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On Generalized (r,s)-numbers

Research Article

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Abstract: In this paper, we investigate the generalized (r, s)-sequence and we deal with, in detail, three special cases which we call them (r, s), Lucas (r, s) and modified (r, s) sequences. We present Binet's formulas, generating functions, Simson formulas, and the summation formulas for these sequences. Moreover, we give some identities and matrices related with these sequences.

MSC: 11B39 • 11B83

Keywords: $(r, s)-$numbers • Lucas (r, s) numbers • Generalized Fibonacci numbers

1. Introduction

In this paper, we investigate the generalized (r, s) sequence and we investigate, in detail, three special cases which we call them (r, s), Lucas (r, s) and modified (r, s) sequences. The sequence of Fibonacci numbers $\{F_n\}$ and the sequence of Lucas numbers $\{L_n\}$ are defined by

$$F_n = F_{n-1} + F_{n-2}, \ \ n \geq 2, \quad F_0 = 0, \ F_1 = 1,$$

and

$$L_n = L_{n-1} + L_{n-2}, \ \ n \geq 2, \quad L_0 = 2, \ L_1 = 1$$

respectively. The generalizations of Fibonacci and Lucas sequences lead to several nice and interesting sequences.

The generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0, W_1; r, s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [15]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \qquad W_0 = a, W_1 = b, \ \ n \geq 2 \tag{1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [14],[[16] and [20]. Now these numbers $\{W_n(a, b; r, s)\}$ are called Horadam numbers.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1) holds for all integer n.

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Table 1. A few special case of generalized Fibonacci sequences.

Name of sequence	$W_n(a,b;r,s)$	Binet Formula	OEIS[[31]]
Fibonacci	$W_n(0,1;1,1)=F_n$	$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2,1;1,1)=L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0,1;2,1)=P_n$	$\frac{(1+\sqrt{2})^n-(1-\sqrt{2})^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2,2;2,1)=Q_n$	$(1+\sqrt{2})^n+(1-\sqrt{2})^n$	A002203
Jacobsthal	$W_n(0,1;1,2)=J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2,1;1,2)=j_n$	$2^n+(-1)^n$	A014551

For some specific values of a,b,r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r,s and initial values.

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

Jacobsthal sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [1],[2],[4],[5],[6],[7],[8],[12],[18],[19],[25],[26],[30],[37],[38].

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [3],[9],[10],[13],[17],[21],[28],[29]]. For higher order Pell sequences, see [[22],[23],[33],[34],[35],[36].

In 1843, Binet gave a formula which is called "Binet's formula" for the usual Fibonacci numbers F_n by using the roots $\alpha_F=\frac{1+\sqrt{5}}{2}$, $\beta_F=\frac{1-\sqrt{5}}{2}$ of the characteristic equation $x^2-x-1=0$:

$$F_n=\frac{\alpha_F^n-\beta_F^n}{\alpha-\beta}.$$

Here α_F is called Golden Proportion (or Golden Number or Golden Section) (for details, see for example [27],[39],[40]).

Binet's formula of generalized Fibonacci sequence can be calculated using its characteristic equation (the quadratic equation) which is given as

$$x^2-rx-s=0.$$

The roots of characteristic equation are

$$\alpha=\frac{r+\sqrt{\Delta}}{2},\quad \beta=\frac{r-\sqrt{\Delta}}{2}.$$

where

$$\Delta=r^2+4s$$

and the followings hold

$$\begin{aligned}\alpha+\beta &= r,\\ \alpha\beta &= -s,\\ (\alpha-\beta)^2 &= (\alpha+\beta)^2-4\alpha\beta=r^2+4s.\end{aligned}$$

Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.1.
Binet's formula of generalized Fibonacci numbers is

$$W_n=\frac{b_1\alpha^n}{\alpha-\beta}+\frac{b_2\beta^n}{\beta-\alpha}=\frac{b_1\alpha^n-b_2\beta^n}{\alpha-\beta} \tag{2}$$

where

$$b_1=W_1-\beta W_0,\ b_2=W_1-\alpha W_0.$$

(2) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n \tag{3}$$

where

$$A_1 = \frac{W_1-\beta W_0}{\alpha-\beta},\ A_2 = \frac{W_1-\alpha W_0}{\beta-\alpha}.$$

Note that

$$A_1A_2 = \frac{(W_1^2-sW_0^2-rW_1W_0)}{-(r^2+4s)}.$$

The Binet's form of a sequence satisfying (2) for non-negative integers is valid for all integers n and we have the following formula

$$W_{-n} = \frac{\beta^n b_1-\alpha^n b_2}{\alpha^n\beta^n(\alpha^n b_1-\beta^n b_2)}W_n.$$

We can also give Binet's formula of the generalized (r,s) numbers (the generalized Fibonacci numbers) for the negative subscripts as follows: for $n=1,2,3,...$ we have

$$W_{-n} = \frac{-r+\alpha}{s}\frac{\alpha^{-n+1}b_1}{\alpha-\beta}+\frac{-r+\beta}{s}\frac{\beta^{-n+1}b_2}{\beta-\alpha}.$$

Next, we give the ordinary generating function $\sum\limits_{n=0}^{\infty}W_nx^n$ of the sequence $\{W_n\}$.

