Int. J. Adv. Appl. Math. and Mech. 7(4) (2020) 64 – 71 (ISSN: 2347-2529)
Journal homepage: www.ijaamm.com

International Journal of Advances in Applied Mathematics and Mechanics

On Fibonacci lacunary statistical convergence of double sequences in intuitionistic fuzzy normed linear spaces

Research Article

Ömer KİŞİ *

Department of Mathematics, Faculty of Science, Bartın University, Bartın, Turkey

Received 23 April 2020; accepted (in revised version) 06 June 2020

Abstract: We investigate the concept of Fibonacci lacunary statistical convergence of double sequences in intuitionistic fuzzy normed linear spaces. We introduce Fibonacci lacunary statistically Cauchy double sequences and establish some inclusion relations. We also introduce here a new concept, that is, Fibonacci lacunary statistical completeness and show that every intuitionistic fuzzy normed linear space is Fibonacci lacunary statistically complete.

MSC: 11B39 • 41A10 • 41A25 • 41A36 • 40A30 • 40G15

Keywords: Lacunary sequence • double sequence • Fibonacci lacunary statistical convergence • intuitionistic fuzzy normed linear space

1. Introduction

The concept of statistical convergence for real number sequences was first originated by Fast [1]. Later, it was further investigated from sequence point of view and linked with summability theory by Fridy [2] and Šalát [3]. Theory of lacunary statistical convergence has become an important working area after the study of Fridy and Orhan [4].

The concept of 2-normed spaces was introduced and studied by S. Gähler [5]. This notion which is nothing but a two dimensional analogue of a normed space got the attention of a wider audience after the publication of a paper by Albert George, White Jr. [6] of USA in 1969 entitled 2-Banach spaces. In the same year Gähler [7] published another paper on this theme. A.H. Siddiqi delivered a series of lectures on this theme in various conferences in India and Iran. His joint paper with S. Gähler and S.C. Gupta of 1975 also provide valuable results related to the theme of this paper. Results up to 1977 were summarized in the survey paper by A.H. Siddiqi [8].

Fuzziness has revolutionized many areas such as mathematics, science, engineering, medicine. This concept was given by Zadeh [9]. The concept of fuzziness are using by many researchers for Cybernetics, Artificial Intelligence, Expert System and Fuzzy control, Pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, Weather forecasting.

Recently, the fuzzy logic became an important area of research in several branches of mathematics like metric and topological spaces, theory of functions etc. It attracted many researchers on sequence spaces and summability theory to introduce various types of sequence spaces and examine their different properties.

The notion of a fuzzy norm on a linear space was first originated by Katsaras [10]. Felbin [11] gave an alternative idea of a fuzzy norm whose concerned metric is of Kaleva and Seikkala [12] type.

* *E-mail address:* okisi@bartin.edu.tr

Intuitionistic fuzzy sets was examined by Atanassov [13] is appropriate for such a situation. The notion of intuitionistic fuzzy metric space has been introduced by Park [14]. Furthermore, the concept of intuitionistic fuzzy normed space is given by Saadati and Park [15]. A lot of improvement has been made in the area of intuitionistic fuzzy normed space after the studies of ([16], [17], [18], [19], [22], [20], [21], [22]).

Fibonacci sequence was initiated in the book Liber Abaci of Fibonacci which was written in 1202. However, the sequence is based on older history. The sequence had been described earlier as Virahanka numbers in Indian mathematics [23]. In Liber Abaci, the sequence starts with 1, nowadays the sequence begins either with $f_0 = 0$ or with $f_1 = 1$.

The numbers in the bottom row are called Fibonacci numbers, and the number sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

is the Fibonacci sequence [24].

The Fibonacci numbers are a sequence of numbers (f_n) for $n = 1, 2, \ldots$ defined by the linear recurrence equation $f_n = f_{n+1} - f_{n-2}$, $n \geq 2$. From this definition, it means that the first two numbers in Fibonacci sequence are either 1 and 1 (or 0 and 0) depending on the chosen starting point of the sequence and all subsequent numbers are the sum of the previous two.

Some properties of Fibonacci numbers are given by

$$\lim_{n\to\infty} \frac{f_{n+1}}{f_n} = \frac{1+\sqrt{5}}{2} = \alpha, \text{ (Golden ratio)}$$
$$\sum_{k=0}^{n} f_k = f_{n+2} - 1 \ (n \in \mathbb{N}),$$
$$\sum_k \frac{1}{f_k} \text{ converges,}$$
$$f_{n-1} f_{n+1} - f_n^2 = (-1)^{n+1}, \ n \geq 1. \text{ (Cassini formula)}$$

The Fibonacci sequence was firstly used in the theory of sequence spaces by Kara and Başarır [25]. Afterward, Kara [26] defined the Fibonacci difference matrix $\widehat{F}$ by using the Fibonacci sequence (f_n) for $n \in \{1, 2, 3, \ldots\}$ and introduced the new sequence spaces related to the matrix domain of $\widehat{F}$.

Following [25] and [26], high quality papers have been produced on the Fibonacci matrix by many mathematicians ([27], [28], [29], [30], [31], [32], [33], [34], [35], [36], [37], [38]).

