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Using Lie Symmetry to solve first and second order Linear Differential Equation

Research Article

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Abstract: In this paper Lie group theory is used to reduce the order of ordinary differential equations. For an ordinary differential equation admitting one parameter Lie group symmetry, order of differential equation, in principle, can always be reduce by one. Ordinary differential equation admitted symmetry group Seperation of variables by canonical coordinates of admitted is also given.

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1. Introduction

Lie Group Theory was originally idea of Sophus Lie who was inspired by lecture of his fellow Norwegian Sylow on Galois Theory of solving algebraic equations, he was curious about how he can develop a similar theory for solving ordinary differential equations(ODE). Historically work of Sophus Lie faded into obscurity until it was re-discovered by his successors Vessiot, E. Cartan, E. and Birkhoff, G. who exploited the applications of Lie Group Theory to ordinary differential equations(ODE). The success of this theory is basically due to the perfection of the necessary tools of analysis and algebra, especially the availability with sufficiently useful hypotheses of the Implicit Function Theorem and the Existence-Uniqueness of Ordinary Differential Equations. In Lie Group Theory the invariance of ordinary differential equations(ODE) is studied under group of transformations called Lie Group transformations which are precisely characterised by their infinitesimals. Once admitted symmetry group for ordinary differential equation is recognized an algorithm can be developed for reducing order of differential equation plus quadratures thus leading to general solution. The beauty of Lie Group Theory Lies in the fact that the complicated non-linear conditions under continuous group action can be reduced to far simpler linear conditions.

Definition 1.1 (Lie point transformation [6]).
A One-parameter Lie group (or a Lie point transformation) is a group G with the set of transformations $A_\lambda : (x, y) \mapsto (\hat{x}, \hat{y}) = X$ depending on the parameter λ in $\mathbb{R}$ that takes points (x, y) to X such that the following conditions are satisfied.

- A_λ is bijective.

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- $A_{\lambda_1} \circ A_{\lambda_2} = A_{\lambda_1+\lambda_2}$
- $A_0 = I$
- For each λ_1 there exists a unique $\lambda_1 = -\lambda_2$ such that $A_{\lambda_1} \circ A_{\lambda_2} = A_0 = I$.

Definition 1.2 (Differential Equation [1]).
A differential equation is an equation between specified derivative on an unknown function, its values, and known quantities and functions, Many physical laws are most simply and naturally formulated as differential equations.

Definition 1.3 (Orbits [2]).
are an essential tool for solving differential equations using symmetry methods. Suppose there is a point A on a solution curve to a differential equation. Under a given symmetry, the orbit of A is the set of all points that A can be mapped to for all possible values of λ.

Definition 1.4 (The Tangent Vectors [2]).
The tangents to the orbit at any point $(\hat{x}, \hat{y})$ are described by the tangent vector in the x direction, denoted $\xi(\hat{x}, \hat{y})$ and the tangent vector in the y direction, denoted $\eta(\hat{x}, \hat{y})$. Thus

$$\frac{\partial \hat{x}}{\partial \lambda} = \xi(x, y),$$

and

$$\frac{\partial \hat{y}}{\partial \lambda} = \eta(x, y)$$

at the initial point (x, y), λ is equal to 0, therefore:

$$\left(\frac{\partial \hat{x}}{\partial \lambda}|_{\lambda=0}, \frac{\partial \hat{y}}{\partial \lambda}|_{\lambda=0}\right) = \big(\xi(x, y), \eta(x, y)\big)$$

Definition 1.5 (Infinitesimal operator [3]).
We define the infinitesimal operator as

$$X = \xi(x, y)\frac{\partial}{\partial x} + \eta(x, y)\frac{\partial}{\partial y}.$$

Definition 1.6 (The Symmetry Condition [4]).
The total derivative operator is important for understanding the symmetry condition. The total derivative operator is:

$$D_x = \frac{\partial}{\partial x} + \frac{dy}{dx}\frac{\partial}{\partial y} + \frac{d^2y}{dx^2}\frac{\partial}{\partial \acute{y}} + \cdots.$$