Lemma 1.1.
Suppose that $f_{W_n}(x)=\sum\limits_{n=0}^{\infty}W_nx^n$ *is the ordinary generating function of the generalized Fibonacci sequence* $\{W_n\}_{n\geq0}$. *Then,* $\sum\limits_{n=0}^{\infty}W_nx^n$ *is given by*

$$\sum_{n=0}^{\infty}W_nx^n = \frac{W_0+(W_1-rW_0)x}{1-rx-sx^2}. \tag{4}$$

Proof. Using the definition of generalized Fibonacci numbers, and substracting $rx\sum_{n=0}^{\infty}W_nx^n$ and $sx^2\sum_{n=0}^{\infty}W_nx^n$ from $\sum_{n=0}^{\infty}W_nx^n$ we obtain

$$\begin{aligned}
(1-rx-sx^2)\sum_{n=0}^{\infty}W_nx^n &= \sum_{n=0}^{\infty}W_nx^n - rx\sum_{n=0}^{\infty}W_nx^n - sx^2\sum_{n=0}^{\infty}W_nx^n\\
&= \sum_{n=0}^{\infty}W_nx^n - r\sum_{n=0}^{\infty}W_nx^{n+1} - s\sum_{n=0}^{\infty}W_nx^{n+2}\\
&= \sum_{n=0}^{\infty}W_nx^n - r\sum_{n=1}^{\infty}W_{n-1}x^n - s\sum_{n=2}^{\infty}W_{n-2}x^n\\
&= (W_0+W_1x)-rW_0x+\sum_{n=2}^{\infty}(W_n-rW_{n-1}-sW_{n-2})x^n\\
&= W_0+(W_1-rW_0)x.
\end{aligned}$$

Rearranging above equation, we obtain (4). □

We next find Binet's formula of generalized Fibonacci numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.2.
(Binet's formula of generalized Fibonacci numbers)

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)}+\frac{d_2\beta^n}{(\beta-\alpha)} \tag{5}$$

where

$$\begin{aligned}
d_1 &= W_0\alpha+(W_1-rW_0),\\
d_2 &= W_0\beta+(W_1-rW_0)\beta.
\end{aligned}$$

Proof. Let

$$h(x) = 1 - rx - sx^2.$$

Then for some α and β we write

$$h(x) = (1-\alpha x)(1-\beta x)$$

i.e.,

$$1 - rx - sx^2 = (1-\alpha x)(1-\beta x). \tag{6}$$

Hence $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ are the roots of $h(x)$. This gives α and β as the roots of

$$h(\frac{1}{x}) = 1 - \frac{r}{x} - \frac{s}{x^2} = 0.$$

This implies $x^2 - rx - s = 0$. Now, by (4) and (6), it follows that

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{(1-\alpha x)(1-\beta x)}.$$

Then we write

$$\frac{W_0 + (W_1 - rW_0)x}{(1-\alpha x)(1-\beta x)} = \frac{B_1}{(1-\alpha x)} + \frac{B_2}{(1-\beta x)}. \tag{7}$$

So

$$W_0 + (W_1 - rW_0)x = B_1(1-\beta x) + B_2(1-\alpha x).$$

If we consider $x = \frac{1}{\alpha}$, we get $W_0 + (W_1 - rW_0)\frac{1}{\alpha} = B_1(1 - \frac{\beta}{\alpha})$. This gives

$$B_1 = \frac{\alpha(W_0 + (W_1 - rW_0)\frac{1}{\alpha})}{(\alpha-\beta)} = \frac{W_0\alpha + (W_1 - rW_0)}{(\alpha-\beta)}.$$

Similarly, we obtain

$$B_2 = \frac{W_0\beta + (W_1 - rW_0)\beta}{(\beta-\alpha)}.$$

Thus (7) can be written as

$$\sum_{n=0}^{\infty} W_n x^n = B_1(1-\alpha x)^{-1} + B_2(1-\beta x)^{-1}.$$

This gives

$$\sum_{n=0}^{\infty} W_n x^n = B_1 \sum_{n=0}^{\infty} \alpha^n x^n + B_2 \sum_{n=0}^{\infty} \beta^n x^n = \sum_{n=0}^{\infty} (B_1\alpha^n + B_2\beta^n)x^n.$$