Kirişçi and Karaisa [39] defined Fibonacci type statistical convergence and investigated some fundamental properties. Afterward, Kirişçi [40] examined Fibonacci statistical convergence on intuitionistic fuzzy normed spaces. Recently, Kişi [41] investigated Fibonacci lacunary statistical convergence on intuitionistic fuzzy normed spaces.

In this paper, we have given a new definition of the notion of Fibonacci lacunary statistical convergence of double sequences in intuitionistic fuzzy normed linear spaces which is different from the one defined in [17] and other related works. This definition removes any ambiguity that could arise from previous definitions. We have developed all our results based on this new definition.

The aim of the present paper is to introduce and investigate the Fibonacci lacunary statistical convergence of double sequences and Fibonacci lacunary statistically Cauchy double sequences on intuitionistic fuzzy normed linear spaces and obtain some important results on them.

Definition 1.1.
([42]) A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

(i) $*$ is commutative and associative,
(ii) $*$ is continuous,
(iii) $a * 1 = a$, for all $a \in [0,1]$,
(iv) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Definition 1.2.
([42]) A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\Diamond$ satisfies the following conditions:

(i) $\Diamond$ is commutative and associative,
(ii) $\Diamond$ is continuous,
(iii) $a \Diamond 0 = a$ for all $a \in [0,1]$,
(iv) $a \Diamond b \leq c \Diamond d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0,1]$.

Using the continuous *t-norm* and t-*conorm*, Saadati and Park [15] have introduced the concept of intuitionistic fuzzy normed space as follows:

Definition 1.3.
([15]) The five-tuple $(\mathscr{X},\phi,\omega,*,\Diamond)$ is said to be intuitionistic fuzzy normed linear space or in short IFNS if $\mathscr{X}$ is a vector space, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and ϕ,ω are fuzzy sets on $\mathscr{X}\times(0,\infty)$ satisfying the following conditions: For every $x,y\in\mathscr{X}$ and $s,t>0$:

(i) $\phi(x,t)+\omega(x,t)\leq 1$,
(ii) $\phi(x,t)>0$,
(iii) $\phi(x,t)=1$ if and only if $x=0$,
(iv) $\phi(cx,t)=\phi\left(x,\frac{t}{|c|}\right)$ if $c\neq 0$,
(v) $\phi(x,t)*\phi(y,s)\leq\phi(x+y,t+s)$,
(vi) $\phi(x,.):(0,\infty)\to[0,1]$ is continuous in t;
(vii) $\lim_{t\to\infty}\phi(x,t)=1$ and $\lim_{t\to 0}\phi(x,t)=0$,
$(viii)$ $\omega(x,t)<1$,
(ix) $\omega(x,t)=0$ if and only if $x=0$,
(x) $\omega(cx,t)=\omega\left(x,\frac{t}{|c|}\right)$ if $c\neq 0$,
(xi) $\omega(x,t)\Diamond\omega(y,s)\geq\omega(x+y,t+s)$,
(xii) $\omega(x,.):(0,\infty)\to[0,1]$ is continuous in t;
$(xiii)$ $\lim_{t\to\infty}\omega(x,t)=0$ and $\lim_{t\to 0}\omega(x,t)=1$.

Definition 1.4.
([15]) Let $(\mathscr{X},\phi,\omega,*,\Diamond)$ be an intuitionistic fuzzy normed linear space. A sequence $x=(x_k)$ is said to convergent to $\xi\in\mathscr{X}$ with respect to the intuitionistic fuzzy norm (ϕ,ω) if for every $\varepsilon>0$ and $t>0$, there exists a positive integer n_0 such that $\phi(x_k-\xi,t)>1-\varepsilon$ and $\omega(x_k-\xi,t)<\varepsilon$ for all $k\geq n_0$. In this case, we write $(\phi,\omega)-\lim x=\xi$ or $x_k\overset{(\phi,\omega)}{\to}\xi$ as $k\to\infty$.

Definition 1.5.
([15]) A sequence $x=(x_k)$ is said to be Cauchy with respect to the intuitionistic fuzzy norm (ϕ,ω) if for every $\varepsilon>0$ and $t>0$, there exists a positive integer m_0 such that $\phi(x_k-x_l,t)>1-\varepsilon$ and $\omega(x_k-x_l,t)<\varepsilon$ for all $k,l\geq m_0$.

Definition 1.6.
([40]) Take an IFNS $(\mathscr{X},\phi,\omega,*,\Diamond)$. A sequence (x_k) is said to be Fibonacci statistical convergence with respect to IFN (ϕ,ω), if there is a number $\xi\in\mathscr{X}$ such that for every $\varepsilon>0$ and $t>0$, the set

$$K_\varepsilon(\widehat{F}):=\left\{k\leq n:\phi\left(\widehat{F}x_k-\xi,t\right)\leq 1-\varepsilon \text{ or } \omega\left(\widehat{F}x_k-\xi,t\right)\geq\varepsilon\right\}$$

has natural density zero, i.e., $d(K_\varepsilon(\widehat{F}))=0$. That is,

$$\lim_{n\to\infty}\frac{1}{n}\left|\left\{k\leq n:\phi\left(\widehat{F}x_k-\xi,t\right)\leq 1-\varepsilon \text{ or } \omega\left(\widehat{F}x_k-\xi,t\right)\geq\varepsilon\right\}\right|=0.$$

In this case, we write $d(\widehat{F})_{IFN}-\lim x_k=\xi$ or $x_k\to\xi(S(\widehat{F})_{IFN})$.