In general, we study the differential equation of the form:

$$\frac{dy}{dx} = w(x, y).$$

In order to satisfy the symmetry condition, the point $(\hat{x}, \hat{y})$ must be a solution to the differential $\frac{dy}{dx} = w(x, y)$. Now written with the derivative operator we get:

$$\begin{aligned}\frac{d\hat{y}}{d\hat{x}} &= \frac{D_x\hat{y}}{D_x\hat{x}}\\ &= \frac{\hat{y}_x + y'\hat{y}_y}{\hat{x}_x + y'\hat{x}_y}\\ &= w(\hat{x}, \hat{y}),\end{aligned}$$

since

$$y' = w(x, y),$$

therefore:

$$\frac{\hat{y}_x + w(x, y)\hat{y}_y}{\hat{x}_x + w(x, y)\hat{x}_y} = w(\hat{x}, \hat{y}).$$

Symmetry Condition.

Definition 1.7 (The Linearized Symmetry Condition [2]).
In order to find a symmetry, it is necessary to solve the symmetry condition

$$\frac{\hat{y}_x + w(x,y)\hat{y}_y}{\hat{x}_x + w(x,y)\hat{x}_y} = w(\hat{x},\hat{y})$$

This equation gives the symmetry $(x,y) \mapsto (\hat{x},\hat{y})$. If we could solve this equation for $\hat{x}$ and $\hat{y}$ then we could find the tangent vectors ξ and η We can linearize the symmetry condition using a Taylor series expansion. We can expand $\hat{x}$, $\hat{y}$ and $w(\hat{x},\hat{y})$ About $\lambda = 0$

$$\begin{aligned}
\hat{x} &= x + \lambda\xi(x,y) + O(\lambda^2),\\
\hat{y} &= x + \lambda\eta(x,y) + O(\lambda^2),\\
w(\hat{x},\hat{y}) &= w(x,y) + \lambda(w_x\xi(x,y) + w_y\eta(x,y)) + O(\lambda^2).
\end{aligned}$$

We wish to seek invariant of

$$\frac{dy}{dx} = w(x,y),$$

under infinitesimal transformation $\hat{x}$ and $\hat{y}$. then we get Lies invariance condition

$$\eta_x + (\eta_y - \xi_x)w - \xi_y w^2 = \xi w_x + \eta w_y,$$

Definition 1.8 (The Extended Operator [5]).
The symbole of the first extended operator is

$$\Gamma^{[1]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y'},$$

in which

$$\begin{aligned}
\eta^{(1)} &= D_x(\eta) - y' D_x(\xi)\\
&= \eta_x + (\eta_y - \xi_x)\omega - \xi_y\omega^2
\end{aligned}$$

is called the Linearized symmetry condition for first ordinary differential equation.
If we denote our ODE as Δ such that,

$$\Delta = \frac{dy}{dx} - w(x,y) = 0,$$

then the invariance condition is

$$\Gamma^{(1)}\Delta\,|_{\Delta=0} = 0,$$

and the symbol of the second extended operator is

$$\Gamma^{[2]} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \eta^{(1)}\frac{\partial}{\partial y'} + \eta^{(2)}\frac{\partial}{\partial y''}$$

in which

$$\begin{aligned}
\eta^{(2)} &= D_x(\eta^{(1)}) - y'' D_x(\xi),\\
&= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3
\end{aligned}$$

is called the Linearized symmetry condition for second order differential Equation.
If we denote our ODE as Δ,

$$\Delta = y'' - w(x,y,y'),$$

then the invariance condition is

$$\Gamma^{(2)}\Delta\,|_{\Delta=0} = 0.$$

Definition 1.9 (The Canonical Coordinates [3]).
In the simplest case we look for coordinates that admits a symmetry:

$$A_\lambda : (r,s) \longmapsto (\hat{r},\hat{s}) = (r, s+\lambda)$$

when

$$(r,s) = (r(x,y), s(x,y)),$$

such that

$$(\hat{r},\hat{s}) = (r(\hat{x},\hat{y}), s(\hat{x},\hat{y})).$$

Then the tangent vector at (r,s) when $\lambda = 0$ is

$$(\frac{\partial \hat{r}}{\partial \lambda}|_{\lambda=0}, \frac{\partial \hat{s}}{\partial \lambda}|_{\lambda=0}) = (0,1)$$