Therefore, comparing coefficients on both sides of the above equality, we obtain

$$W_n = B_1\alpha^n + B_2\beta^n$$

where

$$\begin{aligned} B_1 &= \frac{W_0\alpha + (W_1 - rW_0)}{(\alpha-\beta)}, \\ B_2 &= \frac{W_0\beta + (W_1 - rW_0)\beta}{(\beta-\alpha)}, \end{aligned}$$

and then we get (5). □

Note that from (2) and (5) we have

$$W_1 - \beta W_0 = W_0\alpha + (W_1 - rW_0), \tag{8}$$
$$W_1 - \alpha W_0 = W_0\beta + (W_1 - rW_0)\beta. \tag{9}$$

Now we define three special cases of the sequence $\{W_n\}$. (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$, Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ and modified (r,s) sequence $\{E_n(1,r-1;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$\begin{aligned} G_{n+2} &= rG_{n+1}+sG_n, \quad G_0=0, G_1=1, & (10)\\ H_{n+2} &= rH_{n+1}+sH_n, \quad H_0=2, H_1=r, & (11)\\ E_{n+2} &= rE_{n+1}+sE_n, \quad E_0=1, E_1=r-1. & (12)\end{aligned}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)}, \tag{13}$$

$$H_{-n} = -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)}, \tag{14}$$

$$E_{-n} = -\frac{r}{s}E_{-(n-1)} + \frac{1}{s}E_{-(n-2)}, \tag{15}$$

for $n=1,2,3,\ldots$ respectively. Therefore, recurrences (13), (14) and (15) hold for all integer n.

Next, we present the first few values of the (r,s), Lucas (r,s) and modified (r,s) numbers with positive and negative subscripts:

Table 2. The first few values of the special second-order numbers with positive and negative subscripts.

n	0	1	2	3	4
G_n	0	1	r	$s+r^2$	$r\left(2s+r^2\right)$
G_{-n}	....	$\frac{1}{s}$	$-\frac{r}{s^2}$	$\frac{1}{s^3}\left(s+r^2\right)$	$-\frac{r}{s^4}\left(2s+r^2\right)$
H_n	2	r	$2s+r^2$	$r\left(3s+r^2\right)$	$4r^2s+r^4+2s^2$
H_{-n}	....	$-\frac{r}{s}$	$\frac{2s+r^2}{s^2}$	$-r\frac{3s+r^2}{s^3}$	$\frac{4r^2s+r^4+2s^2}{s^4}$
E_n	1	$r-1$	$-r+s+r^2$	$-s+2rs-r^2+r^3$	$3r^2s-2rs-r^3+r^4+s^2$
E_{-n}	....	$-\frac{1}{s}$	$\frac{r+s}{s^2}$	$-\frac{s+rs+r^2}{s^3}$	$\frac{r^2s+2rs+r^3+s^2}{s^4}$

Some special cases of (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are as follows:

1. $G_n(0,1;1,1)=F_n$, Fibonacci sequence,
2. $H_n(2,1;1,1)=L_n$, Lucas sequence,
3. $G_n(0,1;2,1)=P_n$, Pell sequence,
4. $H_n(2,2;2,1)=Q_n$, Pell-Lucas sequence,
5. $G_n(0,1;1,2)=J_n$, Jacobsthal sequence,
6. $H_n(2,1;1,2)=j_n$, Jacobsthal-Lucas sequence.

For all integers n, (r,s), Lucas (r,s) and modified (r,s) numbers (using initial conditions in (2) or (5)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)},\\ H_n &= \alpha^n+\beta^n,\\ E_n &= \frac{(\alpha-1)\alpha^n}{(\alpha-\beta)} + \frac{(\beta-1)\beta^n}{(\beta-\alpha)},\end{aligned}$$

respectively.

Note that for all n we have

$$E_n = G_{n+1} - G_n,$$

and

$$
\begin{aligned}
G_{-n} &= \frac{-1}{\alpha^n\beta^n}G_n = \frac{-1}{(-s)^n}G_n, \ \ n\geq 1, \\
H_{-n} &= \frac{1}{\alpha^n\beta^n}H_n = \frac{1}{(-s)^n}H_n, \ \ n\geq 1, \\
E_{-n} &= \frac{-(\alpha^n\left(\beta-1\right)-\beta^n(\alpha-1))}{\alpha^n\beta^n(\alpha^n(\alpha-1)-\beta^n\left(\beta-1\right))}E_n, \ \ n\geq 1,
\end{aligned}
$$

Lemma 1.1 gives the following results as particular examples (generated functions of (r,s), Lucas (r,s) and modified (r,s) numbers).