2. MAIN RESULTS

In this section, we examine the concept of Fibonacci lacunary statistically convergent double sequences in intuitionistic fuzzy normed space.

Definition 2.1.
Let $(\mathscr{X},\phi,\omega,*,\Diamond)$ be an intuitionistic fuzzy normed linear space and θ be a lacunary sequence. Then, a sequence $x=(x_{kl})$ is said to be Fibonacci lacunary statistically convergent to $\xi\in\mathscr{X}$ with regards to the intuitionistic fuzzy norm (ϕ,ω), if for every $\varepsilon>0$ and $p>0$,

$$\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi,p\right)\leq 1-\varepsilon \text{ or } \omega\left(\widehat{F}x_{kl}-\xi,p\right)\geq\varepsilon\right\}\right)=0, \tag{1}$$

or equivalently

$$\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi,p\right)>1-\varepsilon \text{ and } \omega\left(\widehat{F}x_{kl}-\xi,p\right)<\varepsilon\right\}\right)=1. \tag{2}$$

In this case, we write $S_\theta^{(\phi,\omega)}(\widehat{F})-\lim x=\xi$ or $x_{kl}\overset{(\phi,\omega)}{\rightarrow}\xi(S_\theta(\widehat{F}))$, where ξ is said to be $S_\theta^{(\phi,\omega)}(\widehat{F})-\lim x$ and we denote the set of all Fibonacci S_θ-convergent sequences with regards to intuitionistic fuzzy norm (ϕ,ω) by $S_\theta^{(\phi,\omega)}(\widehat{F})$.

By using (1) and (2), we easily get the following lemma.

Lemma 2.1.
Let $(\mathscr{X},\phi,\omega,*,\Diamond)$ be an intuitionistic fuzzy normed linear space and θ be a lacunary sequence. For every $\varepsilon>0$ and $p>0$, the following statements are equivalent:

(a) $S_\theta^{(\phi,\omega)}(\widehat{F})-\lim x=\xi$;

(b)

$$\begin{aligned}&\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi,p\right)\le 1-\varepsilon\right\}\right)\\&\quad=\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\omega\left(\widehat{F}x_{kl}-\xi,p\right)\ge \varepsilon\right\}\right)=0;\end{aligned}$$

(c) $\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi,p\right)>1-\varepsilon \text{ and } \omega\left(\widehat{F}x_{kl}-\xi,p\right)<\varepsilon\right\}\right)=1$;

(d)

$$\begin{aligned}&\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi,p\right)> 1-\varepsilon\right\}\right)\\&\quad=\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\omega\left(\widehat{F}x_{kl}-\xi,p\right)< \varepsilon\right\}\right)=1;\end{aligned}$$

(e) $S_\theta(\widehat{F})-\lim\phi\left(\widehat{F}x_{kl}-\xi,p\right)=1$ *and* $S_\theta(\widehat{F})-\lim\omega\left(\widehat{F}x_{kl}-\xi,p\right)=0$.

Theorem 2.1.
Let $(\mathscr{X},\phi,\omega,*,\Diamond)$ be an intuitionistic fuzzy normed linear space and θ be a lacunary sequence. If a sequence $x=(x_{kl})$ is Fibonacci lacunary statistically convergent with regards to the intuitionistic fuzzy norm (ϕ,ω), then $S_\theta^{(\phi,\omega)}(\widehat{F})$-limit is unique.

Proof. Assume that there exist two distinct elements $\xi_1,\xi_2\in X$ such that $S_\theta^{(\phi,\omega)}(\widehat{F})-\lim x_{kl}=\xi_1$ and $S_\theta^{(\phi,\omega)}(\widehat{F})-\lim x_{kl}=\xi_2$. Given $\varepsilon>0$, choose $\gamma>0$ such that $(1-\gamma)*(1-\gamma)>1-\varepsilon$ and $\gamma\Diamond\gamma<\varepsilon$. Hence, for any $p>0$, define the following sets as:

$$\mathscr{K}_{\phi,1}(\gamma,p)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi_1,\tfrac{p}{2}\right)\le 1-\gamma\right\},$$

$$\mathscr{K}_{\phi,2}(\gamma,p)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi_2,\tfrac{p}{2}\right)\le 1-\gamma\right\},$$

$$\mathscr{K}_{\omega,1}(\gamma,p)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\omega\left(\widehat{F}x_{kl}-\xi_1,\tfrac{p}{2}\right)\ge \gamma\right\},$$

$$\mathscr{K}_{\omega,2}(\gamma,p)=\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\omega\left(\widehat{F}x_{kl}-\xi_2,\tfrac{p}{2}\right)\ge \gamma\right\}.$$

