar{\theta}'_{,i} dV. \tag{67}$$

$$\int_V \bar{\theta}'_{,ii} \bar{\theta} dV = \int_A \bar{\theta}'_{,n} \bar{\theta} dA - \int_V \bar{\theta}_{,i} \bar{\theta}'_{,i} dV. \tag{68}$$

Combining the relations (66), (67) and (68) and arranging one can get easily the following relation

$$\begin{aligned} & (ks + k^*) \int_V (\bar{\theta}_{,n} \bar{\theta}' - \bar{\theta}'_{,n} \bar{\theta}) dV + \rho s \int_V (\bar{Q} \bar{\theta}' - \bar{Q}' \bar{\theta}) dV \\ & - \gamma_1 T_0 s^2 \int_V (\bar{e} \bar{\theta}' - \bar{e}' \bar{\theta}) dV - dT_0 s^2 \int_V (\bar{P} \bar{\theta}' - \bar{P}' \bar{\theta}) dV = 0. \end{aligned} \tag{69}$$

First two terms of (69) contains thermal causes, namely, heat sources and heating of the surface A. So, in order to establish reciprocity theorem we have to eliminate last two terms of (69)

Now, eliminating the integral  $\int_V (\bar{e} \bar{\theta}' - \bar{e}' \bar{\theta}) dV$  from (63) and (69) we obtain the following form of reciprocity theorem

$$\begin{aligned} & \rho T_0 s^2 \int_V (\bar{F}_i \bar{u}'_i - \bar{F}'_i \bar{u}_i) dV + T_0 s^2 \int_A (\bar{p}_i \bar{u}'_i - \bar{p}'_i \bar{u}_i) dA \\ & + l_1 L_1 (ks + k^*) \int_V (\bar{\theta}_{,n} \bar{\theta}' - \bar{\theta}'_{,n} \bar{\theta}) dV - \rho s L_1 l_1 \int_V (\bar{Q} \bar{\theta}' - \bar{Q}' \bar{\theta}) dV \\ & = -\gamma_2 T_0 s^2 \int_V (\bar{P}_i \bar{e}'_i - \bar{P}'_i \bar{e}_i) dV - dT_0 s^2 L_1 l_1 \int_V (\bar{P} \bar{\theta}' - \bar{P}' \bar{\theta}) dV. \end{aligned} \tag{70}$$

The terms of left hand sides of (70) contains thermo-mechanical causes. So, we have to eliminate the two terms of right hand side of (70) to establish reciprocity theorem in composite form.

The diffusion equation (13), in Laplace transform domain, for both the systems takes the following form

$$D \bar{P}_{,ii} = \gamma_2 s \bar{e} + ns \bar{P} + ds L_1 \bar{\theta} \tag{71}$$

$$D \bar{P}'_{,ii} = \gamma_2 s \bar{e}' + ns \bar{P}' + ds L_1 \bar{\theta}' \tag{72}$$

Multiplying  $\bar{P}'$  with (71) and  $\bar{P}$  with (72) and subtracting and then integrating over volume V we obtain

$$D \int_V (\bar{P}' \bar{P}_{,ii} - \bar{P} \bar{P}'_{,ii}) dV = \gamma_2 s \int_V (\bar{P}' \bar{e} - \bar{P} \bar{e}') dV + ds L_1 l_1 \int_V (\bar{P}' \bar{\theta} - \bar{P} \bar{\theta}') dV \tag{73}$$

Consider,  $\int_V \bar{P}_{,ii} \bar{P}' dV = \int_A \bar{P}_{,i} \bar{P}' n_i dA - \int_V \bar{P}_{,i} \bar{P}'_{,i} dV$ , and using the notation,

$\bar{P}_{,n} = \bar{P}_{,i} n_i$  and boundary condition, the above equation can be represented in the following form for both system

$$\int_V \bar{P}_{,ii} \bar{P}' dV = \int_A \bar{P}_{,n} \bar{P}'_0 dA - \int_V \bar{P}_{,i} \bar{P}'_{,i} dV \tag{74}$$

Analogous of the above equation for second system can be written as

$$\int_V \bar{P}'_{,ii} \bar{P} dV = \int_A \bar{P}'_{,n} \bar{P}_0 dA - \int_V \bar{P}'_{,i} \bar{P}_{,i} dV. \tag{75}$$

Combining the equations (73), (74) and (75) and arranging, we get

$$\begin{aligned} & D \int_A (\bar{P}'_0 \bar{P}_{,n} - \bar{P}_0 \bar{P}'_{,n}) dA + \gamma_2 s \int_V (\bar{P} \bar{e} - \bar{P}' \bar{e}') dV \\ & + ds L_1 l_1 \int_V (\bar{P} \bar{\theta}' - \bar{P}' \bar{\theta}) dV = 0 \end{aligned} \tag{76}$$

The first term of (76) containing the diffusion causes, namely, chemical potential on the surface A. So, to obtain the final form of reciprocity theorem we have to eliminate the rest two terms of (76).