Corollary 1.1.
Generated functions of (r,s), Lucas (r,s) and modified (r,s) numbers are

$$
\begin{aligned}
\sum_{n=0}^{\infty} G_n x^n &= \frac{x}{1-rx-sx^2}, \\
\sum_{n=0}^{\infty} H_n x^n &= \frac{2-rx}{1-rx-sx^2}, \\
\sum_{n=0}^{\infty} E_n x^n &= \frac{1-x}{1-rx-sx^2},
\end{aligned}
$$

respectively.

Proof. In Lemma 1.1, take $W_n = G_n$ with $G_0 = 1, G_1 = r$, $W_n = H_n$ with $H_0 = 2, H_1 = r$ and $W_n = E_n$ with $E_0 = 1, E_1 = r-1$, respectively. □

2. Simson Formulas

There is a well-known Simson Identity (formula) for Fibonacci sequence $\{F_n\}$, namely,

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n$$

which was derived first by R. Simson in 1753 and it is now called as Cassini Identity (formula) as well. This can be written in the form

$$\begin{vmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{vmatrix} = (-1)^n.$$

The following theorem gives generalization of this result to the generalized (r,s) sequence $\{W_n\}_{n\geq 0}$.

Theorem 2.1 (Simson Formula of Generalized Fibonacci Numbers).
For all integers n, we have

$$\begin{vmatrix} W_{n+1} & W_n \\ W_n & W_{n-1} \end{vmatrix} = (-1)^n s^n \begin{vmatrix} W_1 & W_0 \\ W_0 & W_{-1} \end{vmatrix}. \tag{16}$$

Proof. Eq. (16) is given in Soykan [32]. □
The previous theorem gives the following results as particular examples.

Corollary 2.1.
For all integers n, Simson formula of (r,s), Lucas (r,s) and modified (r,s) numbers are given as

$$
\begin{aligned}
\begin{vmatrix} G_{n+1} & G_n \\ G_n & G_{n-1} \end{vmatrix} &= (-1)^n s^{n-1}, \\
\begin{vmatrix} H_{n+1} & H_n \\ H_n & H_{n-1} \end{vmatrix} &= (-1)^{n+1} s^{n-1}(r^2+4s), \\
\begin{vmatrix} E_{n+1} & E_n \\ E_n & E_{n-1} \end{vmatrix} &= (-1)^{n+1} s^{n-1}(r+s-1),
\end{aligned}
$$

respectively.

3. Some Identities

In this section, we obtain some identities of (r,s), Lucas (r,s) and modified (r,s) numbers. First, we can give a few basic relations between $\{G_n\}$ and $\{H_n\}$.

Lemma 3.1.
The following equalities are true:

$$\begin{aligned}
s^3 H_n &= -(3rs+r^3)G_{n+4} + (4r^2s+r^4+2s^2)G_{n+3}, \qquad (17)\\
s^2 H_n &= (2s+r^2)G_{n+3} - (3rs+r^3)G_{n+2},\\
sH_n &= -rG_{n+2} + (2s+r^2)G_{n+1},\\
H_n &= 2G_{n+1} - rG_n,\\
H_n &= rG_n + 2sG_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
(r^2s^3+4s^4)G_n &= -(3rs+r^3)H_{n+4} + (4r^2s+r^4+2s^2)H_{n+3},\\
(r^2s^2+4s^3)G_n &= (2s+r^2)H_{n+3} - (3rs+r^3)H_{n+2},\\
(r^2s+4s^2)G_n &= -rH_{n+2} + (2s+r^2)H_{n+1},\\
(4s+r^2)G_n &= 2H_{n+1} - rH_n,\\
(4s+r^2)G_n &= rH_n + 2sH_{n-1}.
\end{aligned}$$

Proof. Note that all the identities hold for all integers n. We prove (17). To show (17), writing

$$H_n = a \times G_{n+4} + b \times G_{n+3}$$

and solving the system of equations

$$\begin{aligned}
H_0 &= a \times G_4 + b \times G_3\\
H_1 &= a \times G_5 + b \times G_4
\end{aligned}$$

we find that $a = -\frac{1}{s^3}(3rs+r^3)$, $b = \frac{1}{s^3}(4r^2s+r^4+2s^2)$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we present a few basic relations between $\{G_n\}$ and $\{E_n\}$.