Since $S_\theta^{(\phi,\omega)}(\widehat{F})-\lim x_{kl}=\xi_1$, we have by Lemma 1

$$\delta_\theta\left(\mathscr{K}_{\phi,1}(\gamma,p)\right)=\delta_\theta\left(\mathscr{K}_{\omega,1}(\gamma,p)\right)=0 \text{ for all } p>0.$$

Futhermore, using $S_\theta^{(\phi,\omega)}(\widehat{F})-\lim x_{kl}=\xi_2$, we get

$$\delta_\theta\left(\mathscr{K}_{\phi,2}(\gamma,p)\right)=\delta_\theta\left(\mathscr{K}_{\omega,2}(\gamma,p)\right)=0 \text{ for all } p>0.$$

Now let $\mathscr{K}_{\phi,\omega}(\gamma,p)=\left(\mathscr{K}_{\phi,1}(\gamma,p)\cup\mathscr{K}_{\phi,2}(\gamma,p)\right)\cap\left(\mathscr{K}_{\omega,1}(\gamma,p)\cup\mathscr{K}_{\omega,2}(\gamma,p)\right)$.

Then, observe that $\delta_\theta\left(\mathscr{K}_{\phi,\omega}(\gamma,p)\right)=0$ which implies $\delta_\theta\left(\mathbb{N}\times\mathbb{N}\setminus\mathscr{K}_{\phi,\omega}(\gamma,p)\right)=1$.

If $(k,l)\in\mathbb{N}\times\mathbb{N}\setminus\mathscr{K}_{\phi,\omega}(\gamma,p)$, then we have two possible cases.

Case (i) $(k,l)\in\mathbb{N}\times\mathbb{N}\setminus\left(\mathscr{K}_{\phi,1}(\gamma,p)\cup\mathscr{K}_{\phi,2}(\gamma,p)\right)$ and

case (ii) $(k,l)\in\mathbb{N}\times\mathbb{N}\setminus\left(\mathscr{K}_{\omega,1}(\gamma,p)\cup\mathscr{K}_{\omega,2}(\gamma,p)\right)$.

We first consider that $(k,l)\in\mathbb{N}\times\mathbb{N}\setminus\left(\mathscr{K}_{\phi,1}(\gamma,p)\cup\mathscr{K}_{\phi,2}(\gamma,p)\right)$. Then, we have

$$\phi\left(\xi_1-\xi_2,p\right)\ge\phi\left(\widehat{F}x_{kl}-\xi_1,\frac{p}{2}\right)*\phi\left(\widehat{F}x_{kl}-\xi_2,\frac{p}{2}\right)>(1-\gamma)*(1-\gamma)$$

Since $(1-\gamma)*(1-\gamma)>1-\varepsilon$, it follows that $\phi(\xi_1-\xi_2,p)>1-\varepsilon$. Since $\varepsilon>0$ is arbitrary, we get $\phi(\xi_1-\xi_2,p)=1$ for all $p>0$, which yields $\xi_1=\xi_2$.

On the other hand, if $(k,l)\in\mathbb{N}\times\mathbb{N}\setminus\left(\mathscr{K}_{\omega,1}(\gamma,p)\cup\mathscr{K}_{\omega,2}(\gamma,p)\right)$, then we may write

$$\omega\left(\xi_1-\xi_2,p\right)<\omega\left(\widehat{F}x_{kl}-\xi_1,\frac{p}{2}\right)\Diamond\omega\left(\widehat{F}x_{kl}-\xi_2,\frac{p}{2}\right)<\gamma\Diamond\gamma.$$

Now using the fact that $\gamma\Diamond\gamma<\varepsilon$, we see that $\omega(\xi_1-\xi_2,p)<\varepsilon$. Since arbitrary $\varepsilon>0$, we get $\omega(\xi_1-\xi_2,p)=0$ for all $p>0$. This occurs that $\xi_1=\xi_2$. So, we conclude that $S_\theta^{(\phi,\omega)}(\widehat{F})$-limit is unique. □

Definition 2.2.
A sequence $x = (x_{kl})$ is said to Fibonacci convergent to $\xi \in \mathscr{X}$ with respect to the intuitionistic fuzzy norm (ϕ, ω) if for every $\varepsilon > 0$ and $t > 0$, there exists a positive integer k_0, l_0 such that $\phi\left(\widehat{F}x_{kl} - \xi, t\right) > 1 - \varepsilon$ and $\omega\left(\widehat{F}x_{kl} - \xi, t\right) < \varepsilon$ for all $k \geq k_0$, $l \geq l_0$. In this case, we write $(\phi, \omega)_{(\widehat{F})} - \lim x_{kl} = \xi$ or $x_{kl} \overset{(\phi,\omega)}{\rightarrow} \xi(\widehat{F})$ as $k, l \to \infty$.

Theorem 2.2.
Let $(\mathscr{X}, \phi, \omega, *, \Diamond)$ be an intuitionistic fuzzy normed linear space and θ be a lacunary sequence. If $(\phi, \omega)_{(\widehat{F})} - \lim x = \xi$ then $S_\theta^{(\phi,\omega)}(\widehat{F}) - \lim x = \xi$.