Taking derivatives with respect to λ at $\lambda = 0$, we get

$$\frac{\partial \hat{r}}{\partial \lambda}|_{\lambda=0} = \frac{\partial r}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial \lambda}|_{\lambda=0} + \frac{\partial r}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial \lambda}|_{\lambda=0} = \frac{\partial r}{\partial x}\xi(x,y) + \frac{\partial r}{\partial y}\eta(x,y) = 0$$

$$\frac{\partial \hat{s}}{\partial \lambda}|_{\lambda=0} = \frac{\partial s}{\partial \hat{x}}\frac{\partial \hat{x}}{\partial \lambda}|_{\lambda=0} + \frac{\partial s}{\partial \hat{y}}\frac{\partial \hat{y}}{\partial \lambda}|_{\lambda=0} = \frac{\partial s}{\partial x}\xi(x,y) + \frac{\partial s}{\partial y}\eta(x,y) = 1.$$

Or

$$r_x\xi(x,y) + r_y\eta(x,y) = 0 \tag{1}$$

$$s_x\xi(x,y) + s_y\eta(x,y) = 1 \tag{2}$$

Canonical coordinates can be obtained from Eq.(1) and Eq.(2) by using the method of characteristics . The characteristic equations are

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)} = ds \tag{3}$$

The first integral of a differential equation

$$\frac{dy}{dx} = f(x,y), \tag{4}$$

is a non constant functions whose value is constant along solution curves of equation . Then the general solution as follows :

$$\phi(x,y) = c$$

where c is constant .
If we apply the total derivative operator to the function $\phi(x,y)$,we get the following equation:

$$\phi_x + f(x,y)\phi_y = 0, \phi_y \neq 0. \tag{5}$$

Suppose that $\xi(x,y) \neq 0$, we have divided Eq. (1) by $\xi(x,y)$, we get

$$r_x + \frac{\eta(x,y)}{\xi(x,y)}r_y = 0.$$

Comparing this result to Eq. (5) , then

$$\frac{dy}{dx} = \frac{\eta(x,y)}{\xi(x,y)}. \tag{6}$$

So , r is found by solving Eq. (6) where $r(x,y) = c$ but the coordinates s(x,y) can use Eq. (3) .
Hence ,

$$s = \int \frac{dy}{\eta(x,y)} = \int \frac{dx}{\xi(x,y)}.$$

There is special case when $\xi(x,y) = 0$ and $\eta(x,y) \neq 0$
Then $r = x$ and $s = \int \frac{dy}{\eta(x,y)}$.

The objective of the changing is to write the differential equation in terms of r(x,y) and s(x,y) in order to solve more easier differential equation . Then , we can find $\frac{ds}{dr}$ using the cartesian coordinates .
Apply the total derivative operator to get the following :

$$\frac{ds}{dr} = \frac{s_x + \omega(x,y)s_y}{r_x + \omega(x,y)r_y}. \tag{7}$$

2. Solving First order ODEs using LSM

Example 2.1.
Consider the equation,

$$\frac{dy}{dx} = \frac{y}{x} + x. \tag{8}$$

substituting Eq.(8) into the linearized symmetry condition, Eq. (8) to get

$$\eta_x + (\eta_y - \xi_x)\omega - \xi_y \omega^2 - (\xi\omega_x + \eta\omega_y) = 0$$

$$\eta_x + (\eta_y - \xi_x)(\frac{y}{x} + x) - \xi_y(\frac{y}{x} + x)^2 - (\xi(1 - \frac{y}{x^2}) + \eta(\frac{1}{x})) = 0.$$