Upon elimination of integrals  $\int_V (\overline{P}\overline{\theta}' - \overline{P}'\overline{\theta}) dV$  and  $\int_V (\overline{e}\overline{\theta}' - \overline{e}'\overline{\theta}) dV$  from (70) and (76) and arranging, we obtain the following relation

$$\begin{aligned} & \rho T_0 s^2 \int_V (\overline{F}_i \overline{u}'_i - \overline{F}'_i \overline{u}_i) dV + T_0 s^2 \int_A (\overline{p}_i \overline{u}'_i - \overline{p}'_i \overline{u}_i) dA - DT_0 s \int_V (\overline{P}_{,n} \overline{P}'_0 - \overline{P}'_{,n} \overline{P}_0) dV \\ & = l_1 L_1 (ks + k^*) \int_V (\overline{\theta}_{,n} \overline{\theta}' - \overline{\theta}'_{,n} \overline{\theta}) dV + \rho s L_1 l_1 \int_V (\overline{Q}\overline{\theta}' - \overline{Q}'\overline{\theta}) dV. \end{aligned} \quad (77)$$

This is the required reciprocity theorem in composite form for both the system of causes and effects.

To invert the Laplace transform of (77), we use convolution theorem of the transform, namely,

$L^{-1} [\overline{f}(s) \overline{g}(s)] = \int_0^t f(\tau) g(t-\tau) d\tau = \int_0^t g(\tau) f(t-\tau) d\tau$  to obtain the theorem in space-time domain in integral form

$$\begin{aligned} & T_0 \int_A \int_0^t \left[ p_i(x, t-\tau) \frac{\partial^2 u'_i(x, t)}{\partial \tau^2} - p'_i(x, t-\tau) \frac{\partial^2 u_i(x, t)}{\partial \tau^2} \right] d\tau dA \\ & + \int_V \int_0^t \left[ F_i(x, t-\tau) \frac{\partial^2 u'_i(x, t)}{\partial \tau^2} - F'_i(x, t-\tau) \frac{\partial^2 u_i(x, t)}{\partial \tau^2} \right] d\tau dV \\ & - k^* l_1 \int_A \int_0^t [\vartheta'(x, t-\tau) \theta_{,n}(x, t) - \vartheta(x, t-\tau) \theta'_{,n}(x, t)] d\tau dA \\ & + k l_1 \int_A \int_0^t \left[ \vartheta'(x, t-\tau) \frac{\partial \theta_{,n}(x, t)}{\partial \tau} - \vartheta(x, t-\tau) \frac{\partial \theta'_{,n}(x, t)}{\partial \tau} \right] d\tau dA \\ & + k l_1 \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \int_A \int_0^t \left[ \vartheta'(x, t-\tau) \frac{\partial^{\alpha+1} \theta_{,n}(x, t)}{\partial \tau^{\alpha+1}} - \vartheta(x, t-\tau) \frac{\partial^{\alpha+1} \theta'_{,n}(x, t)}{\partial \tau^{\alpha+1}} \right] d\tau dA \\ & + k^* l_1 \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \int_A \int_0^t \left[ \vartheta'(x, t-\tau) \frac{\partial^\alpha \theta_{,n}(x, t)}{\partial \tau^\alpha} - \vartheta(x, t-\tau) \frac{\partial^\alpha \theta'_{,n}(x, t)}{\partial \tau^\alpha} \right] d\tau dA \\ & + \rho l_1 \int_V \int_0^t \left[ \theta(x, t-\tau) \frac{\partial Q'(x, t)}{\partial \tau} - \theta'(x, t-\tau) \frac{\partial Q(x, t)}{\partial \tau} \right] d\tau dV \\ & + \rho l_1 \int_V \int_0^t \left[ \theta(x, t-\tau) \frac{\partial^{\alpha+1} Q'(x, t)}{\partial \tau^{\alpha+1}} - \theta'(x, t-\tau) \frac{\partial^{\alpha+1} Q(x, t)}{\partial \tau^{\alpha+1}} \right] d\tau dV \\ & = DT_0 \int_A \int_0^t \left[ P'_0(x, t-\tau) \frac{\partial P_{,n}(x, t)}{\partial \tau} - P_0(x, t-\tau) \frac{\partial P'_{,n}(x, t)}{\partial \tau} \right] d\tau dA \end{aligned}$$

### 5.3 Limiting cases:

We now can introduce the following cases

(i) Taking  $\alpha = 1, \tau_1 = 1$ , the proposed model containing equations (7-13) becomes the familiar GNIII model with diffusion. Further, if we take  $k^* = 0$  the model reduces to coupled problem with diffusion.