Proof. Let $(\phi, \omega)_{(\widehat{F})} - \lim x = \xi$. Then, for every $\varepsilon > 0$ and $p > 0$, then there is a number $k_0, l_0 \in \mathbb{N}$ such that

$$\phi\left(\widehat{F}x_{kl} - \xi, p\right) > 1 - \varepsilon \text{ and } \omega\left(\widehat{F}x_{kl} - \xi, p\right) < \varepsilon$$

for all $k \geq k_0$, $l \geq l_0$ Hence, the set

$$\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi\left(\widehat{F}x_{kl} - \xi, p\right) \leq 1 - \varepsilon \text{ or } \omega\left(\widehat{F}x_{kl} - \xi, p\right) \geq \varepsilon\right\}$$

has finite number of terms. Since every finite subset of $\mathbb{N} \times \mathbb{N}$ has density zero and hence

$$\delta_\theta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi\left(\widehat{F}x_{kl} - \xi, p\right) \leq 1 - \varepsilon \text{ or } \omega\left(\widehat{F}x_{kl} - \xi, p\right) \geq \varepsilon\right\}\right) = 0,$$

that is, $S_\theta^{(\phi,\omega)}(\widehat{F}) - \lim x = \xi$. □

Theorem 2.3.
Let $(\mathscr{X}, \phi, \omega, *, \Diamond)$ be an intuitionistic fuzzy normed linear space and θ be a lacunary sequence. Then, for any lacunary sequence θ, $S_\theta^{(\phi,\omega)}(\widehat{F}) - \lim x = \xi$ iff there exists a subset $K = \{(k,l)\} \subseteq \mathbb{N} \times \mathbb{N}$, $k, l = 1, 2, \ldots$, such that $\delta_\theta(K) = 1$ and $(\phi, \omega)_{(\widehat{F})} - \lim_{k,l\to\infty} x_{kl} = \xi$.

Proof. Necessity. Assume that $S_\theta^{(\phi,\omega)}(\widehat{F}) - \lim x = \xi$. Let for any $p > 0$ and $s = 1, 2, \ldots$

$$T_{(\phi,\omega)}(s,p) = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi\left(\widehat{F}x_{kl} - \xi, p\right) > 1 - \frac{1}{s} \text{ and } \omega\left(\widehat{F}x_{kl} - \xi, p\right) < \frac{1}{s}\right\}$$

and

$$R_{(\phi,\omega)}(s,p) = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi\left(\widehat{F}x_{kl} - \xi, p\right) \leq 1 - \frac{1}{s} \text{ or } \omega\left(\widehat{F}x_{kl} - \xi, p\right) \geq \frac{1}{s}\right\}.$$

Then $\delta_\theta\left(R_{(\phi,\omega)}(s,p)\right) = 0$ since $S_\theta^{(\phi,\omega)}(\widehat{F}) - \lim x = \xi$. Also

$$T_{(\phi,\omega)}(s,p) \supset T_{(\phi,\omega)}(s+1,p) \tag{3}$$

and

$$\delta_\theta\left(T_{(\phi,\omega)}(s,p)\right) = 1 \tag{4}$$

for $p > 0$ and $s = 1, 2, \ldots$.

Now we have to show that for $(k,l) \in T_{(\phi,\omega)}(s,p)$, $x_{kl} \overset{(\phi,\omega)}{\rightarrow} \xi(\widehat{F})$. Suppose that for some $(k,l) \in T_{(\phi,\omega)}(s,p)$, $x_{kl} \overset{(\phi,\omega)}{\nrightarrow} \xi(\widehat{F})$. Therefore, there is $\sigma > 0$ and a positive integer k_0, l_0 such that

$$\phi\left(\widehat{F}x_{kl} - \xi, p\right) \leq 1 - \sigma \text{ or } \omega\left(\widehat{F}x_{kl} - \xi, p\right) \geq \sigma$$

for all $k \geq k_0$, $l \geq l_0$. Let

$$\phi\left(\widehat{F}x_{kl} - \xi, p\right) > 1 - \sigma \text{ and } \omega\left(\widehat{F}x_{kl} - \xi, p\right) < \sigma$$

for all $k \geq k_0$, $l \geq l_0$. Then, we get

$$\delta_\theta\left(\left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi\left(\widehat{F}x_{kl} - \xi, p\right) > 1 - \sigma \text{ and } \omega\left(\widehat{F}x_{kl} - \xi, p\right) < \sigma\right\}\right) = 0.$$

Since $\sigma > \frac{1}{s}$, we have $\delta_\theta\left(T_{(\phi,\omega)}(s,p)\right) = 0$, which contradicts (4). Therefore $x_{kl} \overset{(\phi,\omega)}{\rightarrow} \xi(\widehat{F})$.