It is necessary to solve this equation for ξ and η. In its current form, this is a very differential task. Therefore, we can make an ansatz(ansatz!) about ξ and η. Suppose that $\xi = 0$ and η is a function of x only.Then we get

$$\eta_x - \frac{\eta}{x} = 0.$$

The differential equation is easily solved:

$$\int \frac{d\eta}{\eta} = \int \frac{dx}{x}$$

$$\ln \eta = \ln x + c_o$$

$$\eta = cx.$$

Now, we can find the canonical coordinates r and s. Since $\xi(x, y) = 0$, r=x. To find s we can solve

$$ds = \frac{dy}{\eta}.$$

Therefore

$$s = \int \frac{dy}{cx} = \frac{y}{cx}.$$

Now, set c=1 to get

$$(r(x, y), s(x, y)) = (x, \frac{y}{x}).$$

And

$$s_x = \frac{-y}{x^2}, s_y = \frac{1}{x}.$$

Now, we can substituting r and s into Eq. (7), to obtain

$$\frac{ds}{dr} = \frac{\frac{-y}{x^2} + \frac{1}{x}(\frac{y}{x} + x)}{1} = 1.$$

Therefore, s=r+k where k is a constant. substituting x and y back in, we get

$$\frac{y}{x} = x + k.$$

Then the general solution of Eq. (8) is

$$y = x^2 + kx.$$

3. Solving second order ODE using LSM

Example 3.1.
Consider the following second order differential equation:

$$y'' = 0. \tag{9}$$

The linearized symmetry condition for this ODE is

$$\eta^{(2)} = 0 \ \ when\ y'' = 0,$$

that is,

$$\eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy}y'^3 = 0 \tag{10}$$

As ξ and η are independent ofy', the linearized symmetry condition splits into the following system of determining equations:

$$\eta_{xx} = 0,\ 2\eta_{xy} - \xi_{xx} = 0,\ \eta_{yy} - 2\xi_{xy} = 0,\ \xi_{yy} = 0. \tag{11}$$

The general solution of the last of (11) is

$$\xi(x, y) = A(x)y + B(x),$$

for arbitrary functions A and B. The third of (11) gives

$$\eta(x, y) = A'(x)y^2 + C(x)y + D(x),$$

where C and D are also arbitrary functions. Then the remaining equations in (11) amount to

$$A'''(x)y^2 + C''(x)y + D''(x) = 0,\ 3A''(x)y + 2C'(x) - B''(x) = 0. \tag{12}$$

Equating power of y in (12), we obtain a system of ODEs for the unknown function s A, B, C, and D:

$$A''(x) = 0,\ C''(x) = 0,\ D''(x) = 0,\ B''(x) = 2C'(x).$$

These ODEs are easily solved, leading to the following result. For every one-parameter Lie group of symmetries of (9), the functions ξ and η are of the form

$$\xi(x, y) = c_1 + c_3x + c_5y + c_7x^2 + c_8xy,$$

$$\eta(x, y) = c_2 + c_4y + c_6x + c_7xy + c_8y^2,$$

where $c_1, ..., c_8$ are constants. Therefore the most general infinitesimal generator is

$$X = \sum_{i=1}^{8} c_i x_i,$$

where

$$X_1 = \partial x,\ X_2 = \partial y,\ X_3 = x\partial x,\ X_4 = y\partial y,\ X_5 = y\partial x,\ X_6 = x\partial y,\ X_7 = x^2\partial x + xy\partial y,\ X_8 = xy\partial x + y^2\partial y,$$