(ii) If we consider the medium is infinite, assuming body forces and heat sources act in the bounded region only, then the surface integrals are absent and the reciprocity relation becomes

$$\begin{aligned} & T_0 \int_V \int_0^t \left[ F_i(x, t-\tau) \frac{\partial^2 u'_i(x, t)}{\partial \tau^2} - F'_i(x, t-\tau) \frac{\partial^2 u_i(x, t)}{\partial \tau^2} \right] d\tau dV \\ & + l_1 \int_V \int_0^t \left[ \theta(x, t-\tau) \frac{\partial Q'(x, t)}{\partial \tau} - \theta'(x, t-\tau) \frac{\partial Q(x, t)}{\partial \tau} \right] d\tau dV \\ & + l_1 \int_V \int_0^t \left[ \theta(x, t-\tau) \frac{\partial^{\alpha+1} Q'(x, t)}{\partial \tau^{\alpha+1}} - \theta'(x, t-\tau) \frac{\partial^{\alpha+1} Q(x, t)}{\partial \tau^{\alpha+1}} \right] d\tau dV = 0 \end{aligned} \quad (78)$$

(iii) Now we consider the general case where the system of causes are harmonic in time. Hence, we take

$$\begin{aligned} F_i(x, t) &= F_i^*(x) e^{i\omega t}, p_i(x, t) = p_i^*(x) e^{i\omega t}, Q(x, t) = Q^*(x) e^{i\omega t}, \\ \vartheta_i(x, t) &= \vartheta_i^*(x) e^{i\omega t}, P_0(x, t) = P_0^*(x) e^{i\omega t} \end{aligned} \quad (79)$$

Accordingly, the displacement, temperature and chemical potential will also harmonic in time, so

$$\begin{aligned} u_i(x, t) &= u_i^*(x, \omega) e^{i\omega t}, \theta(x, t) = \theta^*(x, \omega) e^{i\omega t}, \\ P(x, t) &= P^*(x, \omega) e^{i\omega t}, e(x, t) = e^*(x, \omega) e^{i\omega t} \end{aligned} \quad (80)$$

Now, applying the above expressions (83) and (84) in case of forced vibration problems, the reciprocity theorem in transformed domain becomes

$$\begin{aligned} & -\rho T_0 \omega^2 \int_V (\overline{F}_i^* \overline{u}'^*_i - \overline{F}'^*_i \overline{u}_i^*) dV - T_0 \omega^2 \int_A (\overline{p}_i^* \overline{u}'^*_i - \overline{p}'^*_i \overline{u}_i^*) dA \\ & - i\omega DT_0 \int_V (\overline{P}_{,n}^* \overline{P}'^*_0 - \overline{P}'^*_{,n} \overline{P}_0^*) dV = l_1 L_1 (i\omega k + k^*) \int_V (\overline{\theta}_{,n}^* \overline{\theta}'^* - \overline{\theta}'^*_{,n} \overline{\theta}^*) dV \\ & + i\omega \rho L_1 l_1 \int_V (\overline{Q}^* \overline{\theta}'^* - \overline{Q}'^* \overline{\theta}^*) dV. \end{aligned} \quad (81)$$

### 6. Variational principle

The principle of virtual work for the diffusive deformable body can be written in the form

$$\int_V (\rho F_i - \rho \ddot{u}_i) \delta u_i dV + \int_A p_i \delta u_i dA = \int_V \sigma_{ji} \delta e_{ij} dV \tag{82}$$

Where, V is the arbitrary volume bounded by closed and bounded surface A,  $F_i$  is the virtual work of the body force per unit mass,  $p_i = \sigma_{ji} n_j$  are the components of surface traction applied to the surface A, where,  $n_j$  are components of a normal vector to the surface A and  $\sigma_{ji} u_{i,j} = \sigma_{ji} \delta e_{ij}$  that is the symmetry of stress tensor.

Now, taking the advantage of the constitutive relation (7) and arranging we obtain

$$\int_V (\rho F_i - \rho \ddot{u}_i) \delta u_i dV + \int_A p_i \delta u_i dA = \delta W - \int_V [\gamma_2 P + \gamma_1 K_\alpha \theta] \delta e dV \tag{83}$$

Where,  $W = \frac{1}{2} \int_V (2\mu e_{ij} e_{ij} + \lambda_0 e^2) dV$  and  $K_\alpha \equiv 1 + \frac{\tau_1^\alpha}{\Gamma(\alpha+1)} \frac{\partial^\alpha}{\partial t^\alpha}$ .