Lemma 3.2.
The following equalities are true:

$$\begin{aligned}
s^3 E_n &= -(s+rs+r^2)G_{n+4} + (r^2s+2rs+r^3+s^2)G_{n+3}\\
s^2 E_n &= (r+s)G_{n+3} - (s+rs+r^2)G_{n+2}\\
sE_n &= -G_{n+2} + (r+s)G_{n+1}\\
E_n &= G_{n+1} - G_n\\
E_n &= (r-1)G_n + sG_{n-1}
\end{aligned}$$

and

$$\begin{aligned}
(rs^3-s^3+s^4)G_n &= (s-2rs+r^2-r^3)E_{n+4} + (3r^2s-2rs-r^3+r^4+s^2)E_{n+3},\\
(rs^2-s^2+s^3)G_n &= (-r+s+r^2)E_{n+3} + (s-2rs+r^2-r^3)E_{n+2},\\
(-s+rs+s^2)G_n &= -(r-1)E_{n+2} + \left(-r+s+r^2\right)E_{n+1},\\
(r+s-1)G_n &= E_{n+1} - (r-1)E_n,\\
(r+s-1)G_n &= E_n + sE_{n-1}.
\end{aligned}$$

Thirdly, we give a few basic relations between $\{H_n\}$ and $\{E_n\}$.

Lemma 3.3.
The following equalities are true:

$$\begin{aligned}
(rs^3-s^3+s^4)H_n &= (4r^2s-3rs-r^3+r^4+2s^2)E_{n+4}+(-5rs^2+4r^2s-5r^3s+r^4+2s^2-r^5)E_{n+3},\\
(rs^2-s^2+s^3)H_n &= (2s-3rs+r^2-r^3)E_{n+3}+(4r^2s-3rs-r^3+r^4+2s^2)E_{n+2},\\
(-s+rs+s^2)H_n &= (-r+2s+r^2)E_{n+2}+(2s-3rs+r^2-r^3)E_{n+1},\\
(r+s-1)H_n &= -(r-2)E_{n+1}+(-r+2s+r^2)E_n,\\
(r+s-1)H_n &= (r+2s)E_n+(2s-rs)E_{n-1},
\end{aligned}$$

and

$$\begin{aligned}
s^3(4s+r^2)E_n &= (r^2s+3rs+r^3+2s^2)H_{n+4}-(3rs^2+4r^2s+r^3s+r^4+2s^2)H_{n+3},\\
s^2(4s+r^2)E_n &= -(2s+rs+r^2)H_{n+3}+(r^2s+3rs+r^3+2s^2)H_{n+2},\\
s(4s+r^2)E_n &= (r+2s)H_{n+2}-(2s+rs+r^2)H_{n+1},\\
(4s+r^2)E_n &= (r-2)H_{n+1}+(r+2s)H_n,\\
(4s+r^2)E_n &= (-r+2s+r^2)H_n-(2s-rs)H_{n-1}.
\end{aligned}$$

4. Sums

The following theorem presents sum formulas of generalized (r,s) numbers.

Theorem 4.1.
For all integers m and j, we have

$$\sum_{k=0}^{n} W_{mk+j} = \frac{((-s)^m-H_m)W_{mn+j}+(-s)^m W_{mn+j-m}+W_j-(-s)^m W_{j-m}}{1+(-s)^m-H_m}. \tag{18}$$

Proof. Note that

$$\begin{aligned}
\sum_{k=0}^{n} W_{mk+j} &= W_{mn+j}+\sum_{k=0}^{n-1} W_{mk+j} = W_{mn+j}+\sum_{k=0}^{n-1}(A_1\alpha^{mk+j}+A_2\beta^{mk+j})\\
&= W_{mn+j}+A_1\alpha^j\left(\frac{\alpha^{mn}-1}{\alpha^m-1}\right)+A_2\beta^j\left(\frac{\beta^{mn}-1}{\beta^m-1}\right).
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (18) as required. □

As special cases of the above theorem, we have the following corollary.

Corollary 4.1.
For all integers m and j, we have

$$\begin{aligned}
\sum_{k=0}^{n} G_{mk+j} &= \frac{((-s)^m-H_m)G_{mn+j}+(-s)^m G_{mn+j-m}+G_j-(-s)^m G_{j-m}}{1+(-s)^m-H_m},\\
\sum_{k=0}^{n} H_{mk+j} &= \frac{((-s)^m-H_m)H_{mn+j}+(-s)^m H_{mn+j-m}+H_j-(-s)^m H_{j-m}}{1+(-s)^m-H_m},\\
\sum_{k=0}^{n} E_{mk+j} &= \frac{((-s)^m-H_m)E_{mn+j}+(-s)^m E_{mn+j-m}+E_j-(-s)^m E_{j-m}}{1+(-s)^m-H_m}.
\end{aligned}$$

The following theorem presents sum formulas of generalized (r,s) numbers.