Sufficiency. Suppose that there exists a subset $K = \{(k,l)\} \subseteq \mathbb{N} \times \mathbb{N}$ such that $\delta_\theta(K) = 1$ and $(\phi, \omega)_{(\widehat{F})} - \lim_{n\to\infty} x_{kl} = \xi$, i.e. there exists $N \in \mathbb{N}$ such that for every $\sigma > 0$ and $t > 0$

$$\phi\left(\widehat{F}x_{kl} - \xi, p\right) > 1 - \sigma \text{ and } \omega\left(\widehat{F}x_{kl} - \xi, p\right) < \sigma.$$

Now

$$\begin{aligned} R_{(\phi,\omega)}(\sigma,p) &:= \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \phi\left(\widehat{F}x_{kl} - \xi, p\right) \leq 1 - \sigma \text{ or } \omega\left(\widehat{F}x_{kl} - \xi, p\right) \geq \sigma\right\} \\ &\subseteq \mathbb{N} \times \mathbb{N} - \left\{\left(k_{N+1}, l_{N+1}\right), \left(k_{N+2}, l_{N+2}\right), \ldots\right\}. \end{aligned}$$

Therefore $\delta_\theta\left(R_{(\phi,\omega)}(\sigma,p)\right) \leq 1 - 1 = 0$. Hence $S_\theta^{(\phi,\omega)}(\widehat{F}) - \lim x = \xi$. This completes the proof of the theorem. □

Now, we define Fibonacci lacunary statistically Cauchy sequences with respect to an intuitionistic fuzzy normed space and introduce a new concept of Fibonacci lacunary statistical completeness.

Definition 2.3.
Let $(\mathscr{X},\phi,\omega,*,\diamondsuit)$ be an intuitionistic fuzzy normed linear space and θ be a lacunary sequence. Then, a sequence $x=(x_{kl})$ is said to be Fibonacci lacunary statistically Cauchy (or $\widehat{F}S_\theta$-Cauchy) with respect to the intuitionistic fuzzy norm (ϕ,ω) if for every $\varepsilon>0$ and $p>0$, there exists $N=N(\varepsilon)$ and $M=M(\varepsilon)$ such that

$$\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\leq 1-\varepsilon \text{ or } \omega\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\geq\varepsilon\right\}\right)=0.$$

Theorem 2.4.
Let $(\mathscr{X},\phi,\omega,*,\diamondsuit)$ be an intuitionistic fuzzy normed linear space and θ be a lacunary sequence. A sequence $x=(x_{kl})$ is Fibonacci lacunary statistically convergent if and only if it is Fibonacci lacunary statistically Cauchy with respect to the intuitionistic fuzzy norm (ϕ,ω).

Proof. Let $x_{kl}\to\xi(S_\theta^{(\phi,\omega)}(\widehat{F}))$. Then, for any $p>0$, we have

$$\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)\leq 1-\varepsilon \text{ or } \omega\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)\geq\varepsilon\right\}\right)=0.$$

In particular, for $k=N$, $l=M$

$$\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{NM}-\xi,\frac{p}{2}\right)\leq 1-\varepsilon \text{ or } \omega\left(\widehat{F}x_{NM}-\xi,\frac{p}{2}\right)\geq\varepsilon\right\}\right)=0.$$

Since

$$\begin{aligned}\phi\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right) &= \phi\left(\widehat{F}x_{kl}-\xi-\widehat{F}x_{NM}+\xi,\tfrac{p}{2}+\tfrac{p}{2}\right)\\ &\geq \phi\left(\widehat{F}x_{kl}-\xi,\tfrac{p}{2}\right)*\phi\left(\widehat{F}x_{NM}-\xi,\tfrac{p}{2}\right)\end{aligned}$$

and since

$$\omega\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\leq\omega\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)\diamondsuit\omega\left(\widehat{F}x_{NM}-\xi,\frac{p}{2}\right),$$

we have

$$\delta_\theta\left(\left\{(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\leq 1-\varepsilon \text{ or } \omega\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\geq\varepsilon\right\}\right)=0,$$

that is, x is Fibonacci lacunary statistically Cauchy with respect to the intuitionistic fuzzy norm (ϕ,ω).

Conversely, let x be Fibonacci lacunary statistically Cauchy but not Fibonacci lacunary statistically convergent with respect to the intuitionistic fuzzy norm (ϕ,ω). Then, there exists N, M such that

$$\delta_\theta\left(A\left(\varepsilon,p\right)\right)=0, \tag{5}$$

$$\delta_\theta\left(B\left(\varepsilon,p\right)\right)=0, \text{ i.e. } \delta_\theta\left(B^C\left(\varepsilon,p\right)\right)=1; \tag{6}$$

where

$$A\left(\varepsilon,p\right)=\left\{\begin{array}{c}(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\leq 1-\varepsilon\\ \text{or } \omega\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\geq\varepsilon\end{array}\right\},$$

$$B\left(\varepsilon,p\right)=\left\{\begin{array}{c}(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)>\frac{1-\varepsilon}{2}\\ \text{or } \omega\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)<\frac{\varepsilon}{2}\end{array}\right\}.$$