The above equation represents the equality of virtual work done by external internal forces.

This is the first part of the variational principle.

Now for coupling deformation field with temperature and chemical potential, there arise necessarily of considering two additional relations characterizing the phenomena of thermal conductivity and of diffusion.

Now, we use a relation of vector **H** connected with entropy by

$$\rho S = -H_{i,i} \tag{84}$$

Eliminating entropy and the heat flux in absence of heat source term from (11) and (84), we get

$$-H_{i,i} = l_1 K_\alpha \theta + \gamma_1 e + dP \tag{85}$$

and from energy balance equation  $\rho Q - q_{i,i} = \rho T \dot{S}$  and (5), we obtain

$$k \dot{\theta}_{,ii} + k^* \theta_{,ii} = -T_0 \dot{H}_{i,i} \tag{86}$$

Now, equation (86) can be rearranged in the following form

$$\frac{T_0 \Phi^2}{\kappa} H_i + \theta_{,i} = 0 \tag{87}$$

where,  $\Phi^1 \equiv \frac{\partial^1}{\partial t^1}$ ,  $Ph_i^2 \equiv \frac{\partial^2}{\partial t^2}$ ,  $\kappa \equiv k \frac{\partial}{\partial t} + k^*$ .

Operating  $\delta H_i$  of equation (87) and then integrating over the volume V and using divergence theorem and finally arranging, we obtain

$$\int_A \theta n_i \delta H_i dA + \gamma_1 \int_V \theta \delta e dV + d \int_V \theta \delta P dV + \delta (\Gamma_P + \Gamma_D) = 0 \tag{88}$$

Here, we have introduced the thermal potential  $\Gamma_P$ , defined by

$$\delta \Gamma_P = l_1 \int_V \theta \delta [K_\alpha \theta] dV \tag{89}$$

and the thermal dissipation  $\Gamma_D$ , defined by

$$\delta \Gamma_D = T_0 \int_V \frac{\Phi^2}{\kappa} H_i \delta H_i dV. \tag{90}$$

This is the second part of the variational principle.

To compute the third part of the theorem, we introduce the vector function **G** such that

$$C = -G_{i,i}. \tag{91}$$

Now, from equations (4), (6), (9) and (91), we obtain after arrangement

$$\frac{\Phi^1}{D} G_i + P_{,i} = 0. \tag{92}$$

Operating  $\delta G_i$  of equation (92) and then integrating over the volume V and using divergence theorem and finally arranging, we obtain

$$\int_A P n_i \delta G_i dA + \gamma_2 \int_V e \delta P dV + d \int_V K_\alpha \theta \delta P dV - \delta (A + B) = 0 \tag{93}$$

Here, we introduce the diffusion potential A, by

$$A = n \int_V P^2 dV, \delta A = 2n \int_V P \delta P dV \tag{94}$$

and diffusion dissipation B, by

$$\delta B = \frac{\Phi^1}{D} \int_V G_i \delta G_i dV. \tag{95}$$

This is the third part of the variational principle.

Now, eliminate the integrals  $\int_V \theta \delta e dV$  and  $\int_V \theta \delta P dV$  among (83), (88) and (93) and arranging we get,

$$\begin{aligned} & \delta (W + A + B + K_\alpha \Gamma_P + K_\alpha \Gamma_D - \gamma_2 \int_V e \delta P dV \\ & = \int_V (\rho F_i - \rho \ddot{u}_i) \delta u_i dV + \int_A p_i \delta u_i dA + \int_A P n_i \delta G_i dA - \int_A K_\alpha \theta n_i \delta H_i dA. \end{aligned} \tag{96}$$

This is the final general form of variational principle of the theorem.

## 7. Conclusion

- (i) A new model on generalized fraction order thermo-elastic diffusion theory is established.
- (ii) This model is an extension of Green and Naghdi model excluding fractional parameter and obviously of classical coupled thermo-elasticity.
- (iii) The uniqueness theorem of the solution of an initial boundary value problem is proved using Laplace transform technique on the assumption of the positivity of thermo-elastic and diffusive parameter and the symmetry of stress tensor.
- (iv) The reciprocity theorem of the new model is derived and discussed different associated cases on it.
- (v) A convolution variational principle is also established for this new model.
- (vi) The model is valid only for isotropic elastic solids. This new model may be helpful for the researchers in material science as well as for those working on the development of a theory of hyperbolic thermo-elasticity with diffusion and fractional order theory.

## 8. Conflict of interest

None declared.

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