We have chosen

$$X_2 = \partial y.$$

Then, we have the tangent vectors

$$(\xi(x, y), \eta(x, y)) = (0, 1).$$

The simplest canonical coordinates are

$$r(x, y) = x,$$

$$s(x, y) = y.$$

which prolong to

$$\frac{ds}{dr} = y', \frac{d^2s}{dr^2} = \frac{D_x(y')}{D_x(x)} = y''.$$

Let

$$v = \frac{ds}{dr} = y'.$$

Thus

$$\frac{dv}{dr} = y'y'' = 0.$$

Hence, the Eq. (9) reduces to the following equation:

$$\frac{dv}{dr} = 0$$

The general solution of (9) is

$$y = c_1x + c_2.$$

4. Riccati Equations [5]

The general from of a Riccati ODE is

$$\frac{dy}{dx} = P(x)y^2 + Q(x)y + R(x) \tag{13}$$

Our goal is to find $\xi(x, y)$ and $\eta(x, y)$ that satisfies lie's invariance condition. We will assume $\xi(x, y) = 0$, giving Lie's invariance condition as

$$\eta_x + (P(x)y^2 + Q(x)y + R(x))\eta_y = (2P(x)y + Q(x))\eta. \tag{14}$$

One solution of 14 is

$$\eta = (y - y_1)^2 F(x),$$

where y_1 is one solution to Eq.13 and F satisfies

$$F' + (2Py_1 + Q)F = 0. \tag{15}$$

In order to solve for the canonical variables r and s. It is necessary to solve

$$(y - y_1)^2 F(x) r_y = 0,$$
$$(y - y_1)^2 F(x) s_y = 1,$$

Hence, we obtain

$$r = R(x), s = S(x) - \frac{1}{(y - y_1 F},$$

where R and S are arbitrary functions. setting

$$R(x) = x, \quad and \quad S(x) = 0$$

yields

$$x = r, \tag{16}$$

$$y = y_1 - \frac{1}{sF(r)}, \tag{17}$$

there by transforming the original Riccati Eq. 13 to

$$\frac{ds}{dr} = \frac{a(r)}{F(r)}.$$

It is interesting that the usual linearizing transformation is recovered using Lie's method.

Example 4.1.
Consider the Riccte equation:

$$\frac{dy}{dx} = xy^2 - \frac{2y}{x} - \frac{1}{x^3}, \qquad x \neq 0 \tag{18}$$

It has the following symmetry :

$$(\hat{x}, \hat{y}) = (e^{\lambda} x, e^{-2\lambda} y). \tag{19}$$

The tangent vectors are

$$\xi(x, y) = \frac{\partial \hat{x}}{\partial \lambda}|_{\lambda=0} = x,$$

and

$$\eta(x, y) = \frac{\partial \hat{y}}{\partial \lambda}|_{\lambda=0} = -2y. \tag{20}$$

Now we find r using the following:

$$\frac{dy}{dx} = \frac{\xi(x,y)}{\eta(x,y)} = \frac{-2y}{x}, \tag{21}$$

this E.21 is separable ODE. Hence, we can integrate it:

$$\int \frac{1}{y} dy = -2 \int \frac{1}{x} dx.$$

Therefore $y = cx^{-2}$ then,

$$r = c = x^2 y,$$

and

$$s = \ln x.$$

Then the equation reduces to

$$\frac{ds}{dr} = \frac{1}{r^2 - 1},$$

the general solution is

$$y = \frac{1}{2} \ln\left(\frac{r-1}{r+1}\right) + c,$$

which is equivalent to

$$y = \frac{-k - x^2}{x^4 - kx^2}, \tag{22}$$

the invariant solution curve $y = x^{-2}$ can be regarded as the limit of as k approaches infinity. The other invariant solution $y = -x^{-2}$ is obtained by setting $k = 0$ in Eq.22

5. Conclusion

The Lie symmetry method is finest of all techniques developed so far, based no systematic algorithm, equipped with Lie algebra and begin based on strong principles of group theory, Lie symmetry method can be applied to any kind of differential equation for symmetry reduction and thereby exact solution, many symbolic programmes are also developed for generating and solving determining equations for this method, symbolic manipulation packages are written in Maple for easy application of symmetry method.

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