Since

$$\phi\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\geq 2\phi\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)>1-\varepsilon,$$

and

$$\omega\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)\leq 2\omega\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)<\varepsilon,$$

if $\phi\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)>\frac{1-\varepsilon}{2}$ and $\omega\left(\widehat{F}x_{kl}-\xi,\frac{p}{2}\right)<\frac{\varepsilon}{2}$. Therefore

$$\delta_\theta\left(\left\{\begin{array}{c}(k,l)\in\mathbb{N}\times\mathbb{N}:\phi\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)>1-\varepsilon\\ \text{or } \omega\left(\widehat{F}x_{kl}-\widehat{F}x_{NM},p\right)<\varepsilon\end{array}\right\}\right)=0,$$

that is, $\delta_\theta\left(A\left(\varepsilon,p\right)\right)=1$, which contradicts (5), since x was Fibonacci lacunary statistically Cauchy with respect to the intuitionistic fuzzy norm (ϕ,ω). Hence x must be Fibonacci lacunary statistically convergent with respect to the intuitionistic fuzzy norm (ϕ,ω). □

Definition 2.4.
An intuitionistic fuzzy normed linear space $(\mathscr{X}, \phi, \omega, *, \Diamond)$ is said to be $\widehat{F}S_\theta$-complete if every $\widehat{F}S_\theta$-Cauchy sequence with respect to the intuitionistic fuzzy norm (ϕ, ω) is $\widehat{F}S_\theta$-convergent with respect to the intuitionistic fuzzy norm (ϕ, ω).

Theorem 2.5.
Let θ be a lacunary sequence. Then every intuitionistic fuzzy normed space $(\mathscr{X}, \phi, \omega, *, \Diamond)$ is $\widehat{F}S_\theta$-complete.

Proof. Let $x = (x_{kl})$ $\widehat{F}S_\theta$-Cauchy but not $\widehat{F}S_\theta$-convergent with respect to the intuitionistic fuzzy norm (ϕ, ω). For given $\varepsilon > 0$, choose $\gamma > 0$ such that $(1-\gamma) * (1-\gamma) > 1-\varepsilon$ and $\gamma \Diamond \gamma < \varepsilon$. Now

$$\phi\left(\widehat{F}x_{kl} - \widehat{F}x_{NM}, p\right) \geq \phi\left(\widehat{F}x_{kl} - \xi, \frac{p}{2}\right) * \phi\left(\widehat{F}x_{NM} - \xi, \frac{p}{2}\right) > (1-\varepsilon) * (1-\varepsilon) > 1-\gamma$$

and

$$\omega\left(\widehat{F}x_{kl} - \widehat{F}x_{NM}, p\right) \leq \omega\left(\widehat{F}x_{kl} - \xi, \frac{p}{2}\right) \Diamond \omega\left(\widehat{F}x_{NM} - \xi, \frac{p}{2}\right) < \varepsilon \Diamond \varepsilon < \gamma,$$

since x is not $\widehat{F}S_\theta^{(\phi,\omega)}$-convergent. Therefore, $\delta_\theta\left(B^c(\varepsilon, p)\right) = 0$, where

$$B(\varepsilon, p) = \left\{(k,l) \in \mathbb{N} \times \mathbb{N} : \omega_{x_{kl} - x_{NM}} \leq 1 - r\right\}$$

and so $\delta_\theta\left(B(\varepsilon, p)\right) = 1$, which is a contradiction, since x was $\widehat{F}S_\theta$-Cauchy with respect to the intuitionistic fuzzy norm (ϕ, ω). So, x must be $\widehat{F}S_\theta$-convergent with respect to the intuitionistic fuzzy norm (ϕ, ω). Hence, every intuitionistic fuzzy normed space is $\widehat{F}S_\theta$-complete. □

References

[1] H. Fast, Sur la convergence statistique, Colloq. Math. 2 (1951) 241-244.
[2] J.A. Fridy, On statistical convergence, Analysis, 5 (1985) 301-313.
[3] T. Šalát, On statistical convergence of real numbers, Math. Slovaca, 30 (1980) 139-150.
[4] J.A. Fridy, C. Orhan, Lacunary statistical convergence, Pacific. J. Math. 160 (1993) 43-51.
[5] S. Gähler, Linear-2-normierte Raume, Math. Nachr. 28 (1965) 1-43.
[6] A. White, Jr. George, 2-Banach spaces, Math. Nachr., 42 (1969) 43-60.
[7] S. Gähler, Untersuchungen uber verallgemeinerte m-metrische Raume, I, II, III, Math. Nachr. 40 (1969) 165-189.
[8] A. H. Siddiqi, 2-normed spaces, Aligarh Bull. Math. (1980) 53-70.
[9] L.A. Zadeh, Fuzzy sets, Inform. Control, 8 (1965) 338-353.
[10] A.K. Katsaras, Fuzzy topological vector spaces, Fuzzy Sets and Systems, 12 (1984) 143-154.
[11] C. Felbin, Finite dimensional fuzzy normed linear spaces, Fuzzy Sets and Systems, 48 (1992) 239-248.
[12] O. Kaleva, S. Seikkala, On fuzzy metric spaces, Fuzzy Sets and Systems, 12 (1984) 215-229.
[13] K.T. Atanassov, Intuitionistic fuzzy sets, Fuzzy Sets and Systems, 20(1) (1986) 87-96.
[14] J.H. Park, Intuitionistic fuzzy metric spaces, Chaos, Solitons and Fract. 22 (2004) 1039-1046.
[15] R. Saadati, J.H. Park, On the intuitionistic fuzzy topological spaces, Chaos, Solitons and Fract. 27 (2006) 331-344.
[16] S. Karakuş, K. Demirci, O. Duman, Statistical convergence on intuitionistic fuzzy normed spaces, Chaos, Solitons and Fract. 35 (2008) 763-769.
[17] A. Alotaibi, On lacunary statistical convergence of double sequences with respect to the intuitionistic fuzzy normed spaces, Int. J. Contemp. Math. Sciences, 5(42) (2010), 2069-2078.
[18] M. Mursaleen , S.A. Mohiuddine, Statistical convergence of double sequences in intuitionistic fuzzy normed space, Chaos, Solitons and Fract. 41(5) (2009) 2414-2421.
[19] M. Mursaleen, S.A. Mohiuddine, On lacunary statistical convergence with respect to the intuitionistic fuzzy normed space, J. Comp. Appl. Math. 233(2) (2009) 142-149.
[20] E. Savaş, M. Gürdal, Certain summability methods in intuitionistic fuzzy normed spaces, J. Intell. Fuzzy Syst. 27(4) (2014) 1621-1629.
[21] E. Savaş, M. Gürdal, A generalized statistical convergence in intuitionistic fuzzy normed spaces, Science Asia, 41 (2015) 289-294.
[22] P. Debnath, Lacunary ideal convergence in intuitionistic fuzzy normed linear spaces, Comput. Math. Appl. 63(3) (2012) 708-715.
[23] S. Goonatilake, Toward a Global Science, p. 126. Indiana University Press, (1998).