Theorem 4.2.
For all integers m and j with $(r^2+4s)(1+(-s)^{2m}-H_{2m})((-s)^m-1)\neq 0$, we have

$$\sum_{k=0}^{n} W_{mk+j}^2 = \frac{\Omega}{(r^2+4s)(1+(-s)^{2m}-H_{2m})((-s)^m-1)}. \tag{19}$$

where

$\Omega = (r^2+4s)((-s)^m-1)((-s)^{2m}-H_{2m})W_{mn+j}^2+(r^2+4s)(-s)^{2m}((-s)^m-1)W_{mn-m+j}^2+(r^2+4s)((-s)^m-1)W_j^2-(r^2+4s)(-s)^{2m}((-s)^m-1)W_{j-m}^2+2(W_1^2-sW_0^2-rW_1W_0)(-s)^j((-s)^{mn}-1)(H_{2m}-2(-s)^m)$.

Proof. *Note that*

$$\sum_{k=0}^{n-1} \alpha^{mk+j} = \alpha^j \left(\frac{(\alpha^m)^n - 1}{\alpha^m - 1} \right)$$

and

$$\begin{aligned} \sum_{k=0}^{n} W_{mk+j}^2 &= W_{mn+j}^2 + \sum_{k=0}^{n-1} W_{mk+j}^2 = W_{mn+j}^2 + \sum_{k=0}^{n-1} (A_1\alpha^{mk+j} + A_2\beta^{mk+j})^2 \\ &= W_{mn+j}^2 + \sum_{k=0}^{n-1} (A_1^2\alpha^{2mk+2j} + A_2^2\beta^{2mk+2j} + 2A_1A_2\alpha^{mk+j}\beta^{mk+j}). \end{aligned}$$

Simplifying the last equalities in the last two expression imply (19) as required. □

As special cases of the above theorem, we have the following corollary (take $r = s = 1$).

Corollary 4.2. **(a)** *For all integers m and j with* $((-1)^m - 1)(L_{2m} - 2) \neq 0$ *(i.e m is odd or −m is odd or* $m \neq 0$*), we have*

$$\sum_{k=0}^{n} F_{mk+j}^2 = \frac{5(L_{2m}-1)F_{j+mn}^2 - 5F_{j-m+mn}^2 - 5F_j^2 + 5F_{j-m}^2 + (-1)^j (L_{2m} - 2(-1)^m)((-1)^{mn} - 1)}{5(L_{2m}-2)}.$$

(b) *For all integers m and j with* $((-1)^m - 1)(L_{2m} - 2) \neq 0$ *(i.e m is odd or −m is odd or* $m \neq 0$*), we have*

$$\sum_{k=0}^{n} L_{mk+j}^2 = \frac{(L_{2m}-1)L_{mn+j}^2 - L_{j-m+mn}^2 - L_j^2 + L_{j-m}^2 - (-1)^j (L_{2m} - 2(-1)^m)((-1)^{mn} - 1)}{(L_{2m}-2)}.$$

(c) *For all integers m and j with* $((-1)^m - 1)(Q_{2m} - 2) \neq 0$ *(i.e m is odd or −m is odd or* $m \neq 0$*), we have*

$$\sum_{k=0}^{n} P_{mk+j}^2 = \frac{8(Q_{2m}-1)P_{mn+j}^2 - 8P_{mn-m+j}^2 - 8P_j^2 + 8P_{j-m}^2 + (-1)^j (Q_{2m} - 2(-1)^m)((-1)^{mn} - 1)}{8(Q_{2m}-2)}.$$

(d) *For all integers m and j with* $((-1)^m - 1)(Q_{2m} - 2) \neq 0$ *(i.e m is odd or −m is odd or* $m \neq 0$*), we have*

$$\sum_{k=0}^{n} Q_{mk+j}^2 = \frac{(Q_{2m}-(-1)^{2m})Q_{mn+j}^2 - Q_{mn-m+j}^2 - Q_j^2 + Q_{j-m}^2 - (-1)^j (Q_{2m} - 2(-1)^m)((-1)^{mn} - 1)}{(Q_{2m}-2)}.$$

As special cases of the above corollary, we have the following identities.