[24] T. Koshy, Fibonacci and Lucas Numbers with Applications. Wiley, New York (2001).

[25] E.E. Kara, M. Başarir, An application of Fibonacci numbers into infinite Toeplitz matrices, Caspian J. Math. Sci. 1(1) (2012) 43-47.

[26] E.E. Kara, Some topological and geometrical properties of new Banach sequence spaces, J. Ineq. Appl. (38) (2013) 15. doi: 10.1186/1029-242X-2013-38.

[27] V.A. Khan, E.E. Kara, H. Altaf, N. Khan, M. Ahmad, Intuitionistic fuzzy $\mathscr{I}$-convergent Fibonacci difference sequence spaces, J. Inequal. Appl. 202 (2019) 7 pages.

[28] M. İlkhan, E. E. Kara, A new Banach space defined by Euler totient matrix operator, Operators and Matrices, 13(2) (2019) 527-544.

[29] N. Turan, E.E. Kara, M. İlkhan, Quasi statistical convergence in cone metric spaces, Facta Univ. (NIS) Ser. Math. Inform. 33(4) (2018) 613-626.

[30] M. İlkhan, E. E. Kara, A new type of statistical Cauchy sequence and its relation to Bourbaki completeness, Cogent Mathematics and Statistics 5 (2018) 9 pages.

[31] E. E. Kara, M. İlkhan, Lacunary $\mathscr{I}$-convergent and lacunary $\mathscr{I}$-bounded sequence spaces defined by an Orlicz function, Electr. J. Math. Anal. Appl. 4(2) (2016) 150-159.

[32] E. E. Kara, M. İlkhan, On some Banach sequence spaces derived by a new band matrix, British J. Math. Comput. Sci. 9 (2) (2015) 141-159.

[33] E. E. Kara, M. Başarır, M. Mursaleen, Compactness of matrix operators on some sequence spaces derived by Fibonacci numbers, Kragujev. J. Math. 39 (2) (2015) 217-230.

[34] E. E. Kara, S. Demiriz, Some new paranormed difference sequence spaces derived by Fibonacci numbers, Miskolc Math. Notes, 16(2) (2015) 907-923.

[35] E. E. Kara, M. Ilkhan, Some properties of generalized Fibonacci sequence spaces, Linear Multilinear Algebra, 64(11) (2016), 2208-2223.

[36] A. Alotaibi, M. Mursaleen, BAS Alamri, S.A Mohiuddine, Compact operators on some Fibonacci difference sequence spaces, J. Inequal. Appl. 203 (2015) doi:10.1186/s13660-015-0713-5

[37] M. Candan, A new approach on the spaces of generalized Fibonacci difference null and convergent sequences, Math. Æterna, 5(1) (2015) 191-210.

[38] S. Demiriz, E. E. Kara, M. Başarır, On the Fibonacci almost convergent sequence space and Fibonacci core, Kyungpook Math. J. 55 (2015) 355-372.

[39] M. Kirişçi, A. Karaisa, Fibonacci statistical convergence and Korovkin type approximation theorems, J. Ineq. Appl. 229 (2017) doi: doi:10.1186/s13660-017-1503-z

[40] M. Kirişçi, Fibonacci statistical convergence on intuitionistic fuzzy normed spaces, J. Intell. Fuzzy Systems, 36 (2019) 5597-5604.

[41] Ö. Kişi, Fibonacci lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, under review.

[42] B. Schweizer, A. Sklar, Statistical metric spaces, Pacific J. Math. 10 (1960) 314-334.