Corollary 4.3.
The following identities hold:

1. $m = 1,\ j = 0.$

(a)

$$\sum_{k=0}^{n} F_k^2 = 2F_n^2 - F_{n-1}^2 + (-1)^n.$$

(b)

$$\sum_{k=0}^{n} L_k^2 = 2L_n^2 - L_{n-1}^2 - 5(-1)^n + 2.$$

(c)

$$\sum_{k=0}^{n} P_k^2 = \frac{1}{32}(40P_n^2 - 8P_{n-1}^2 + 8(-1)^n).$$

(d)

$$\sum_{k=0}^{n} Q_k^2 = \frac{1}{4}(5Q_n^2 - Q_{n-1}^2 - 8(-1)^n + 8).$$

2. $m=-1,\ j=0.$

(a)

$$\sum_{k=0}^{n} F_{-k}^{2}=-F_{-n+1}^{2}+2F_{-n}^{2}+(-1)^{n}.$$

(b)

$$\sum_{k=0}^{n} L_{-k}^{2}=-L_{-n+1}^{2}+2L_{-n}^{2}-5(-1)^{n}+2.$$

(c)

$$\sum_{k=0}^{n} P_{-k}^{2}=\frac{1}{4}(-P_{-n+1}^{2}+5P_{-n}^{2}+(-1)^{n}).$$

(d)

$$\sum_{k=0}^{n} Q_{-k}^{2}=\frac{1}{4}(-Q_{-n+1}^{2}+5Q_{-n}^{2}-8(-1)^{n}+8).$$

3. $m=3,\ j=1.$

(a)

$$\sum_{k=0}^{n} F_{3k+1}^{2}=\frac{1}{16}(17F_{3n+1}^{2}-F_{3n-2}^{2}-4(-1)^{3n}+4).$$

(b)

$$\sum_{k=0}^{n} L_{3k+1}^{2}=\frac{1}{16}(17L_{3n+1}^{2}-L_{3n-2}^{2}+20(-1)^{3n}-12).$$

(c)

$$\sum_{k=0}^{n} P_{3k+1}^{2}=\frac{1}{196}(197P_{3n+1}^{2}-P_{3n-2}^{2}-25(-1)^{3n}+28).$$

(d)

$$\sum_{k=0}^{n} Q_{3k+1}^{2}=\frac{1}{196}(197Q_{3n+1}^{2}-Q_{3n-2}^{2}+200(-1)^{3n}-168).$$

4. $m=-3,\ j=-1.$

(a)

$$\sum_{k=0}^{n} F_{-3k-1}^{2}=\frac{1}{16}(-F_{-3n+2}^{2}+17F_{-3n-1}^{2}-4(-1)^{3n}+4).$$

(b)

$$\sum_{k=0}^{n} L_{-3k-1}^{2}=\frac{1}{16}(-L_{-3n+2}^{2}+17L_{-3n-1}^{2}+20(-1)^{3n}-12).$$

(c)

$$\sum_{k=0}^{n} P_{-3k-1}^{2}=\frac{1}{196}(-P_{-3n+2}^{2}+197P_{-3n-1}^{2}-25(-1)^{3n}+28).$$

(d)

$$\sum_{k=0}^{n} Q_{-3k-1}^{2}=\frac{1}{196}(-Q_{-3n+2}^{2}+197Q_{-3n-1}^{2}+200(-1)^{3n}-168).$$

5. $m=5,\ j=-2.$

(a)

$$\sum_{k=0}^{n} F_{5k-2}^2 = \frac{1}{121}(122F_{5n-2}^2 - F_{5n-7}^2 + 25(-1)^{5n} + 143).$$

(b)

$$\sum_{k=0}^{n} L_{5k-2}^2 = \frac{1}{121}(122L_{5n-2}^2 - L_{5n-7}^2 - 125(-1)^{5n} + 957).$$

(c)

$$\sum_{k=0}^{n} P_{5k-2}^2 = \frac{1}{6724}(6725P_{5n-2}^2 - P_{5n-7}^2 + 841(-1)^{5n} + 27716).$$

(d)

$$\sum_{k=0}^{n} Q_{5k-2}^2 = \frac{1}{6724}(6725Q_{5n-2}^2 - Q_{5n-7}^2 - 6728(-1)^{5n} + 235176).$$

6. $m = -5,\ j = -2.$

(a)

$$\sum_{k=0}^{n} F_{-5k-2}^2 = \frac{1}{121}(-F_{-5n+3}^2 + 122F_{-5n-2}^2 + 25(-1)^{5n} - 22).$$

(b)

$$\sum_{k=0}^{n} L_{-5k-2}^2 = \frac{1}{121}(-L_{-5n+3}^2 + 122L_{-5n-2}^2 - 125(-1)^{5n} + 132).$$

(c)

$$\sum_{k=0}^{n} P_{-5k-2}^2 = \frac{1}{6724}(6725P_{-5n-2}^2 - P_{3-5n}^2 + 841(-1)^{5n} - 820).$$

(d)

$$\sum_{k=0}^{n} Q_{-5k-2}^2 = \frac{1}{6724}(6725Q_{-5n-2}^2 - Q_{3-5n}^2 - 6728(-1)^{5n} + 6888).$$

5. Matrices related with Generalized (r,s) numbers

Matrix formulation of W_n can be given as

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}. \tag{20}$$

For matrix formulation (20), see [24]. In fact, Kalman gave the formula in the following form

$$\begin{pmatrix} W_n \\ W_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ r & s \end{pmatrix}^n \begin{pmatrix} W_0 \\ W_1 \end{pmatrix}.$$

We define the square matrix A of order 2 as:

$$A = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}$$

such that $\det A = -s$. From (1) we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n \\ W_{n-1} \end{pmatrix} \tag{21}$$

and from (20) (or using (21) and induction) we have

$$\begin{pmatrix} W_{n+1} \\ W_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix}^n \begin{pmatrix} W_1 \\ W_0 \end{pmatrix}.$$

If we take $W_n = G_n$ in (21) we have

$$\begin{pmatrix} G_{n+1} \\ G_n \end{pmatrix} = \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} G_n \\ G_{n-1} \end{pmatrix}. \tag{22}$$

We also define

$$B_n = \begin{pmatrix} G_{n+1} & sG_n \\ G_n & sG_{n-1} \end{pmatrix}$$

and

$$C_n = \begin{pmatrix} W_{n+1} & sW_n \\ W_n & sW_{n-1} \end{pmatrix}$$

Theorem 5.1.
For all integer $m, n \geq 0$, we have

(a) $B_n = A^n$

(b) $C_1 A^n = A^n C_1$

(c) $C_{n+m} = C_n B_m = B_m C_n$.

Proof.

(a) By expanding the vectors on the both sides of (22) to 2-colums and multiplying the obtained on the right-hand side by A, we get

$$B_n = AB_{n-1}.$$

By induction argument, from the last equation, we obtain

$$B_n = A^{n-1} B_1.$$

But $B_1 = A$. It follows that $B_n = A^n$.

(b) Using (a) and definition of C_1, (b) follows.

(c) We have

$$\begin{aligned} AC_{n-1} &= \begin{pmatrix} r & s \\ 1 & 0 \end{pmatrix} \begin{pmatrix} W_n & sW_{n-1} \\ W_{n-1} & sW_{n-2} \end{pmatrix} \\ &= \begin{pmatrix} rW_n + sW_{n-1} & W_{n-2}s^2 + rW_{n-1}s \\ W_n & sW_{n-1} \end{pmatrix} = C_n \end{aligned}$$

i.e. $C_n = AC_{n-1}$. From the last equation, using induction we obtain $C_n = A^{n-1} C_1$. Now

$$C_{n+m} = A^{n+m-1} C_1 = A^{n-1} A^m C_1 = A^{n-1} C_1 A^m = C_n B_m$$

and similarly

$$C_{n+m} = B_m C_n.$$

□

Some properties of matrix A^n can be given as

$$A^n = rA^{n-1} + sA^{n-2}$$

and

$$A^{n+m} = A^n A^m = A^m A^n$$

and

$$\det(A^n) = (-s)^n$$

for all integer m and n.

Theorem 5.2.
For $m, n \geq 0$ we have

$$W_{n+m} = W_n G_{m+1} + s W_{n-1} G_m. \tag{23}$$

Proof. From the equation $C_{n+m} = C_n B_m = B_m C_n$ we see that an element of C_{n+m} is the product of row C_n and a column B_m. From the last equation we say that an element of C_{n+m} is the product of a row C_n and column B_m. We just compare the linear combination of the 2nd row and 1st column entries of the matrices C_{n+m} and $C_n B_m$. This completes the proof. □

Remark 5.1.
By induction, it can be proved that for all integers $m, n \leq 0$, (23) holds. So for all integers m, n, (23) is true.

Corollary 5.1.
For all integers m, n, we have

$$\begin{aligned} G_{n+m} &= G_n G_{m+1} + s G_{n-1} G_m, \\ H_{n+m} &= H_n G_{m+1} + s H_{n-1} G_m, \\ E_{n+m} &= E_n G_{m+1} + s E_{n-1} G_m. \end{aligned}$$

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